Geometry

&

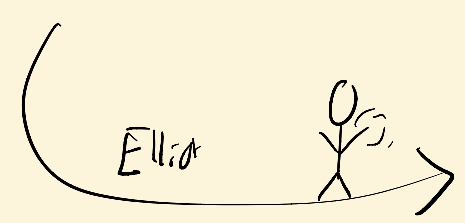
Physics

MATH 299G

Day 1

Math 298G

Geometry And Physics



ellIot . Kienzle @ gmail.com

Purpose of class:

not: - teach how to do problems
- prepare for future class

do: Show something cool
perspective you don't see often

STIC: I am peer
- Name is Dr danas, go through me
- Not scary math professor
- more laid back
- Qs

- What is ppls bkg?

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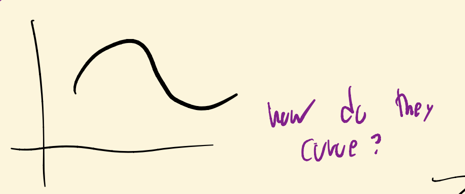
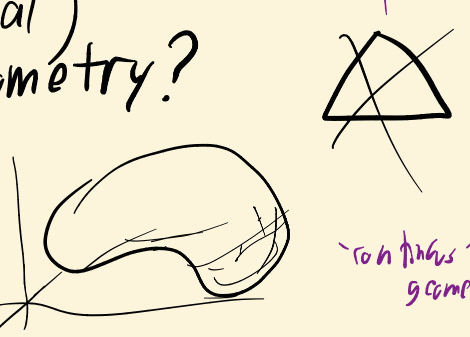
shapes
triangles? not in this house!

What is (Differential) Geometry?



Curvy shapes

how do they curve?



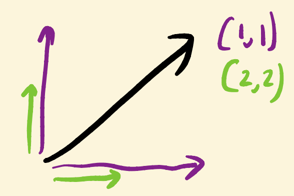
"continuous / differentiable geometry"

modern Diff geo: central themes

- Coordinate invariant

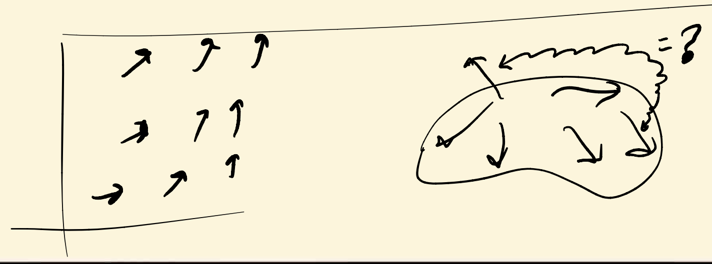
Properties of the underlying object independent of how you write it

Abstract
↓
Coordinate



- Local

data attached to each pt



esp important on blobs, no obv way to compare vectors @ different pts

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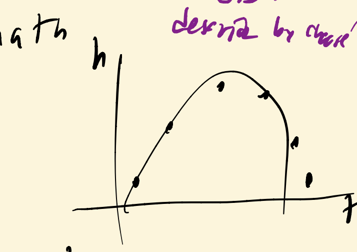
what is Physics?

- The 'real' world

example: throw in air

goal: capture experiments w/ math

e.g.: throw in air, describe by these rules



- Coordinate invariant
- General Covariance

The universe is not drawn on graph paper

h $d^2h/dt^2 = g$
$h/2$ $\frac{\ddot{h}}{2} = g$??

- Local
- field excitations propagate w/ pen

$\boxed{\text{Physics is geometry}}$ geometry is physics
Physics is geometry

If I wanted to be glib, call class "Geometry is physics"

Part 1: differential forms & E&M

Crown Jewel of 19th century physics

forces btwn charges are fields

$\leftarrow \oplus \quad \ominus \rightarrow$

Maxwell's eqs:

Magnetic field has no sources/sinks; Electric field sources/sinks are charges

$$\boxed{\begin{array}{ll}\nabla\cdot B = 0 & \nabla\cdot E = \rho \\ \nabla\times E - \frac{\partial B}{\partial t} = 0 & \nabla\times B + \frac{\partial E}{\partial t} = J\end{array}}$$

two types of E; Faraday's law; B entwined with E

Take whole course to learn these & implications PHYS 411

hard to see

Magnetic field B
Electric field E
charge density ρ
Current density J

$dF = 0 \qquad d{*}F = \mathcal{J}$

$ddA = 0 \qquad d{*}dA = \mathcal{J}$

trivial!

Electromagnetic field (both are 1!)

Potential

field 2-form $F = dA$
exterior derivative d

Captures $\nabla\cdot$, $\nabla\times$

Maxwell's Eqs!

$$\boxed{d{*}dA = \mathcal{J}}$$

Aesthetic

All the physical complexities (quantum) captured by simple PDE!

not sweeping complexity under rug - real physical

Seperate wheat from chaff - see what's going on

Appreciation Beyond just # of symbols

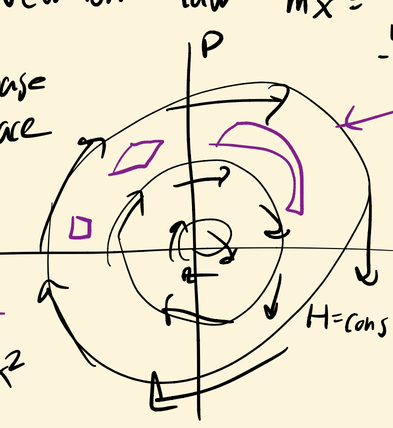
Part 2: Classical Mechanics & Symplectic geo

Newton's law $m\ddot{x} = F = -V' \Rightarrow \dot{x} = p/m, \ \dot{p} = F$

kinetic, potential

Hamiltonian $H = \frac{p^2}{2m} + V$
energy

Phase space



Particle flows w/ phase space vec field

$V = x^2$

$\left\{\begin{array}{l}\dot{x} = \partial H/\partial p \\ \dot{p} = -\partial H/\partial x\end{array}\right. \Rightarrow$ Symplectic structure X_H: $\omega(X_H, -) = dH$

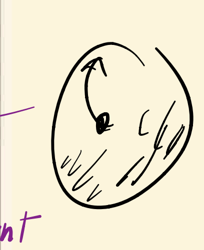
Hamiltonian dynamics

do this part later

right angle rotation

Q: How to do curvy?

A: use geom the invariant language



S^2 Phase space T^*S^2

Canonical Symplectic form

Liouville's thm: volume preserved

not trivially prove

$\nabla\cdot\left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial x}\right) = \frac{\partial H}{\partial p\partial x} - \frac{\partial H}{\partial x\partial p} = 0$

not enlightening

volume form ω^n

$\mathcal{L}_{X_H}(\omega^n) = \mathcal{L}_{X_H}(\omega)\,\omega^{n-1} = 0$, with $\mathcal{L}_{X_H}(\omega) = 0$

how does classical mech structure change under its flow? it doesn't! (by def)

new understanding of why

Throw eraser

Symmetries

Noether's Thm: Continuous symmetry ⟺ conserved quantity

Lie Groups ⟷ Lie algebras

Hamiltonians!
Momentum maps!

Boring logistics

- Assignments:
  - Don't want to be stressful! no grade worry
  - lots of dependency, to make no one fall behind:
    - weekly quiz on main pts (ez)
      - Due monday 2:00 pm
      - encourage procrastination
    - Exercises
    - Page references
  - Final project: Summarize a paper in couple of pages
    Discuss more later
- Libgen

- Office hours: After class, & later in the week
  - detail, exercises, any math-Phys thing
  - encourage
  - Math building
- Bonus hours? informal, cool stuff
  if interest
  - time
  - intro physics class?

don't know what I'm doin,

feedback more than welcomed!

- How was physics level??

Day 2

1-forms

please do ask Qs.

start @ the beginning

1-forms in E&M

Last time: Maxwells eqs $d \star d A = J$ (hodge star, exterior derivative, 1-form)

E-field is intrinsically the integrand

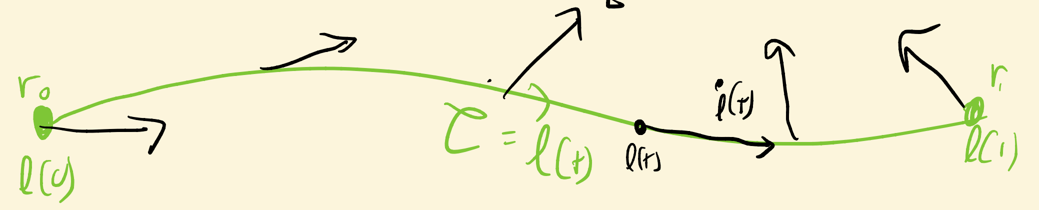
what is the integrand intrinsically?

Electric field: ① Force Vector $q \to E$ $F = -qE$

② $\text{Work} = -q\int_{\mathcal{C}} \underbrace{E \cdot d\ell}_{\mathcal{E}}$

Energy required to move along path

"one-form" minimal structure needed to line integrate



$$\boxed{\text{A 1-form is the integrand of a line } \int}$$

to evaluate integral

$\mathcal{C} = \ell(t)$ $t \in [0,1]$ $W = -q\int_0^1 dt\, E \cdot \dot\ell$, where $E \cdot \dot\ell = \mathcal{E}(\dot\ell)$

$\mathcal{E}$ rule sending vec. $\dot\ell\big|_{\ell(t)} \to \mathbb{R}$

need $\int_{\mathcal{C}} \mathcal{E}$ independent of choice of ℓ

$\int_0^1 dt\, \mathcal{E}(\dot\ell) = \int_0^1 \frac{dt}{2}\, \mathcal{E}(2\dot\ell) \Rightarrow \mathcal{E}(2\dot\ell) = 2\mathcal{E}(\dot\ell)$

$\mathcal{E}_p : T_p\mathbb{R}^n \to \mathbb{R}$ linear (just like $\vec{E}_p \cdot \_$!)

(note: every $\mathcal{E}$ is $E \cdot \_$ for some E)

$\mathcal{E} = E_x dx + E_y dy + E_z dz$

$E \cdot d\ell = E_x dx + E_y dy + E_z dz$ w/ $dx(\dot\ell) = \frac{\partial \ell_x}{\partial t}$, etc.

dual vector space: $\mathcal{E}_p \in T_p\mathbb{R}^{n*}$

$$\boxed{\text{A 1-form is a linear fn } \mathbb{R}^n \to \mathbb{R}}$$

Potential

$\vec{E} = \nabla\phi$ $\int_{\mathcal{C} = \ell(t)} \nabla\phi \cdot \dot\ell = \phi(\ell(1)) - \phi(\ell(0))$

"differential" $d\phi = \nabla\phi \cdot \_$ more fundamental than $\nabla\phi$

$d\phi = \partial_x\phi\, dx + \partial_y\phi\, dy + \partial_z\phi\, dz$

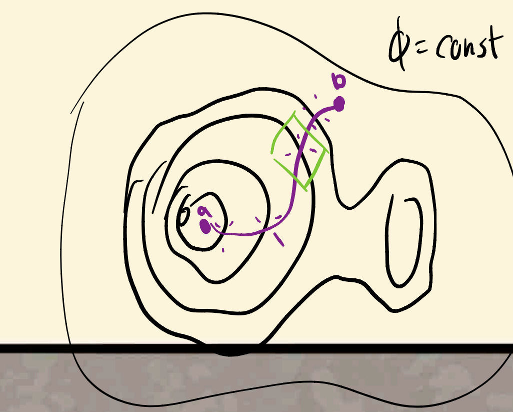
$d\phi$ exterior derivative

$\int_0^1 dt\, d\phi(\dot\ell) = \phi(\ell(1)) - \phi(\ell(0))$

$\int_0^1 dt\, \frac{d}{dt}\phi(\ell(t)) = \int_0^1 dt\, \nabla\phi \cdot \dot\ell$ directional derivative

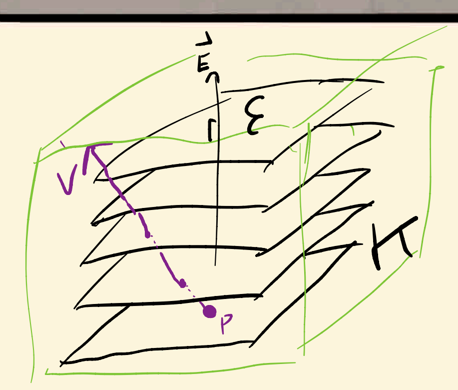
$$\boxed{d\phi \text{ holds directional derivatives of } \phi}$$

Visualizing 1-forms:



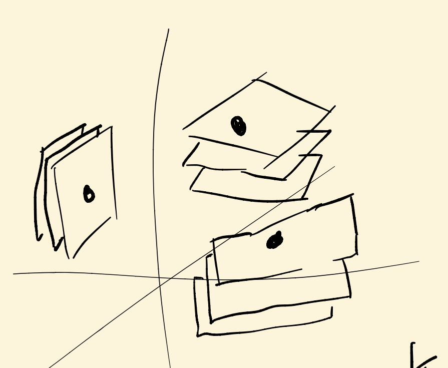
$\phi = \text{const}$

$\int_a^b d\phi$ counts # of hypersurfaces passed



$\mathcal{E}_p$ stack of planes K @ p

$\mathcal{E}_p(v)$ counts planes pierced by v



$K = \ker \mathcal{E}_p$

What planes? $v \in K \iff \mathcal{E}_p(v) = 0$

note: PTWISE picture, Not Local!!!

$$\boxed{\text{1-forms are like stacks of hyperplanes}}$$

Day 3

Last time:

Electric field $\vec{E} \Rightarrow$ 1-form $\mathcal{E}$

$\mathcal{E}$ integrand of line $\int$

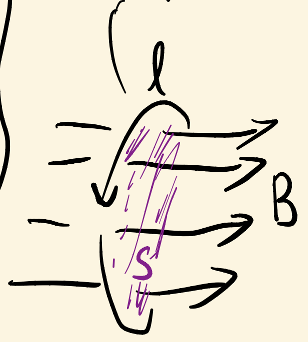
$\mathcal{E}_p$ linear map $T_p\mathbb{R}^n \to \mathbb{R}$

for potential ϕ, $d\phi = \mathcal{E}$

1-form stack of hyperplanes

2-forms in M & E

Magnetic field: B



induction!!

$I = \frac{d}{dt} \iint_S \underbrace{B \cdot \hat{n}}_{\mathcal{B}}$

2-form!

A 2-form is the integrand of a surface integral

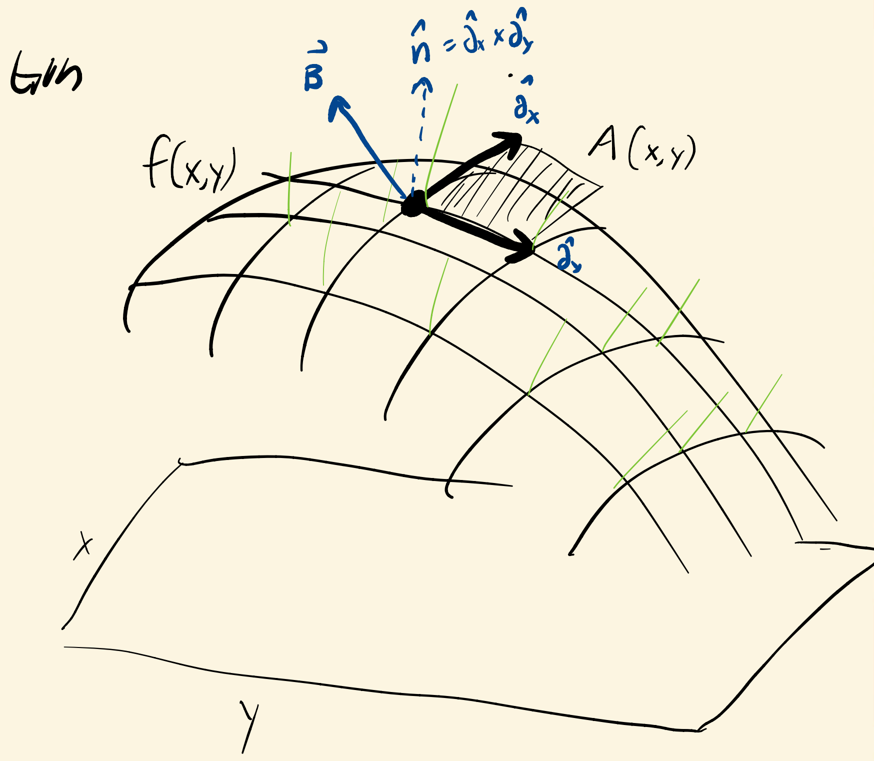
how do you take surface integrals? e.g.: surface area:

Surface Area

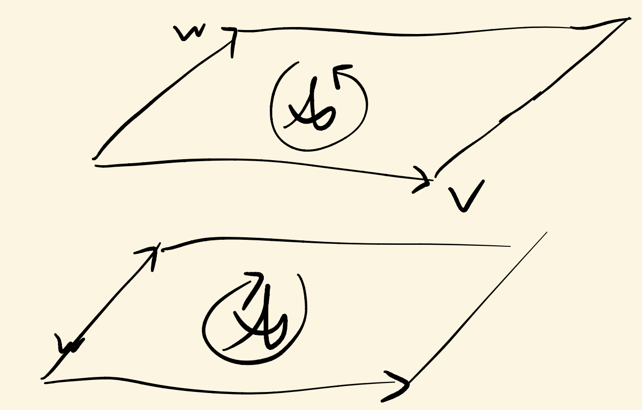
1) Choose parametrization

2) split into tesserae

3) add up areas



oriented area $\mathcal{A}(v,w)$



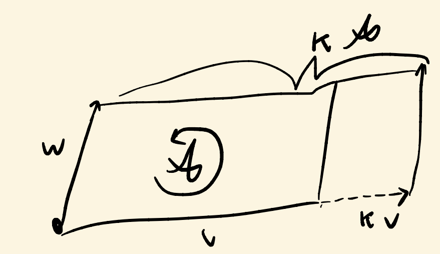
Antisymmetric

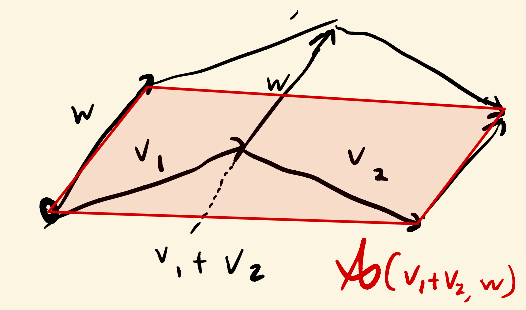
1) $\mathcal{A}(v,w) = -\mathcal{A}(w,v)$

Bilinear

2) $\mathcal{A}(kv,w) = k\mathcal{A}(v,w)$

3) $\mathcal{A}(v_1+v_2, w) = \mathcal{A}(v_1,w) + \mathcal{A}(v_2,w)$





note: in 2D, $\mathcal{A}(v,w) \propto \det\begin{pmatrix} v_1 & v_1 \\ v_2 & w_2 \end{pmatrix}$ antisymmetric ✓ bilinear ✓

in fact, det is the unique such fn

general 2-D integral $\iint_S \mathcal{B}$ modeled on S.A.:

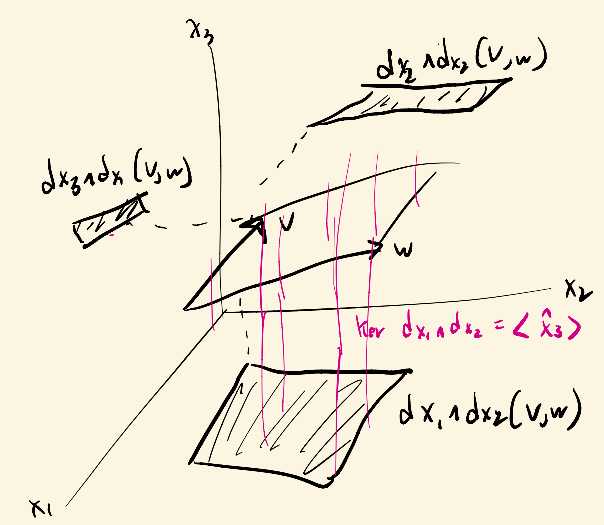
integrand "2-form" is Bilinear, antisymmetric map $\mathcal{B}_p: T_p\mathbb{R}^n \times T_p\mathbb{R}^n \to \mathbb{R}$

flux integral: $\iint_S \underbrace{B \cdot d\vec{S}}_{\Phi_B} = \iint_S (B \cdot \hat{n})\, dS = \iint_S B \cdot \hat{\partial}_x \times \hat{\partial}_y$

2-form is $\Phi_B(v,w) = \vec{B} \cdot (v \times w)$ antisym ✓ bilinear ✓

wedge product: $\alpha_1 \wedge \alpha_2 (v,w) := \alpha_1(v)\alpha_2(w) - \alpha_2(v)\alpha_1(w)$ antisym ✓ bilinear ✓

Wedge product $\wedge$ combines 1-forms into 2-forms



basis 2-forms

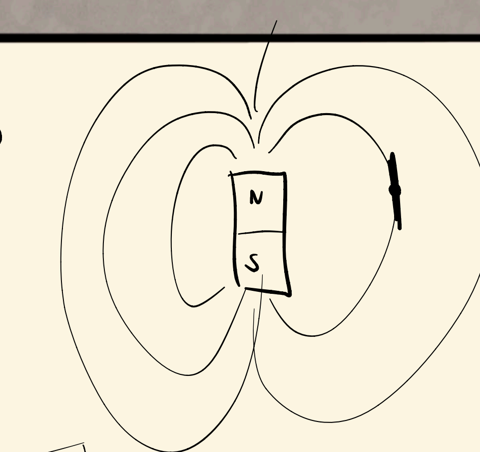
visualize 2-form $\mathcal{B}$: $\ker \mathcal{B} = \{v \mid \mathcal{B}(v,-)=0\}$

following 1-form rule

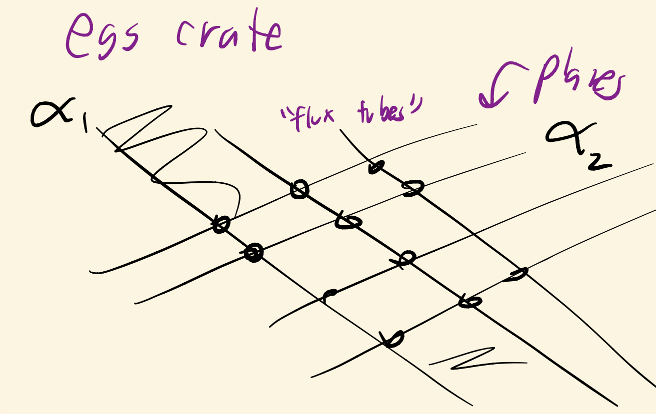
"flux lines"

- Direction
- magnitude (spacing)

$\alpha_1 \wedge \alpha_2$: $\ker(\alpha_1 \wedge \alpha_2) = \ker(\alpha_1) \cap \ker(\alpha_2)$



egg crate



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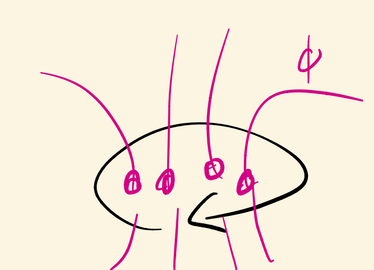
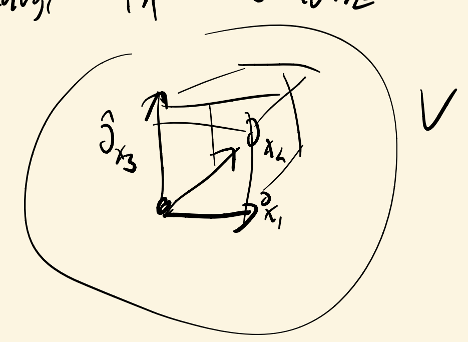
integrate: count piercing flux lines



Charge density ρ: want charge in volume $\iiint_V \rho$

1) Choose parametrization
2) split into tessar
3) sum $\rho \circ \mathrm{vol}(\partial_x, \partial_y, \partial_z)$



$\mathrm{Vol}(v_1, v_2, v_3)$ totally antisymmetric, multilinear

ρ "3-form"

$\boxed{\text{a } \underline{\text{k-form}} \text{ on } \mathbb{R}^n \text{ is a totally antisym, multilinear fn } (T_p\mathbb{R}^n)^k \to \mathbb{R}}$

note: in n dims, $>n+1$ forms always zero, by lin alg
can't have any 4D volume in 3D.

Day 4:

Last time:

=> multilinear, antisymmetric

$\beta(v,w) = -\beta(w,v)$

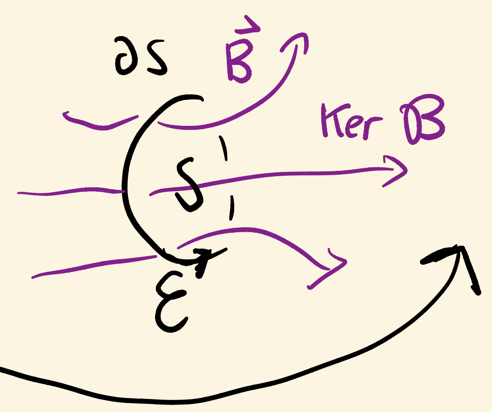
k-form: integrand over k-surface

Faraday's law:

integral

$\int_{\partial S} \vec{E}\cdot d\ell = -\int_S \dot{\vec{B}}\cdot\hat{n}\,ds$

$\int_{\partial S} \mathcal{E} = -\int_S \dot{\mathcal{B}}$



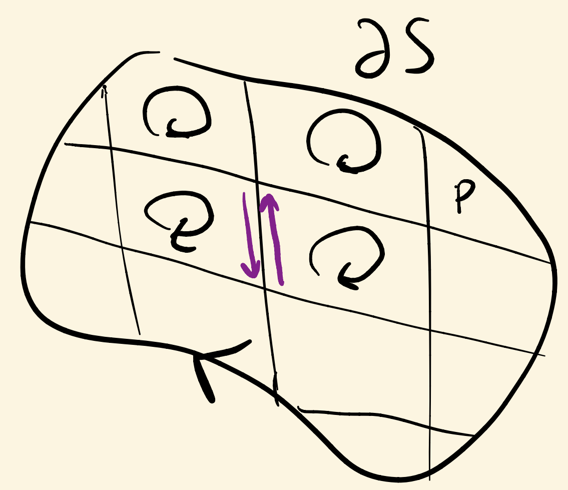
Differential:

$\nabla\times\vec{E} = -\dot{\vec{B}}$

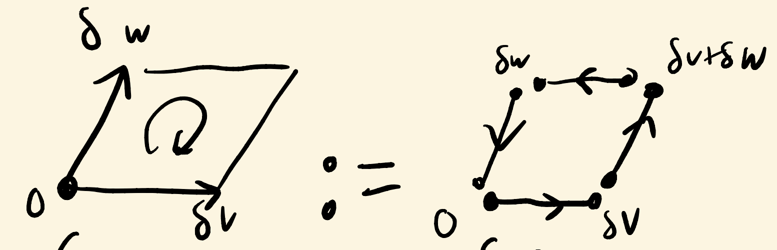
Stokes thm: $\int_S \nabla\times\vec{E}\cdot\hat{n}\,ds = \int_{\partial S}\vec{E}\cdot d\ell$

Goal: want 2-form $d\mathcal{E}$ s.t $\int_{\partial S}\mathcal{E} = \int_S d\mathcal{E}$

($d\mathcal{E}$ is exterior derivative)



$\int_{\partial S}\mathcal{E} = \sum_p \int_{\partial p}\mathcal{E}$



$\int_p d\mathcal{E} \quad := \quad \int_{\partial p}\mathcal{E}$

as $\delta\to 0$, $\frac{1}{\delta^2}\int_p d\mathcal{E} \to d\mathcal{E}|_0(v,w)$

$d\mathcal{E}(v,w) = \lim_{\delta\to 0}\frac{1}{\delta^2}\left(\int_0^{\delta v}\mathcal{E} + \int_{\delta v}^{\delta v+\delta w}\mathcal{E} + \int_{\delta v+\delta w}^{\delta w}\mathcal{E} + \int_{\delta w}^{0}\mathcal{E}\right)$

$\frac{1}{\delta}\left(\mathcal{E}_0(v) + \mathcal{E}_{\delta v}(w) + \mathcal{E}_{\delta w}(-v) + \mathcal{E}_0(-w)\right)$

$= \lim_{\delta\to 0}\frac{\mathcal{E}_0(v)-\mathcal{E}_{\delta w}(v)}{\delta} - \frac{\mathcal{E}_0(w)-\mathcal{E}_{\delta v}(w)}{\delta}$

$d\mathcal{E}(v,w) := \nabla_{\vec{w}}\mathcal{E}_x(v) - \nabla_{\vec{v}}\mathcal{E}_x(w)$

antisym ✓

bilinear ✓

in coords: $\mathcal{E} = E_x dx + E_y dy + E_z dz$

$$\left.\begin{aligned} d\mathcal{E}(\hat{y},\hat{z}) &= \partial_y E_z - \partial_z E_y \\ d\mathcal{E}(\hat{z},\hat{x}) &= \partial_z E_x - \partial_x E_z \\ d\mathcal{E}(\hat{x},\hat{y}) &= \partial_x E_y - \partial_y E_x \end{aligned}\right\} \quad \begin{aligned} d\mathcal{E} &= (\partial_y E_z - \partial_z E_y)\,dy\wedge dz + \dots \\ &= (\nabla\times E)_y\,dz\wedge dx + (\nabla\times E)_z\,dx\wedge dy + (\nabla\times\vec{E})_x\,dy\wedge dz \end{aligned}$$

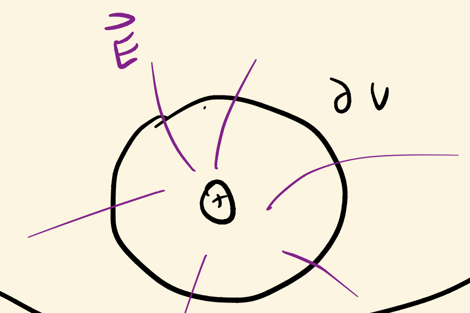
$\boxed{\mathcal{E} : \vec{E} \;::\; d\mathcal{E} : \nabla\times\vec{E}}$

$\int_{\partial S}\mathcal{E} = -\int_S\dot{\mathcal{B}} \Rightarrow \boxed{d\mathcal{E} = -\dot{\mathcal{B}}}$

Gauss's law

integral

$\int_V \rho = \int_{\partial V}\vec{E}\cdot\hat{n}\,ds$



diff.

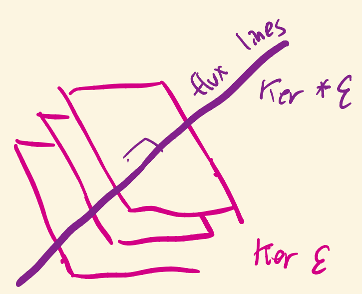
$\rho = \nabla\cdot\vec{E}$

Divergence thm

$\int_V \nabla\cdot E = \int_{\partial V} E$

$\int_V \overset{\text{3-form}}{\rho} = \int_{\partial V} \overset{\text{hodge}}{*}\mathcal{E}$ ✪

$*\mathcal{E}(v,w) = \mathcal{E}(v\times w)$



$*: \Omega^k \to \Omega^{n-k}$ (linear)

$* dx = dy\wedge dz$

$* dy = dz\wedge dx$

$* dz = dx\wedge dy$

$* 1 = dx\wedge dy\wedge dz$

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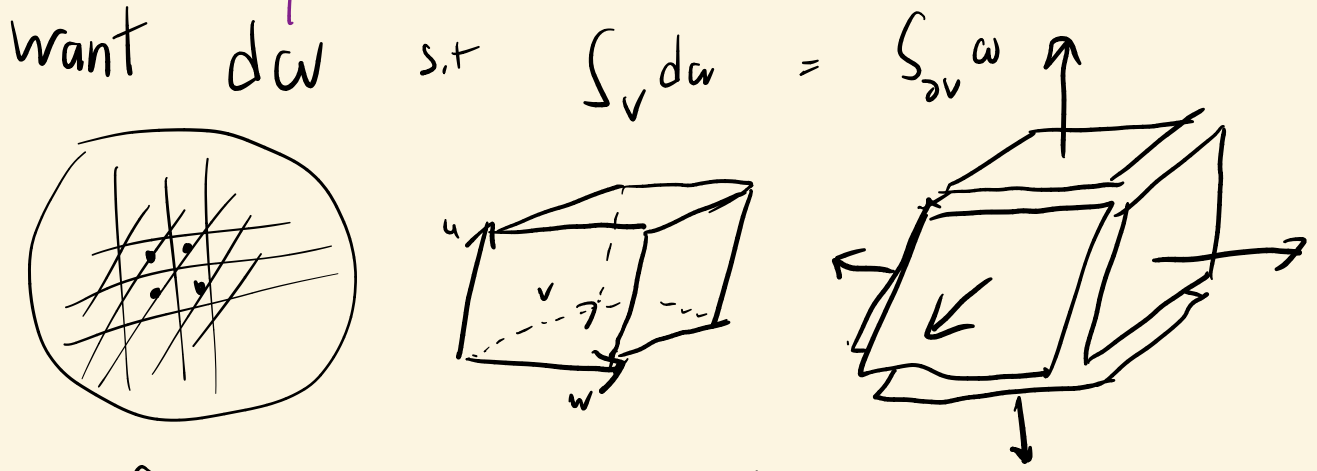
want $\overset{\text{2-form}}{d\omega}$ s.t $\int_V d\omega = \int_{\partial V}\omega$



$d\omega(u,v,w) = \nabla_u\,\omega(v,w) - \nabla_v\,\omega(w,u) + \nabla_w\,\omega(u,v)$

$*\mathcal{E} = \mathcal{E}_x\,dy\wedge dz + \mathcal{E}_y\,dz\wedge dx + \mathcal{E}_z\,dx\wedge dy$

$d*\mathcal{E} = (\partial_x\mathcal{E}_x + \partial_y\mathcal{E}_y + \partial_z\mathcal{E}_z)\,dx\wedge dy\wedge dz = (\nabla\cdot\vec{E})\,dx\wedge dy\wedge dz$

$\boxed{*d*\mathcal{E} = \vec{\nabla}\cdot\vec{E}}$

Day 5:

last time: exterior derivative,

$$\boxed{\text{Generalized Stokes Thm: } \int_M d\omega = \int_{\partial M} \omega}$$

Exterior derivative algebraic properties:

exterior algebra $\Omega^\bullet$, $\wedge: \Omega^q \times \Omega^{p-q} \to \Omega^p$ $d: \Omega^{k} \to \Omega^{k+1}$ (k-form)

1) $d: \Omega^0 \to \Omega^1$ sends fn ϕ to differential $d\phi$

2) linear: $d(\omega + \mu) = d\omega + d\mu$, $d(c\omega) = c\,d\omega$

3) product rule: $d(\omega \wedge \mu) = d\omega \wedge \mu + (-1)^k \omega \wedge d\mu$ $\omega = \sum \omega_I dx_{I_1} \wedge \dots \wedge dx_{I_k}$

need $d(\omega \wedge \mu) = -d(\mu \wedge \omega)$ for 1-forms

4) $d(d\omega) = 0$

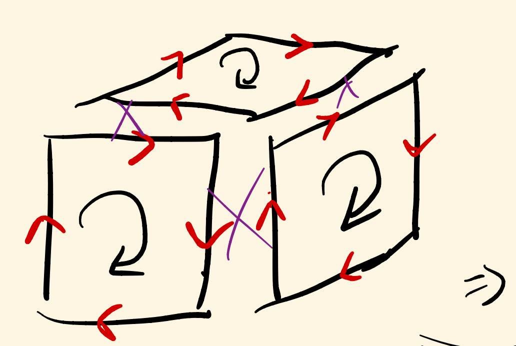
$\underline{d^2 = 0}$: $d\phi = \partial_x\phi\, dx + \partial_y\phi\, dy + \partial_z\phi\, dz$

$d\,d\phi = (\partial_x\partial_x\phi\, dx + \partial_y\partial_x\phi\, dy + \partial_z\partial_x\phi\, dz) \wedge dx + \dots$

$= \partial_x\partial_y\phi\, dx \wedge dy + \partial_y\partial_x\phi\, dy \wedge dx + \dots$

$= (\partial_x\partial_y\phi - \partial_y\partial_x\phi)\, dx \wedge dy + \dots = 0$ as mixed partials commute

geometrically:



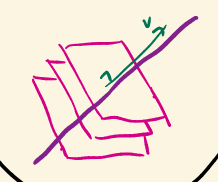
Stokes:

$0 = \int_M d^2\phi = \int_{\partial M} d\phi = \int_{\partial\partial M} \phi$.

$\Rightarrow \partial\partial M$ is empty

$$\boxed{\text{Bdry of a Bdry is empty}}$$

$$\begin{tikzcd}
0 \arrow[r, "d"] & \Omega^0 \arrow[r, "d"] & \Omega^1 \arrow[rr, "d"] & & \Omega^2 \arrow[r, "d"] & \Omega^3 \arrow[r, "d"] \arrow[llll, bend right, leftrightarrow, "\star \text{ hodge star}"'] & 0 \\
& & V\cdot_ & & V\cdot(_\times_) \text{ flux} \arrow[ll, bend right, leftrightarrow, phantom] & & \\
& & & V & & &
\end{tikzcd}$$



Vectors

$\phi \longmapsto \nabla\phi \longmapsto \nabla \times \nabla\phi = 0$

$E \longmapsto \nabla \times E \longmapsto \nabla \cdot \nabla \times E = 0$

$B \longmapsto \nabla \cdot B$

$\nabla \times E = d\mathcal{E}$ $\nabla \times B = d \star \mathcal{B}$

$\nabla \cdot E = d \star \mathcal{E}$ $\nabla \cdot B = d\mathcal{B}$

Maxwell's eqs

$\mathcal{E} = E_x dx + E_y dy + E_z dz \in \Omega^1(\mathbb{R}^3)$

$\mathcal{B} = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy \in \Omega^2(\mathbb{R}^3)$

$\star_1 \begin{cases} \nabla \cdot B = 0 \\ \nabla \times E + \partial_t B = 0 \end{cases}$ $\nabla \cdot \mathcal{E} = 0$, $\nabla \times B - \partial_t E = 0$

$\Downarrow$ $\Downarrow$

$\star_2 \begin{cases} d_s \mathcal{B} = 0 \\ d_s \mathcal{E} + \dot{\mathcal{B}} = 0 \end{cases}$ $d_s \star \mathcal{E} = \rho \in \Omega^3(\mathbb{R}^3)$, $d_s \star \mathcal{B} - \dot{\mathcal{E}} = J \in \Omega^2(\mathbb{R}^3)$

Define Faraday 2-form $F \in \mathcal{E} \wedge dt + \mathcal{B} \in \Omega^2(\mathbb{R}^4)$

$F = F_{\mu\nu}\, dx^\mu \wedge dx^\nu$ $F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ & 0 & B_x & -B_y \\ & & 0 & B_z \\ -\star & & & 0 \end{pmatrix}$

wax poetic about significance of combining $\mathcal{E}$ & $\mathcal{B}$ into field tensor?

$d = d_s + dt \wedge \partial_t$ $d(\mathcal{E} \wedge dt) = d_s\mathcal{E} \wedge dt + \cancel{\partial_t \mathcal{E}\, dt \wedge dt}$

$dF = d(\mathcal{E} \wedge dt) + d\mathcal{B}$ $d\mathcal{B} = d_s\mathcal{B} + dt \wedge \dot{\mathcal{B}}$

$= d_s\mathcal{B} + (d_s\mathcal{E} + \dot{\mathcal{B}})dt$

so, $dF = 0 \iff \begin{matrix} d_s\mathcal{B} = 0 \\ d_s\mathcal{E} + \dot{\mathcal{B}} = 0 \end{matrix}$!!!

$$\boxed{(\star_{\text{source-free}}) \iff dF = 0}$$

$\star F = \star\mathcal{B} \wedge dt - \star\mathcal{E}$

note: $\star dt$ gets extra - sign

"maxwell tensor"

$(\star F)_{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ & 0 & E_x & -E_y \\ - \star & & 0 & E_z \\ & & & 0 \end{pmatrix}$ $F \updownarrow \star F \sim \vec{E}, \vec{B} \updownarrow -\vec{B}, \vec{E}$

4-current $\mathcal{J} = -\rho + J \wedge dt \in \Omega^3(\mathbb{R}^4)$

$d \star F = d \star \mathcal{B} \wedge dt - d \star \mathcal{E}$ $d \star \mathcal{B} \wedge dt = d_s \star \mathcal{B} \wedge dt + \cancel{\star\dot{\mathcal{B}}\, dt \wedge dt}$

$d \star \mathcal{E} = d_s \star \mathcal{E} + \star\dot{\mathcal{E}} \wedge dt$

$= -d_s \star \mathcal{E} + (d_s \star \mathcal{B} - \star\dot{\mathcal{E}}) \wedge dt$

$d \star F = \mathcal{J} \iff \begin{matrix} d_s \star \mathcal{E} = \rho \\ d_s \star \mathcal{B} - \star\dot{\mathcal{E}} = J \end{matrix}$

$$\boxed{\star_{\text{source}} \iff d \star F = \mathcal{J}}$$

Potentials: $\vec{B} = \nabla \times A$, $\vec{E} = -\nabla\phi - \dot{A}$ $\Rightarrow$ $\mathcal{E} = -d_s\phi - \dot{A}$, $\mathcal{B} = d_s A$ ($A \in \Omega^1$)

4-potential $\mathcal{A} = \phi \wedge dt + A$

$d\mathcal{A} = dt \wedge d_s\phi + \cancel{\dot{\phi}\, dt \wedge dt} + d_s A + dt \wedge \dot{A} = d_s A + (-d_s\phi - \dot{A}) \wedge dt = \mathcal{B} + \mathcal{E} \wedge dt$

or, $F = d\mathcal{A}$

$$\boxed{\text{Maxwells eqs} \iff dd\mathcal{A} = 0 \quad d \star d\mathcal{A} = \mathcal{J}}$$

Gauge theory for HT

Day 6:

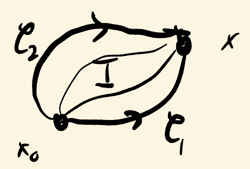
Last time: Maxwell's equations

$F = \mathcal{E} \wedge dt + \mathcal{B}$
$dF = 0$
$d * F = J$

Potentials: 1-form $\mathcal{A}$ s.t $F = d\mathcal{A}$ $\quad \mathcal{A} = \phi\, dt + A$ $\quad d\phi = \mathcal{E}$, $dA = \mathcal{B}$

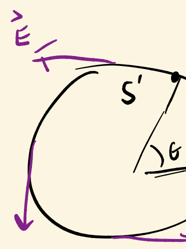
always exists for $d\mathcal{E} = 0$ on $\mathbb{R}^3$: $\phi(x) = \int_{\gamma} \mathcal{E}$ $\quad \gamma(0) = x_0$, $\gamma(1) = x$

ϕ well defined:

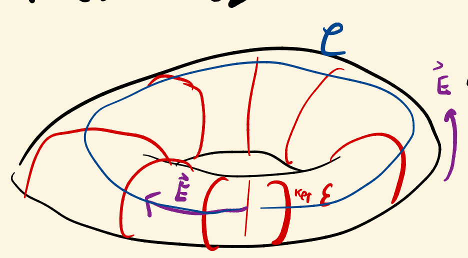


$\int_{\gamma_1 - \gamma_2} \mathcal{E} = \int_{I} d\mathcal{E} = 0 \Rightarrow \int_{\gamma_1} \mathcal{E} = \int_{\gamma_2} \mathcal{E}$

in general, $\overset{\text{"closed"}}{d\omega = 0}$ on $\mathbb{R}^n \Leftrightarrow \overset{\text{"exact"}}{\omega = d\nu}$



$\phi(\theta) = \int_0^\theta |\vec{E}|$
$\phi(2\pi) \neq \phi(0)$
ϕ not well defined!!



T^2 $\quad$ or other way...

consider $\mathcal{E} = \frac{d\theta}{2\pi}$ on torus

$\int_{\gamma} \frac{1}{2\pi} d\theta = \int_0^{2\pi} \frac{1}{2\pi} d\theta(\hat{\theta}) = 1$

so $\phi(x)$ not uniquely defined

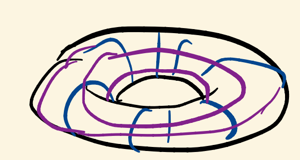
Fact: every closed form has potential Locally (Poincare lemma)
can fail to exist globally (Topology!!)

$\exists\ \mathcal{E}$ w/o potential $\Longleftrightarrow \exists$ non contractible loop

$$\frac{\{\mathcal{E} \in \Omega^1 \mid d\mathcal{E} = 0\}}{\{\mathcal{E} \in \Omega^1 \mid \mathcal{E} = d\phi\}} = \frac{\text{closed}}{\text{exact}} = H^1 \qquad \text{"de-Rahm cohomology"}$$

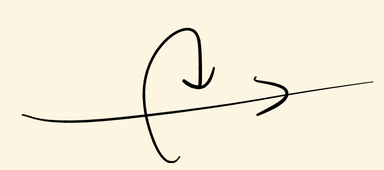
global potentials

counts holes



$H^1(T^2) = \mathbb{R}^2$

Maxwells eqs



wire

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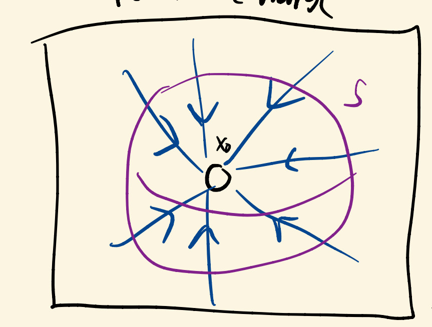
magnetic Point charge



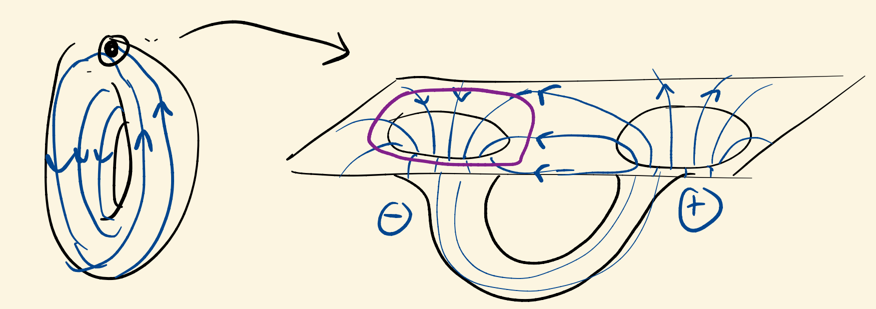
Analytic singularity: $\nabla \cdot \vec{B} = \rho_B \delta_{x_0}$

or, $M = \mathbb{R}^3 - \{x_0\}$ $\quad \nabla \cdot \vec{B} = 0$ $\quad d\mathcal{B} = 0$

$\int_S \mathcal{B} \neq 0 \Rightarrow \mathcal{B} \neq dA$, S non contractible

$$\frac{\mathcal{B}\text{-solutions}}{\text{global potentials}} = \frac{\text{closed}}{\text{exact}} = \frac{\{\mathcal{B} \mid d\mathcal{B} = 0\}}{\{\mathcal{B} \mid \mathcal{B} = dA\}} = H^2$$

wormhole



analytic & topological defects locally indistinguishable
Atiyah singer index thm

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maxwell's eqs: $dF = d^*F = 0 \Longleftrightarrow \overset{\text{hodge laplacian}}{\Delta F = (d^*d + dd^*)F = 0}$

pf: $\Rightarrow$ is clear

$\Leftarrow$) $d^*dF = -dd^*F \Rightarrow \|d^*dF\|^2 = -\langle d^*dF, dd^*F\rangle = -\langle dF, \cancel{d}dd^*F\rangle = 0 \Rightarrow d^*dF = 0,\ dd^*F = 0$

$d^*dF = 0 \Rightarrow 0 = \langle d^*dF, F\rangle = \langle dF, dF\rangle = \|dF\|^2 \Rightarrow dF = 0$

$dd^*F = 0 \Rightarrow d^*F = 0$

Q: potential $\mathcal{A}$ ambiguous up to $d\mathcal{A} = 0$. How to pick one potential in the class H^1?

inner product on Ω^p $\quad \langle \alpha, \beta \rangle := \int_M \overset{k}{\alpha} \wedge * \overset{n-k}{\beta}$

Non degerate: $\langle \alpha, \beta \rangle = 0\ \forall \beta \Rightarrow \alpha = 0$

$\{\text{Closed}\} = \{\text{exact}\} \oplus \underline{\{\text{exact}\}^\perp}$? ☆ hard

$\int d\alpha \wedge *\beta = \int d(\alpha \wedge *\beta) - \int \alpha \wedge d*\beta = \int \alpha \wedge *(*d*\beta)$

$\langle d\alpha, \beta \rangle = \langle \alpha, *d*\beta \rangle := \langle \alpha, d^*\beta \rangle$

$\beta \in \{\text{exact}\}^\perp \Rightarrow \underset{\forall \alpha}{\langle d\alpha, \beta \rangle = 0} \Longrightarrow \langle \alpha, *d*\beta \rangle = 0 \Rightarrow *d*\beta = 0$

Hodge Theorem: $H^p = \frac{\{\text{Closed}\}}{\{\text{exact}\}} = \left\{ \alpha \in \Omega^p \;\middle|\; \begin{matrix} d\alpha = 0 \\ d*\alpha = 0 \end{matrix} \right\}$ harmonic forms

$d^*: \Omega^{p+1} \to \Omega^p$ $\quad \overset{p}{*}\ \overset{n-p}{d}\ \overset{n-(p+1)}{*}\ \overset{p+1}{\beta}$

$d\alpha = d*\alpha = 0 \Longleftrightarrow \underset{\text{Laplacian}}{(d^*d + dd^*)}\alpha = 0$

$$\cdots \underset{d^*}{\overset{d}{\rightleftarrows}} \Omega^{p-1} \underset{d^*}{\overset{d}{\rightleftarrows}} \Omega^p \underset{d^*}{\overset{d}{\rightleftarrows}} \Omega^{p+1} \underset{d^*}{\overset{d}{\rightleftarrows}} \cdots$$

$d^*d + dd^*$

Vacuum maxwells eqs: $\begin{matrix} dF = 0 \\ d*F = J = 0 \end{matrix} \Rightarrow F$ harmonic

$\boxed{\text{On compact spacetime, } \{\text{solutions to vacuum Maxwells eqs}\} = H^2(M)}$

space of solutions to diffeq $\Longleftrightarrow$ topology

- Atiyah-Singer index thm :o

$M = \underset{\text{space}}{S} \times \underset{\text{time}}{\mathbb{R}}$ $\quad F$ time independent

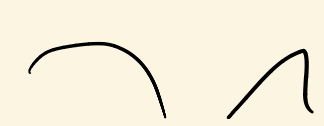
$\begin{matrix} dF = 0 \\ d^*F = 0 \end{matrix} \Rightarrow \begin{matrix} d_s \mathcal{E} = 0 \\ d_s^* \mathcal{E} = 0 \end{matrix}$ & $\begin{matrix} d_s^* B = 0 \\ d_s B = 0 \end{matrix}$

- Yang-Mills eqs: electromagnetism $\Longleftrightarrow$ weak force
higher rank electromagnetism

$D_A * \overset{\text{matrices}}{F_A} = 0$ $\quad$ non linear

space of solutions is hard & weird, but reflects topology of M

revolutionized 4-manifold topology



@ end:

- Next Half: symplectic geometry
- Final project
  - read a paper & give a short (~2 pg) report on what you understand from it
  - lots of cool stuff I wish I could talk abt
  - Motivation: very helpful skill = extracting story from hard stuff
    - See the forest w/o understanding the trees
    - Not graded on accuracy
  - Challenge yourself!
  - if you have specific interests in Math or physics, talk to me
  - due ~ last day of class
- Office Hours (for real this time)
- Feedback
  - Survey on elms

Classical Mechanics & Geometry!!!

2^{nd} Newton's law $F = m\ddot{q}$

Called 1st order: $\dot{q} = mv/m$, $m\dot{v} = F$ $\Rightarrow$ $\dot{q} = p/m$, $\dot{p} = F$

$F = -U'$ ← Potential

$H = \frac{p^2}{2m} + U$ 'Hamiltonian' total energy

$\partial H/\partial q = U' = -F$ $\partial H/\partial p = p/m$

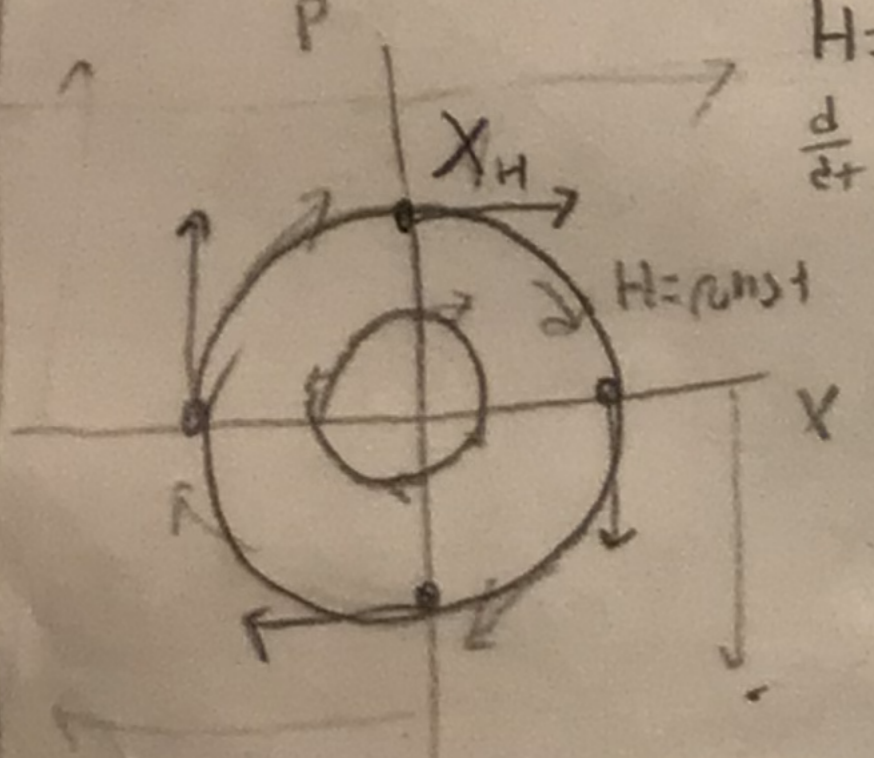
$F = m\ddot{q} \Leftrightarrow \begin{cases} \dot{q} = \partial H/\partial p \\ \dot{p} = \partial H/\partial q \end{cases}$ Hamilton's Equations

evolution uniquely determined by point on phase space

Phase space: e.g. $U(x) = q^2$, $m=1$

$H = p^2 + q^2$

$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} p \\ -q \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}$ ∇H rotate 90 deg $\Rightarrow$ energy constant



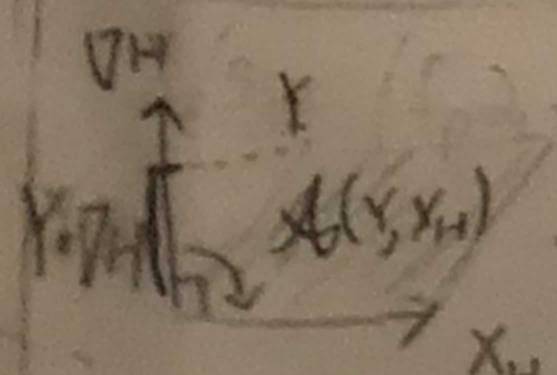
$\boxed{X_H = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right)}$

$X\_$ takes fn to vect field

"Hamiltonian vect. field"

Differential forms!!

$\omega = dq \wedge dp$ symplectic form



$A(X_H, Y) = \omega(X_H, Y) = Y \cdot \nabla H = dH(Y)$

$\boxed{i_{X_H}\omega := \omega(X_H, -) = dH}$ def of X_H

"interior derivative"

ω chooses X_H s.t. dH

$dH(X_H) = \omega(X_H, X_H) = 0$

Higher dimensions: $(q_1, \dots, q_n, p_1, \dots, p_n) \in \mathbb{R}^{2n}$

$X_H = \left(\frac{\partial H}{\partial p_1}, \dots, \frac{\partial H}{\partial p_n}, \frac{\partial H}{\partial q_1}, \dots, \frac{\partial H}{\partial q_n}\right)$

$i_{X_H}\omega = dH$ $\omega = dq_1 \wedge dp_1 + \dots + dq_n \wedge dp_n$

ω measures total area projected to each q_i, p_i plane

no kernel (sorry)

$X_H = \nabla H$ rotated $90°$ in each q_i, p_i plane

H generates time evolution thru X_H

$\frac{d}{dt} f(q(t), p(t)) = \underbrace{(\dot{q}, \dot{p})}_{X_H} \cdot \nabla f = df(X_H) = \omega(X_f, X_H) =: \{f, H\}$

Observables smooth fn on $\mathbb{R}^{2n}$ (energy, position, momentum, etc)

change in f under g evolution: $df(X_g) = \omega(X_f, X_g) = \{f, g\}$

Poisson bracket $\{f, g\} = \omega(X_f, X_g) = -\omega(X_g, X_f) = -\{g, f\}$

$= \sum f_{q_i} g_{p_i} - f_{p_i} g_{q_i}$

$H = q^2 + p^2$ $X_H = (p, -q)$ rotation

$H = p$ $X_H = (1, 0)$ translation

$H = q$ $X_H = (0, -1)$ momentum translation

Say $H = p^2 + U(q)$ U constant...

$X_p \cdot \nabla H = 0 \Rightarrow \{p, H\} = 0 \Rightarrow \{H, p\} = 0 \Rightarrow X_H \cdot \nabla p = 0$

So, p is conserved!!

Last time: "Phase space" $\mathbb{R}^{2n}$ "Hamiltonian" $\mathbb{R}^{2n} \to \mathbb{R}$ $\omega = \sum dq \wedge dp$

"time evolution" X_H satisfies $\omega(X_H, -) = dH$

Today: Manifolds!

configuration spaces: $\mathbb{R}$, S^1, S^2, $S^{1,2}$, $\mathbb{R}^2 \times S^1$

M

path $x: \mathbb{R} \to M$ or $x(t) \in M$ needs to be continuous: $d(x(t+\varepsilon), x(t)) \to 0$ as $\varepsilon \to 0$

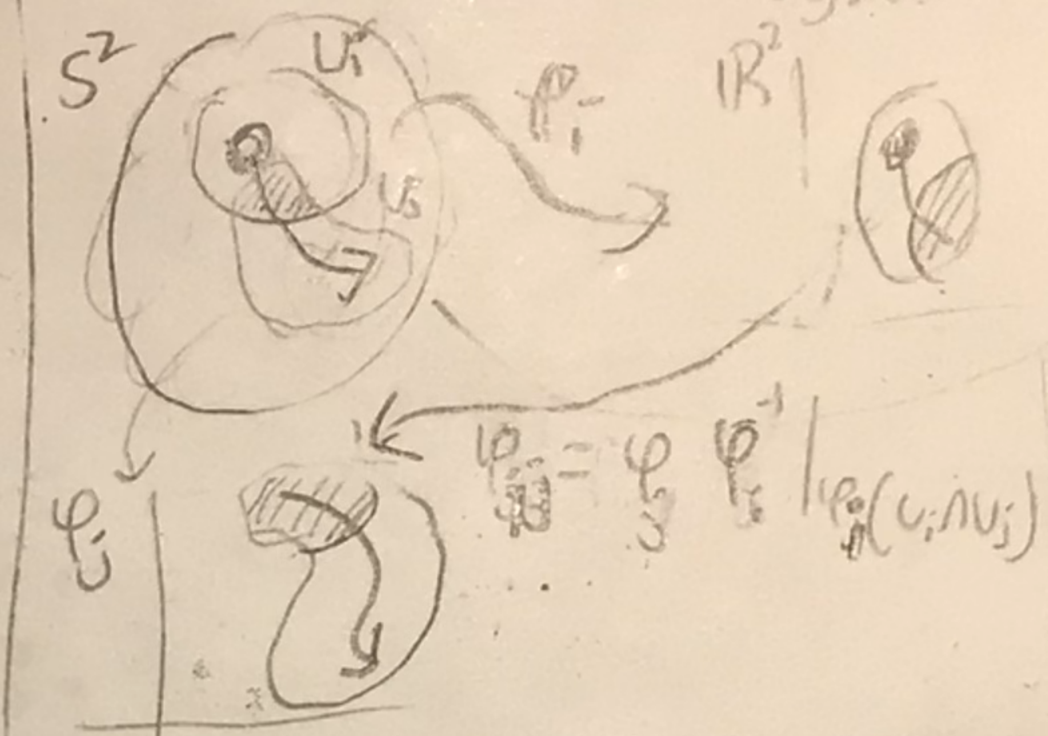
endowed w/ distance ("metric") $\Rightarrow$ M is topological space

needs to have velocity vector $\dot{x}$

How do you actually describe pts $x(t)$ on M??

e.g. $M = S^2$ ϕ θ weird!! $\subset \mathbb{R}^3$ weird

Ao Lucas Janszoon Waghenaer, "The Mariner's Mirror" 1584



S^2 U_i φ_i $\mathbb{R}^2$ U_j φ_j

$\varphi_{ij} = \varphi_j \varphi_i^{-1} |_{\varphi_i(U_i \cap U_j)}$

local charts $\varphi_i : U_i^M \longleftrightarrow \varphi(U_i)$

1-1, continuous, φ^{-1} continuous

Move between charts: φ_{ij}

Def: a differentiable Manifold is topological space M w/ "atlas" $\{\varphi_i, U_i\}$ of 'charts' $\varphi_i : U_i \to \varphi(U_i)$ s.t. φ_i homeo

- $M = \bigcup U_i$
- φ_{ij} differentiable

$\boxed{\text{Manifolds are where you do calculus}}$

what is $\dot{x}|_{x(t)=p}$? Use $\mathbb{R}^n$! $V_i := \frac{d}{dt} \varphi_i(x(t))$ need diffs

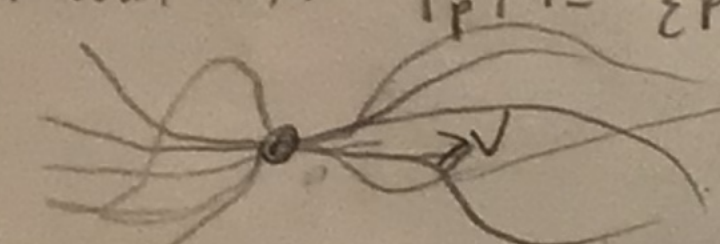
$V_j = \frac{d}{dt} \varphi_{ij}(\varphi_i(x(t))) = D\varphi_{ij} \frac{d}{dt} \varphi_i(x(t)) = D\varphi_{ij} V_i$

jacobian

So set of 'compatible' vectors $\{V_i\}$ s.t. $V_j = D\varphi_{ij} V_i$ "transforms like a vector"

$\{$vectors @ $p\} = T_pM$ "Tangent space @ p" dim n vector space

alternatively: $T_pM = \{\text{paths } x(t) \mid x(0) = p\} / x \sim x'$ if they agree to 1st order



v tangent to path $x(t)$

v is all paths $x(t)$ w/ tangent v

$v \in T_pM$ tangent vector $\omega: T_pM \to \mathbb{R}$ linearly 1-form

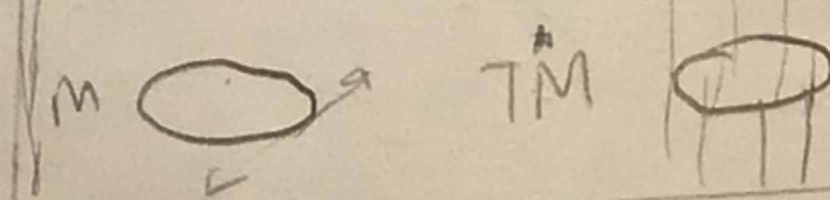
$\omega \in T_p^*M$ co tangent vector

local coords: $\omega_j = D\varphi_{ij}^{-1} \omega_i$ "Transforms like a covector"

likewise, can have $\Lambda^k T_p^*M$ k-form @ p

Space of pairs $(p, \omega \in T_p^*M) = \bigsqcup_{p \in M} T_p^*M = T^*M$ cotangent bundle

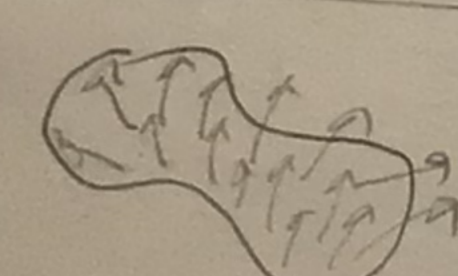
glue together all T_p^*M



M T^*M

T^*M itself a manifold!!

Vector field: $V: M \to TM$
$p \mapsto T_pM$



likewise, k-form field

Maxwells eqns on manifold M

$d \star d A = J$ w/ A 1-form field, J 3-form field

$\star$ needs "metric"

Day 8: manifolds

Last time: Hamiltonian mechanics H: "energy function" $H: \mathbb{R}^{2n} \to \mathbb{R}$

time evolution: $X_H = \left(\frac{\partial H}{\partial q_1}, \dots, \frac{\partial H}{\partial q_n}, \frac{\partial H}{\partial p_1}, \dots, \frac{\partial H}{\partial p_n}\right)$

Symplectic form $\omega = dq_1 \wedge dp_1 + \dots dq_n \wedge dp_n$

$\boxed{\omega(X_H, -) = dH}$

spherical pendulum S^2

Energy $= \frac{1}{2} m v^2 + U$

??? How to define v

Consider path $x: \mathbb{R} \to S^2$, then $v = \dot{x}(t) \dots ?$

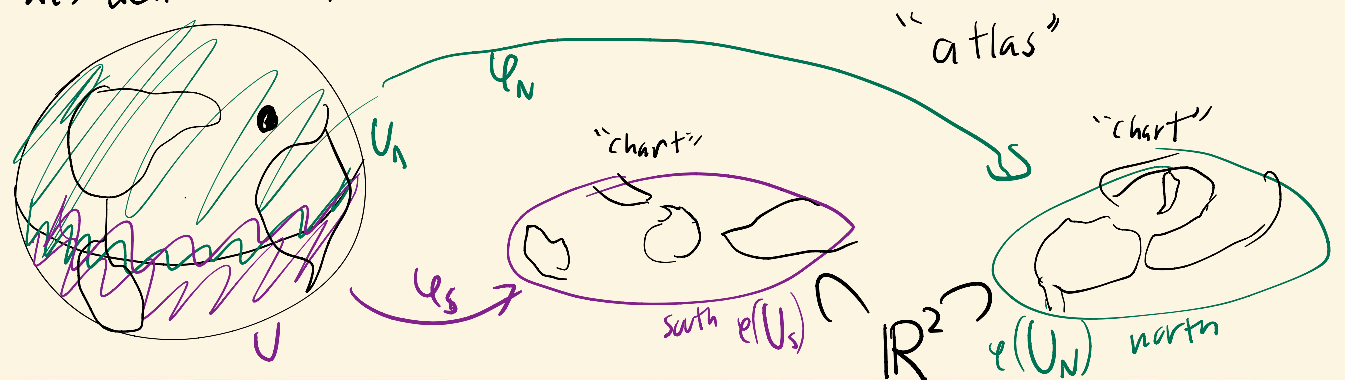
option 1: extrensic

$S^2 \subset \mathbb{R}^3 \quad x(t) \in \mathbb{R}^3 \Rightarrow \dot{x}(t) \in \mathbb{R}^3, \quad \|\dot{x}\|^2$ defined

option 2: intrensic

$\dot{x}(t)$ defined on $\mathbb{R}^n$... so just map S^2 to $\mathbb{R}^2$ locally!!

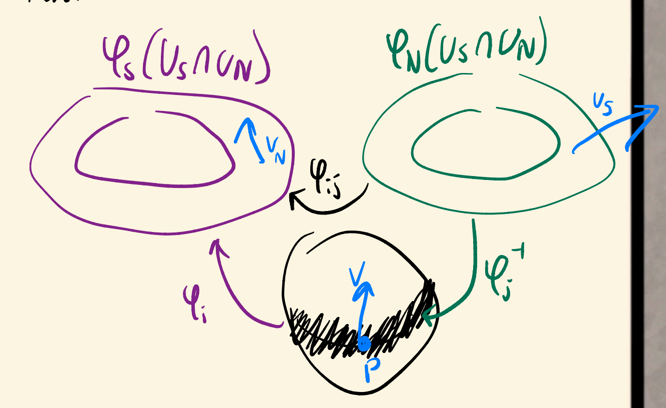
"atlas"



Manifold

1) local charts $\varphi_i: U_i \subset M \xrightarrow{1\text{-}1} \varphi_i(U_i) \subset \mathbb{R}^n$ (n: dimension of mfld)

2) way to move between charts $\varphi_{ij} = \varphi_i \varphi_j^{-1}$



Def: a differentiable Manifold is a space M w/ an atlas $\{U_i, \varphi_i\}$ s.t. $M = \bigcup U_i$ & φ_{ij} & φ_{ij}^{-1} is differentible "local coordinates"

$\boxed{\text{Manifolds are where you can do calculus}}$

What is velocity? $\dot{x}\big|_{p = x(t)}$

option 1:

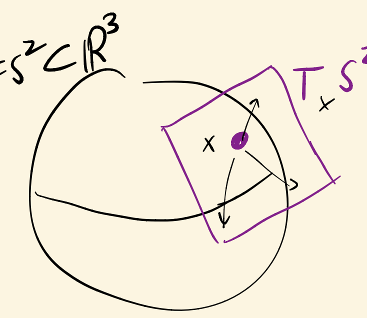
a collection $v_i \in \mathbb{R}^n$ @ $\varphi_i(p)$ s.t. $v_i = D\varphi_{ij} v_j$ ($D\varphi_{ij}$: Jacobian)

Day 9: cotangent spaces

Last time: Manifolds!! M "looks like" $\mathbb{R}^n$
can do calculus!!

Tangent vector @ p: 1st derivative of path $x(t)$

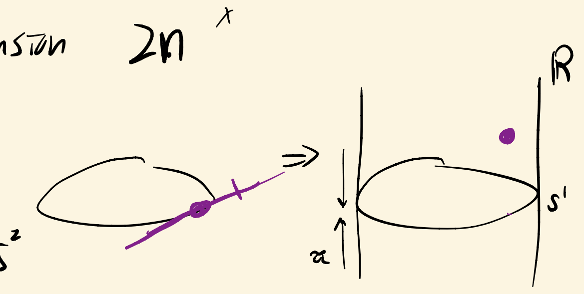
$M = S^2 \subset \mathbb{R}^3$



tangent plane to surface

"Tangent bundle" $TM = \bigsqcup_x T_x M$
manifold of dimension $2n$

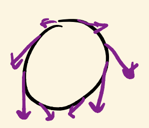
Examples: $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ $\quad TS^1 = S^1 \times \mathbb{R}$



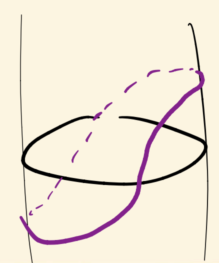
$TS^2 = \{(\vec{x}, \vec{v}) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \|\vec{x}\| = 1 \ \& \ \vec{v} \cdot \vec{x} = 0\}$ $\underline{\text{not}} =$ $\mathbb{R}^2 \times S^2$

but $\pi: TS^2 \to S^2 \quad (x, v) \mapsto x$

vector field = function $V(x) \in T_x M$, or $V: M \to TM$



w/ $\pi \circ V: M \to M$
identity



Differential forms: "cotangent vector"

$p \in T_q^* M \Rightarrow p: T_q M \to \mathbb{R}$. eats velocity, spits out #

cotangent bundle: $T^*M = \bigsqcup_q T_q^* M$

1 form field: $p(q) \in T_q^* M$, $\quad p: M \to T^*M$ $\quad$ $p(q) \cdot V(x) \in \mathbb{R}$
$p \cdot v: M \to \mathbb{R}$

works like $\mathbb{R}^n$: wedge products, integrals, stokes thm, etc

Mechanics on a manifold: configuration space M, Phase space $\mathcal{P} = T^*M$

$\boxed{\text{Momentum is a 1-form}}$

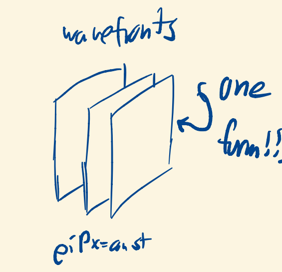
position $q \in M$, momentum $p \in T_q^* M$

velocity $\in T_p M$, momentum and velocity $\underline{\text{not}}$ equivalent

$H = \frac{p^2}{2m} + V(x)$ $\quad$ Hamiltons eqs: $\dot{x} = \frac{p}{m}$ so $p = mv$

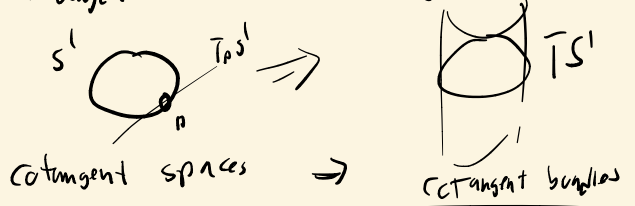
$\underline{\text{only}}$ happens w/ this type of hamiltonian!!

Cute reason: deBroglie principle $\overset{\text{classical}}{\text{momentum } p} \Rightarrow \overset{\text{quantum}}{\text{wave } e^{ipx}}$



Day 10

Last time: Tangent spaces $\Rightarrow$ tangent bundles

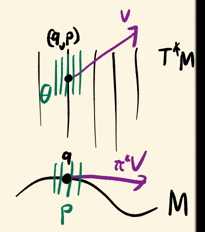


cotangent spaces $\rightarrow$ cotangent bundles

$\boxed{\text{momentum is a 1-form}}$

Tautological 1-form: a point (q,p) on $\mathcal{P} = T^*M$ is 1-form @ $\pi(x) \in M$

Choose 1-form $\theta_{(q,p)} \in T^*_{(q,p)}\mathcal{P}$: p itself! for $v \in T_{(q,p)}\mathcal{P}$, $\theta(v) := p(\pi^* v)$

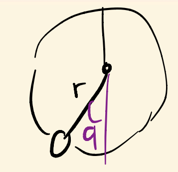


Coords $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ on T^*M: $\boxed{\theta = p_1 dq_1 + \ldots + p_n dq_n}$

"symplectic form" $\omega = dp_1 \wedge dq_1 + \ldots + dp_n \wedge dq_n = d\theta$ ← encodes phase space structure

Hamiltonian mechanics: time evolution X_H satisfies $\omega(X_H, -) = dH$

Example: Pendulum:



configuration space $M = S^1$

$$H = \frac{p^2}{2m} + mgr\cos q$$

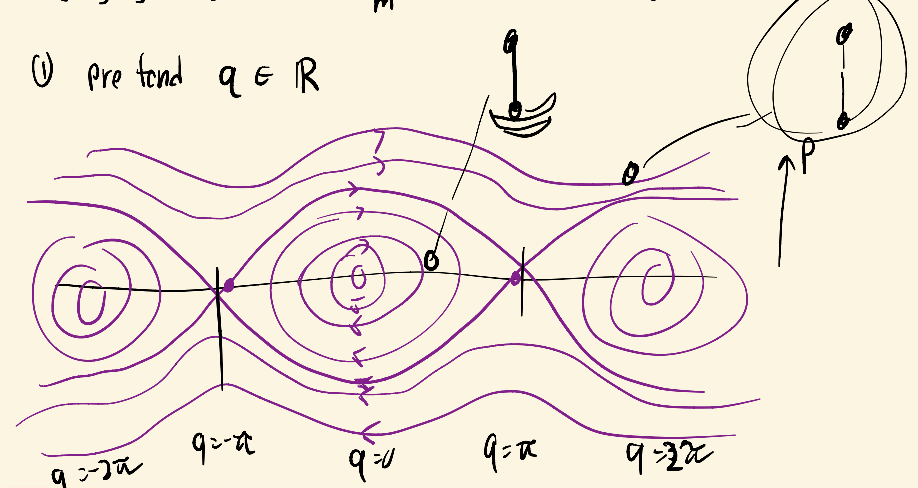
$dH = \frac{p}{m}dp - mgr\sin q\, dq$

$\omega = dq \wedge dp$ $\quad \omega((\dot q, \dot p), (\dot q', \dot p')) = \dot q \dot p' - \dot p \dot q'$

$\Rightarrow \omega((\dot q, \dot p), -) = \dot q\, dp - \dot p\, dq$

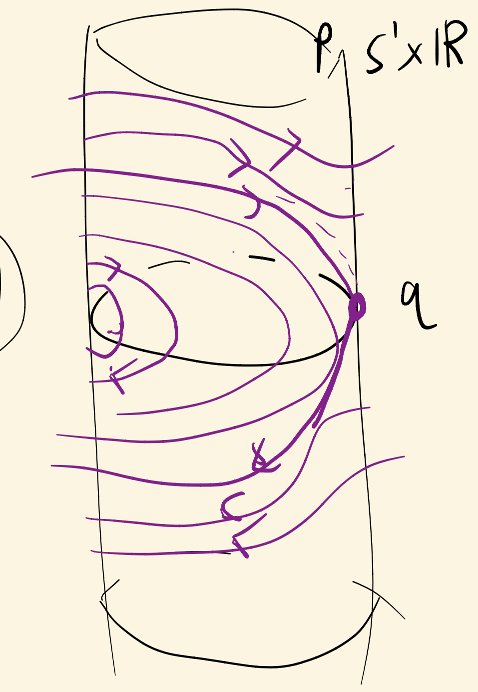
$\omega((\dot q, \dot p), -) = dH \Rightarrow \dot q = \frac{p}{m} \quad \dot p = mgr\sin q$

① pretend $q \in \mathbb{R}$



② $q \in S^1$, $p \in T^*_q S^1$

$\mathcal{P} = T^*S^1 = S^1 \times \mathbb{R}$



Abstract symplectic manifolds 2n-dimensional manifold $\mathcal{P}$

ω 2-form, $d\omega = 0$, ("closed") $\omega_x(X, -) = 0 \Leftrightarrow X = 0$ ("nondegenerate") $\omega_p = \omega|_{\wedge T^*_x \mathcal{P}}$ $X \in T_x\mathcal{P}$

Pointwise: $\omega_x \Rightarrow$ matrix $W : \underset{\mathbb{R}^{2n}}{T_x\mathcal{P}} \to \underset{\mathbb{R}^{2n}}{T_x\mathcal{P}}$ $\quad W^T = -W$

e.g. $W = \begin{bmatrix} 0 & -5 & 9 & 0 \\ 5 & 0 & -1 & 0 \\ -9 & 1 & 0 & -9 \\ 0 & 0 & 9 & 0 \end{bmatrix}$

W "normal form": basis $e_1, e_1', \ldots, e_n, e_n'$ s.t. $W = \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{bmatrix}$ ($\omega_x(e_1, e_1')$, $\omega_x(e_1', e_1)$)

Pf: W diagonalizable, has basis v_i s.t. $Wv_i = \lambda_i v_i$

but, $\bar{v}_i^T W v_i = \lambda_i \|v_i\|^2$, $-\lambda_i \|v_i\|^2 = \bar{v}_i^T W^T v_i = \overline{Wv_i}^T v_i = \bar\lambda_i \|v_i\|^2$, $-\lambda_i = \bar\lambda_i$

so, $W = \begin{bmatrix} i\lambda_1 & & & & \\ & -i\lambda_1 & & & \\ & & \ddots & & \\ & & & i\lambda_n & \\ & & & & -i\lambda_n \end{bmatrix} \cong \begin{bmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \lambda_n \\ & & & -\lambda_n & 0 \end{bmatrix}$ nondegenerate: each $\lambda_i \neq 0$

nondegenerate

$\det W \neq 0$

Darboux's Theorem: every symplectic form locally looks like $\sum dp_i \wedge dq_i$

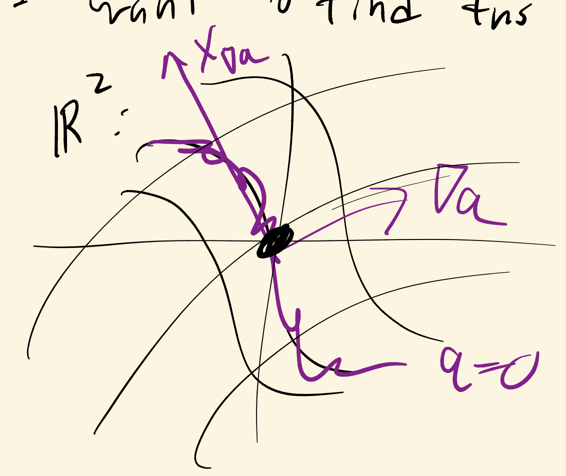
i.e. there are "coordinates" $\phi : U \to \mathbb{R}^{2n}$ s.t. $\omega|_U = \phi^* \sum dq_i \wedge dp_i$

"normal form" of ω extends from pt x to all U!

pf: later

Pf: want to find fns $p_i(x)$, $q_i(x)$ s.t. $\omega = \sum dp_i \wedge dq_i$

i) $\mathbb{R}^2$:



define base pt @ 0,0

$\omega(X_p, X_a) = da(X_a) = 1$

$(t,s) = \phi^p_t \phi^a_s (0,0)$

vector field flow

$X \Rightarrow \phi_t$ $\quad$ w/ $x(t) = \phi_t(x)$, $\dot x = X$

lie derivative $\mathcal{L}_X$ "fisherman's derivative"

Cartan's magic formula $\mathcal{L}_X \alpha = i_X d\alpha + d i_X \alpha$

$\mathcal{L}_{X_H} \omega = 0$ $\quad \frac{d}{dt}\phi^*_{X_H}\omega$

$\int_C \omega = \int_{\phi_{X_H} C} \omega$

$\phi^{t*}_{X_H} \omega = \omega$

Day 11

Last time:

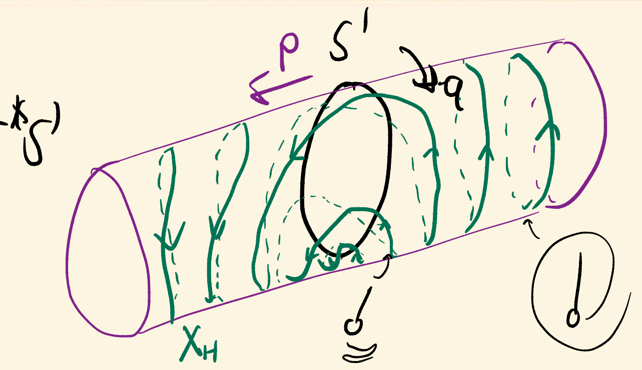
$P = T^*S^1$

$\omega = dq \wedge dp$

symplectic form on cotangent bundles

$H = P^2/2m - mgr\cos q$

$\omega(X_H, -) = dH$



abstract symplectic manifold 2n-dimensional manifold P

ω 2-form, $d\omega = 0$, $\omega_x(X, -) = 0 \Leftrightarrow X = 0$

"closed" "nondegenerate"

nondegenerate $\Rightarrow$ X_H unique: if $\omega(X_H, -) = \omega(X_H', -) = dH$, then $\omega(X_H - X_H', -) = 0$ $\Rightarrow X_H = X_H'$

X_H exists: map $T_x^*P \to T_xP$, $dH \mapsto X_H$ linear, injective, $\dim T_x^*P = \dim T_xP$ $\Rightarrow$ map is isomorphism!

Darboux's theorem. every symplectic form locally looks like $\sum dp_i \wedge dq_i$

i.e. there are "coordinates" $\phi: U \to \mathbb{R}^{2n}$ s.t. $\omega|_U = \phi^* \sum dq_i \wedge dp_i$

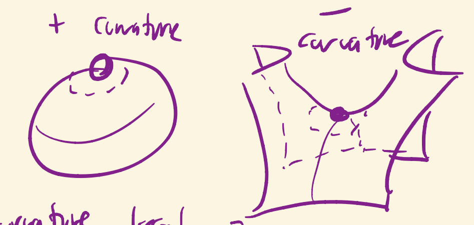
$\omega = \sum dq \wedge dp$ on whole open set U !!

Symplectic manifold: locally modeled on phase space

all symplectic forms locally identical

↳ symp. geo. "rigid"

contrast:



curvature local invariant

Vector field flow:

$\vec{v}: P \to TP$

trajectory of vector field $\vec{V}(x)$: $x(t)$ s.t. $\dot{x}(t) = V(x(t))$, $x(0) = x_0$

track all trajectories at once:

$\phi^t(x_0) = x(t)$

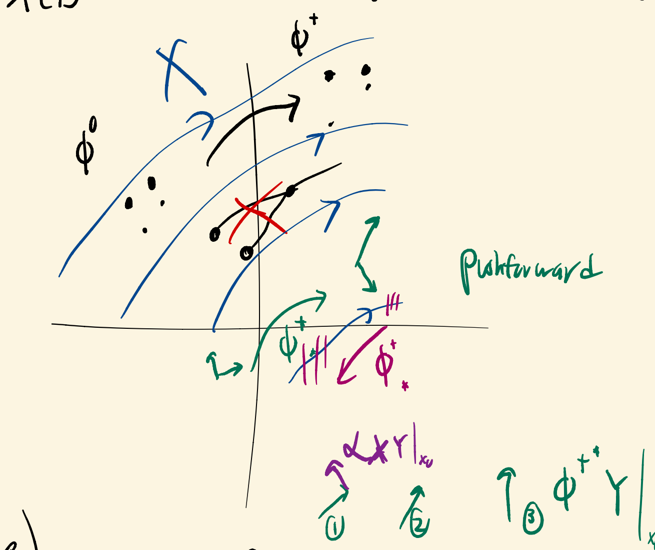
start @ x_0, flow for time t

so, $\phi^t: P \to P$

X differentiable $\Rightarrow$ flow reversible!

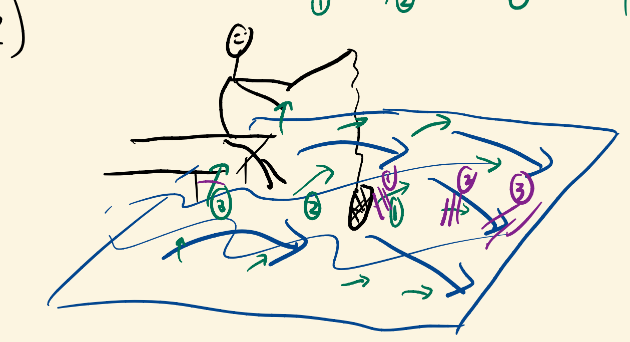
flow ϕ^{-t} of $-X$ reverses flow of X

so, $\phi^t \circ \phi^{-t} = \phi^{t-t} = \phi^0 = id$



Lie derivative (fisherman's derivative)

$\mathcal{L}_X \omega = \frac{d}{dt} \phi^{t*} \omega$



Day 12

Last time: Abstract simplectic mfld $(\mathcal{P}, \omega)$ ω closed, non degenerate 2-form

vector field $X \Rightarrow$ flow $\phi^t(x_0)$ s.t $x(t) = \phi^t(x_0)$ gives trajectory of X

Lie derivative / "fisherman's derivative"

Change @ point wrt flow

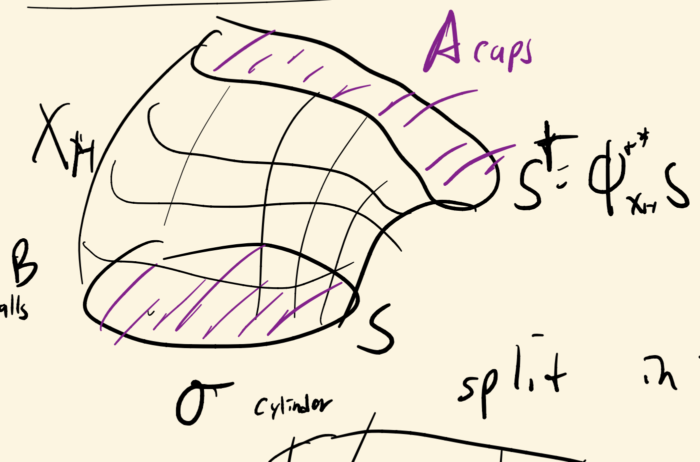
e.g $\int_C \mathcal{L}_X \mathcal{E} = \frac{d}{dt}\int_{\phi^t C} \mathcal{E}$

$\mathcal{L}_X : \Omega^p \to \Omega^p$

or, $\mathcal{L}_X \mathcal{E} = \frac{d}{dt} \phi^{t*} \mathcal{E}$ pullback

$\int_C \mathcal{L}_X \mathcal{E} = \lim_{t\to 0} \frac{1}{t} \int_{C - \phi^{t*}C} \mathcal{E}$

ω preserved under hamiltonian flow:

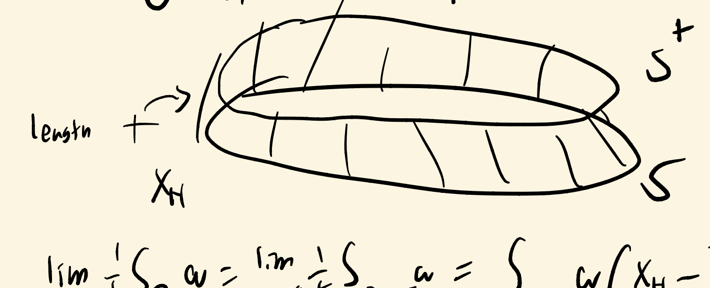


WTS: $\int_S \omega = \int_{S^t} \omega$

$\partial\sigma = A + B$

$\int_{\partial\sigma} \omega = \int_\sigma d\omega = 0 \Rightarrow \int_A \omega = -\int_B \omega$

split into infetesimal

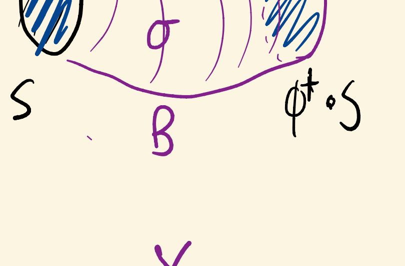


$\lim_{t\to 0} \frac{1}{t} \int_A \omega = \lim_{t\to 0} \frac{\int_S \omega - \int_{S^t} \omega}{t} = \int_S \mathcal{L}_{X_H} \omega$

$\lim_{t\to 0} \frac{1}{t} \int_B \omega = \lim_{t\to 0} \frac{1}{t} \int_{\partial S \times I} \omega = \int_{\partial S} \omega(X_H, -) = \int_{\partial S} dH = \int_S ddH = 0$

Thus, $\int_S \mathcal{L}_{X_H} \omega = 0 \quad \forall S \Rightarrow \mathcal{L}_{X_H} \omega = 0$, $\int_S \omega = \int_{\phi^{t*}_{X_H} S} \omega$, $\omega = \phi^{t*}_{X_H} \omega$

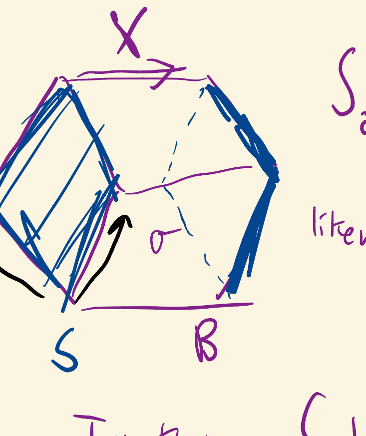
"Cartan's magic formula" $\mathcal{L}_X \omega = i_X d\omega + d i_X \omega$



$A = \int_S \mathcal{L}_X \omega = \frac{d}{dt} \int_{\phi^t S} \omega = \lim_{\delta\to 0} \int_{\phi^\delta S} \omega - \int_S \omega$

$\partial\sigma = A + B \Rightarrow A = \partial\sigma - B$

infetesimal picture:



$\int_{\partial\sigma} \omega = \int_\sigma d\omega = \int_{S\times I} d\omega = \int_S d\omega(X, -, -) = \int_S i_X d\omega$

likewise, $B = \int_{\partial S \times I} \omega = \int_{\partial S} i_X \omega = \int_S d i_X \omega$

Together, $\int_S \mathcal{L}_X \omega = \int_S i_X d\omega + d i_X \omega \Rightarrow \mathcal{L}_X = i_X d + d i_X$

Thm (Louville): $\mathcal{L}_{X_H} \omega = i_{X_H} \cancel{d\omega}^{0} + d \underbrace{i_{X_H} \omega}_{dH} = ddH = 0$

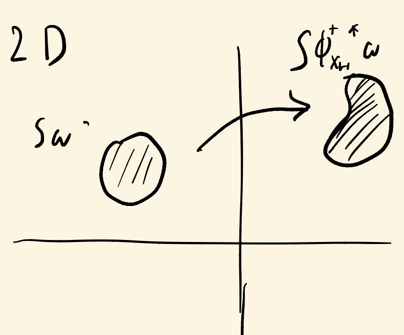
$\Rightarrow \boxed{\mathcal{L}_{X_H} \omega = 0}$

or, the symplectic structure is preserved under Hamiltonian flow

$\phi^{t*}_{X_H} \omega = \omega$ as it should be!!!

(converse: $\mathcal{L}_X \omega = 0 \Rightarrow X$ (locally) hamiltonian)

2D



$\int \omega = \int dq \wedge dp$ measures area

area in phase space is preserved!

2n-D volume: # assigned to 2n-D submfld

= integral of 2n form "volume form"

$\Omega = \frac{\omega^n}{n!} = \frac{1}{n!} \overbrace{\omega \wedge \dots \wedge \omega}^{n}$ note $\Omega \neq 0$

$(dq_1 \wedge dp_1 + dq_2 \wedge dp_2)(dq_1 \wedge dp_1 + dq_2 \wedge dp_2)$ $\times 0$

$dq_1 \wedge dp_1 \wedge dq_2 \wedge dp_2$

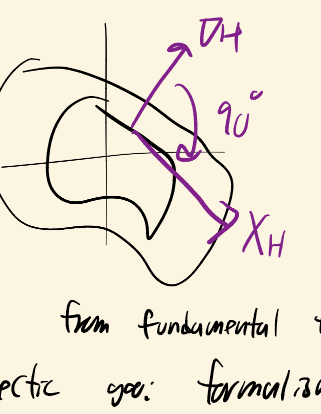
$(-1)^2 dq_1 \wedge dp_1 \wedge dq_2 \wedge dp_2$

in local coords, $\Omega = dq_1 \wedge dp_1 \wedge \dots \wedge dq_n \wedge dp_n$

$\int_{\phi^{t*} V} \frac{\omega^n}{n} = \int_V \phi^*\left(\frac{\omega^n}{n}\right) = \int_V \frac{(\phi^*\omega)^n}{n} = \int_V \frac{\omega^n}{n}$

Volume is preserved!!

Example (2D)



$\nabla \cdot X_H = \nabla \times \nabla H = 0$

Louville theorem immediate from fundamental fact $\mathcal{L}_{X_H} \omega = 0$

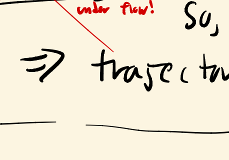
sucess of symplectic geo: formalism!!!

Poincare recurrence

THM: if trajectory lives in finite volume region, any trajectory will return arbitrarly close to starting pt

Pf: Suppose the trajectory never gets within B_ε, ball radius ε about origin

consider "noodle" $\bigcup_t \phi^t B_\varepsilon$: noodle self-intersection invariant under ϕ^t



$\Rightarrow$ any self-intersection implies a trajectory hitting $B_\varepsilon \Rightarrow$ no self-intersection

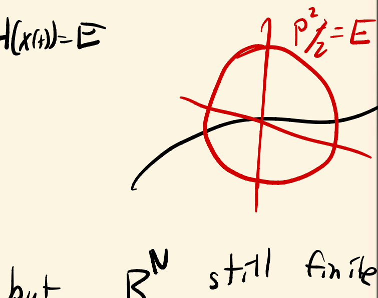
So, noodle is exclusive zone of ever increasing volume

$\Rightarrow$ trajectory cannot be bounded

e.g $\mathcal{P} = T^*M$, $H = \|p\|^2/2 + V(x)$, min $V = 0$, & $x(t)$ lives on $H(x(t)) = E$

then $x(t) \subset$ sphere size E in T^*M

M finite volume $\Rightarrow$ trajectory bounded



E.g: gas on a ball $\mathcal{P} = T^*(B^N)$ $N \approx 10^{23}$ but B^N still finite volume!!

$\Rightarrow$ If you wait long enough the gas in this room will return arbitrarily close to its starting pt!!

Day 13

"Last time": lie derivative $\mathcal{L}_X$

measures change advecting along X

$\mathcal{L}_{X_H}\omega = 0$ $\quad \omega$ symplectic form $\quad$ Hamiltonian $H : \mathcal{P} \to \mathbb{R}$

$\omega(X_H, -) = dH$

$\boxed{\text{Phase space preserved under Hamiltonian evolution}}$

$\Rightarrow$ Liouville theorem

ω^n volume form, volume is preserved (divergence free)

Symmetries:

Consider a particle moving through free space:

$\bullet\!\longrightarrow$ Q: which direction does it go

Left? or Right? Neither! else, breaks symmetry

Q: How fast is it?

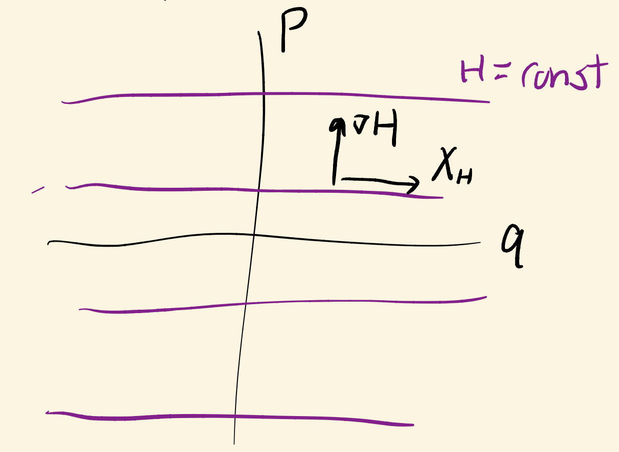
Slower? or Faster? Neither! else, breaks symmetry

$\Rightarrow$ velocity is constant! (Newton's 1st)

Formally:

1D: $\mathcal{P} = \mathbb{R} \times \mathbb{R}$ $\quad (q, p)$ $\quad q$

Translation symmetry: $\quad H(q,p) = H(q+A, p) \Rightarrow H(q,p) = f(p)$



$\nabla H = \begin{pmatrix} 0 \\ f' \end{pmatrix}$ $\quad X_H = \begin{pmatrix} f' \\ 0 \end{pmatrix}$

Velocity $= q$- part of X_H $= f'(p) = \text{const}!$

really, $f = \frac{p^2}{2m}$ so $v = f' = p/m$

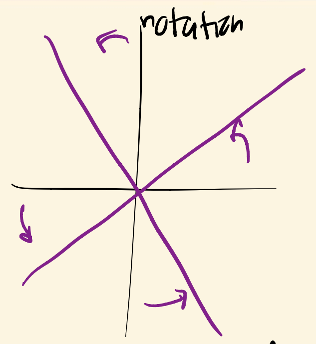
Translation "generated by" p $\quad$ or $\quad dq \wedge dp(X_H, -) = dp$

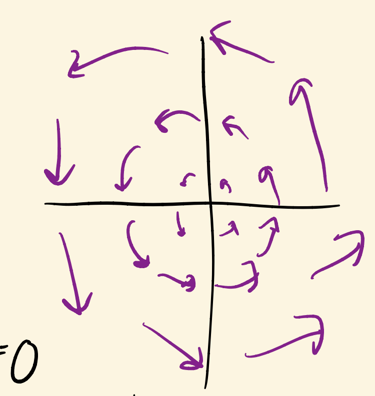
for $H = p$, $\nabla p = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad X_p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \Rightarrow X_H = \dot{q}$

flow of X_p $\quad \phi^t_{X_p}(q,p) = (q+t, p)$

General symmetry:

vector field Y gives symmetric direction on $\mathcal{P}$:





Classical mechanics system (ω, H) symmetric $\Rightarrow$
- $\mathcal{L}_Y \omega = 0$
- directional derivative $Y(H) = dH(Y) = 0$

$\mathcal{L}_Y \omega = 0 \Rightarrow d\omega(Y, -) = 0 \Rightarrow Y = X_f$ $\quad$ Y is (locally) Hamiltonian, "generated" by fn f $\quad Y = X_f$

Noether Thm: Every continuous symmetry has a conserved quantity

Thm: if $X_f(H) = 0$, then $X_H(f) = 0$ (f is conserved)

$X_f(H) = dH(X_f) = \omega(X_H, X_f) = -\omega(X_f, X_H) = -df(X_H) = -X_H(f)$

note: $X_H(f) = \{H, f\}$ poisson bracket

Day 14

Last time: Symmetries of Hamiltonian systems

vector field Y symmetry of $(\mathcal{P}, \omega, H)$

ω symmetric $\Rightarrow \mathcal{L}_Y \omega = 0 \Rightarrow Y = X_f$ (locally) i.e. Y is flow of hamiltonian f

Y "generated by" f

H symmetric $\Rightarrow Y(H) = 0$

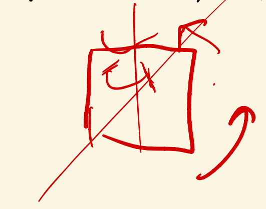
Noether theorem: every symmetry has conserved quantity

$0 = Y(H) = dH(Y) = \omega(X_H, Y) = -\omega(Y, X_H) = -df(X_H) = -X_H(f)$

$X_H(f) = 0 \Rightarrow f$ conserved under H

Multiple symmetries: Suppose $f^\tau : \mathcal{P} \to \mathcal{P}$ is symmetry, $\tau \in G$ space parametrizing symmetries

f family of symmetries parametrized by M
full



Properties:

1) $f^\tau \circ f^{\tau'}$ is also symmetry, so $f^\tau \circ f^{\tau'} = f^{\tau + \tau'}$ $\tau + \tau' \in G$
closure

2) identity is a symmetry, so $\exists\, 0 \in M$ s.t. $f^0 = id$
identity

3) Symmetries are reversible: $f^{\tau^{-1}}$ is a symmetry, call it $f^{-\tau}$ $-\tau \in G$
inverse

4) $f^\tau \circ (f^{\tau'} \circ f^{\tau''}) = (f^\tau \circ f^{\tau'}) \circ f^{\tau''} \Rightarrow \tau + (\tau' + \tau'') = (\tau + \tau') + \tau''$
associative

Symmetries are a group! (or groups are symmetries ...)

e.g. if $\phi^t_{X_H}$ symmetry, $t \in (\mathbb{R}, +)$ ← additive group

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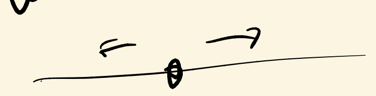
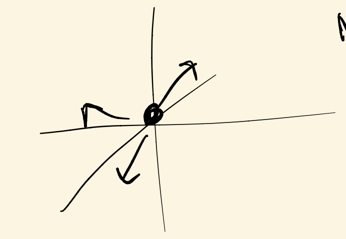
Continuous symmetries: G is itself a manifold

"Lie group"

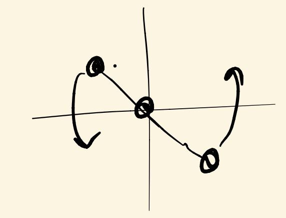
Examples:



1D particle: group $(\mathbb{R}, +)$



ND particle: group $(\mathbb{R}^n, +)$



top down

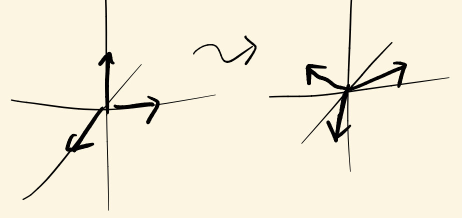
rotational symmetry: group $(U(1), \cdot)$

$U(1) = 1$ by 1 unitary matrix

$= a \in \mathbb{C}$ s.t. $\|a\|^2 = a\bar{a} = 1$

unit circle in $\mathbb{C}$

3D rotational symmetry: $SO(3)$



3 vectors @ origin, each mutually norm 1 & orthogonal

3 by 3 matrix $[\vec{v}_1, \vec{v}_2, \vec{v}_3]$

$$[\vec{v}_1, \vec{v}_2, \vec{v}_3] \cdot [v_1, v_2, v_3]^T = \begin{bmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \langle v_1, v_3 \rangle \\ & \langle v_2, v_2 \rangle & \langle v_2, v_3 \rangle \\ & & \langle v_3, v_3 \rangle \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$SO(3)$ = matrices M s.t. $MM^T = I$ & $\det M = 1$

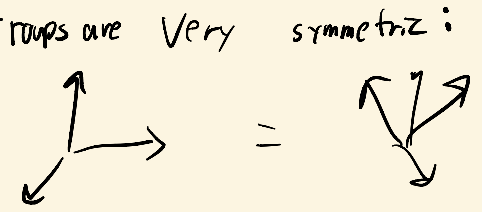
(orientation: $O(3)$ has reflections)

nonabelian!!

generally, lie groups $\approx$ groups of matrices

$SO(3)$ is archetypical liegrp

Groups are very symmetric: each point "the same"



from different perspective

idea: describe lie groups by local structure

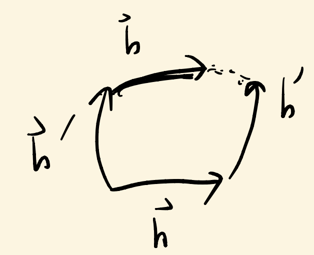
Lie algebras: direction on G

$\underline{g} \cong T_e G$

for $h \in \underline{g}$, define vector field $\vec{h}$ by $\vec{h}|_k = (DK\cdot)\,\vec{h}$

pushforward h to all G

lie bracket $[h, h'] = \mathcal{L}_{\vec{h}} \vec{h}'$



measures noncommutativity of G

Lie Groups $\longleftrightarrow$ Lie algebras

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for group G of symmetries of $(\mathcal{P}, \omega)$

each $g \in G$ acts on $\mathcal{P}$ $g^* \omega = \omega$

generated by infinitesimal symmetries: $h \in \underline{g} \Rightarrow \vec{h}$ gives vect. field on $\mathcal{P}$

$\vec{h}$ symmetry $\Rightarrow \mathcal{L}_{\vec{h}} \omega = 0 \Rightarrow \vec{h} = X_{\mu(h)}$ ← some hamiltonian

$X_{\mu(h+h')} = \overrightarrow{(h+h')} = \vec{h} + \vec{h}' = X_{\mu(h)} + X_{\mu(h')} = X_{\mu(h) + \mu(h')}$

$\Rightarrow \mu(h+h') = \mu(h) + \mu(h')$

at each point, μ is linear map $\underline{g} \to \mathbb{R} \Rightarrow \mu_p \in \underline{g}^*$ (dual space)

<u>Def</u> Momentum map $\mu : \mathcal{P} \to \underline{g}^*$

μ gives family of hamiltonians to generate families of symmetries