Final project topics possibilities

- Knots in electromagnetic fields
  - https://cdn.intechopen.com/pdfs/33435/InTech-Topological_electromagnetism_knots_and_quantization_rules.pdf
  - Can you tie knots with eclectic field lines? Yes! These knots are in fact stable if you have the electric and magnetic field perpendicular from one another, and are described by a map from the three sphere to the two sphere. This map has topological invariants, which translates into topological invariants of the knots. This even describes the quantization of electric charges.
- Electromagnetic fields as U(1) principle bundle
  - John Baez "Gauges theories, knots and gravity"
  - With the hindsight of a century of physics, we can see that electromagnetism is best treated not as a differential form, but as a connection on a line bundle. This is the precursor to principal bundles of more complicated groups, which gives the mathematical formalism describing the gauge theories underlying the universe.
- Geometric quantization (differential geometry aspects)
  - Classical mechanical systems are described by a Poisson algebra of classical observables. The question is, can we replace this with some algebra of operators, with the poisson bracket replaced by commutation? This is called quantization. One approach is geometric quantization, which builds these operators from line bundles on your symplectic manifold.
- Geometric quantization (algebraic geometry aspects)
  - Arnold "mathematical methods of classical mechanics" appendix 3
  - https://johncarlosbaez.wordpress.com/2018/12/30/geometric-quantization-part-5/
  - Classical algebraic geometry and polynomials play a nice role in geometric quantization. All the geometric structures you need are very natural on projective space, which is the space that parametrizes solutions of homogeneous polynomial equations. This leads to very explicit descriptions of the quantization of a system of noninteraction spin 1/2 particles.
- Hamilton-Jacobi equations and WKB
  - Lectures on geometry of quantization, Weinstein, chapter 2
  - While Hamiltonian mechanics is nice, it interfaces best with quantum mechanics and other areas of mathematical physics through the Hamilton Jacobi equations. In the semiclassical limit, quantum particles can be described hydrodynamically as submanifolds moving through classical phase space via the hamilton-jacobi equation, which is known as the WKB approximation.
- Frobenius theorem, integrable hamiltonian systems, and action angle coordinates
  - Arnold "mathematical methods of classical mechanics chapter 10
  - Sometimes Hamitlonian systems are very easy to describe. With enough conserved quantities, the evolution is entirely predictable. It simply travels around a torus!
- Contact geometry and time dependent mechanics
  - Arnold "mathematical methods of classical mechanics appendix 4

    - Contact geometry is symplectic geometry's odd dimensional cousin. This is useful for studying time dependent hamiltonian systems, where the manifold is phase space times time, so is odd dimensional. It also lets you describe the mechanics on energy level sets, which are of one dimension lower than phase space so are odd.
- Symplectic approach to thermodynamics
    - https://arxiv.org/pdf/2104.13009.pdf (I need to read this)
    - The formula relating the state variables in thermodynamics (like pressure, volume, temperature...) fit into an odd dimensional manifold. Theses have a natural contact structure, and carry many of the same properties as classical mechanics systems
- Lagrangians and lagrangian submanifolds
    - A lagrangian submanifold is a half dimensional submanifold of phase space, on which the symplectic form is trivial. These are very useful: The symplectic creed is, “everything is a lagrangian submanifold”. In particular, their name comes from their role in transforming between the hamiltonian and lagrangian formulations of mechanics. After extending phase space to include momentum and velocity, the lagrancian submanifold defines which velocity to assign to which momentum.
- Geometric optics
    - Light, as a set of rays, has a natural phase space of positions and directions. This carries a natural symplectic structure, and the laws of optics all come from symplectic geometry
- Lagrangian singularities, caustics
    - You might recognize caustics from the light playing off the inside of a mug, or the dancing lights on the bottom of a pool. These are general occurrences from folding some manifold back on itself. In this case, the wavefront of light is a lagrangian submanifold of the optics phase space, and is bent by the mug before being projected onto the table. The sorts of caustics you see can be classified very nicely. Remarkably, this classification is identical to that describing lie groups, and the symmetry groups of platonic solids.
- Principle of the symplectic camel, classical mechanics, and Heisenberg uncertainty
    - Liouville's theorem says that volume in phase space is preserved by hamiltonian flows, but hamiltonian flows are more special. Starting with a sphere, no hamiltonian can deform the sphere to a shape with less than 1/4 the diameter. In other words, you can't squeeze a symplectic camel through a needle. This means you can never really pin down the position of your ensemble to a single place. This is very reminiscent of the Heisenberg uncertainty principle, and is like a classical shadow.
- Shrodinger’s equation via symplectic mechanics
    - Quantum mechanics lives on projective space, which is naturally a symplectic manifold. The quantum hamiltonian defines a hamiltonian function on this space, and the flow of this gives schrodinger's equation! Moreover, projective space carries a metric, which encodes the heisenberg uncertainty principle.