# **4.** Vectors, Dot Products, Cross Products, Lines and Planes

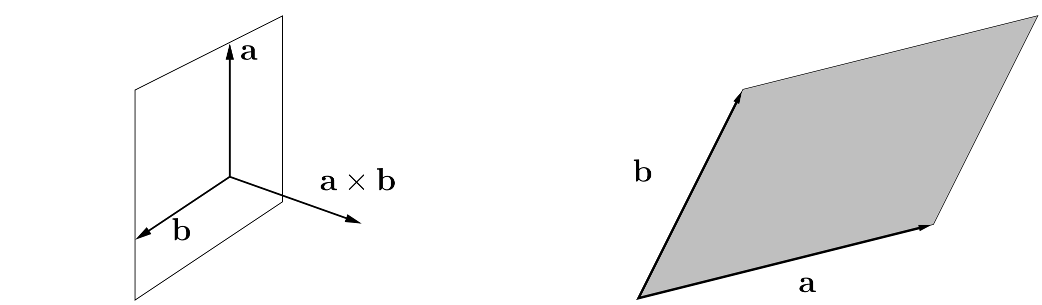
In engineering and the physical sciences, a vector is any quantity possessing both magnitude and direction. Force, displacement, velocity, and acceleration are all examples of vectors.

One way of multiplying two vectors together is the *dot product.* If $\mathbf{a} = \langle a_1, a_2, \ldots, a_n \rangle$ and $\mathbf{b} = \langle b_1, b_2, \ldots, b_n \rangle$ then $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$. Note that the result of the dot product is a number, not a vector. The dot product gives an easy way of computing the angle between two vectors: the relationship is given by the formula $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$. In particular, $\mathbf{a}$ and $\mathbf{b}$ are perpendicular if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Another way of multiplying vectors in $\mathbf{R}^3$ is the *cross product.* If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2)\mathbf{i} + (a_3 b_1 - a_1 b_3)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}.$$

Note that the result of the cross product is a vector, not a number. The cross product reflects several interesting geometric quantities. First, it gives the angle between the vectors by the formula $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$. Second, the vector $\mathbf{a} \times \mathbf{b}$ is perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$. Third, $|\mathbf{a} \times \mathbf{b}|$ is the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.




(a) Vector perpendicular to a plane (b) Parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$

Figure 3: The cross product

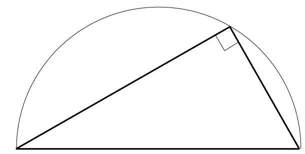
## Questions

1. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbf{R}^3$. Can $\mathbf{u} \times \mathbf{v}$ be a non-zero scalar multiple of $\mathbf{u}$?

2. Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero vectors in $\mathbf{R}^3$. Show that $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if and only if $|\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2$. What is the name of this famous theorem?

3. Find a vector perpendicular to $\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ in $\mathbf{R}^3$. Draw a picture to illustrate that there are many correct answers.

4. (a) Suppose that one side of a triangle forms the diameter of a circle and the vertex opposite this side lies on a circle. Use the dot product to prove that this is a right triangle.



   (b) Now, do the same in $\mathbf{R}^3$.
   (**Hint:** Let the center of the circle be the origin.)

# Problems

1. Here we find parametric equations for the line in $\mathbf{R}^3$ passing through the points $\mathbf{a} = (1, 0, 1)$ and $\mathbf{b} = (2, 1, -1)$.
   (a) Find a vector $\mathbf{u}$ pointing in the same direction as the line.
   (b) Let $\mathbf{c}$ be any point on the line. Explain why $\mathbf{c} + t\mathbf{u}$ gives parametric equations for the line. Write down these equations
   (c) Can you get more than one parametrization of the line from these methods?
2. Consider the plane $x + y - z = 4$.
   (a) Find any point in the plane and call it $\mathbf{a}$. Let $\mathbf{x} = (x, y, z)$ and show that $(\mathbf{x} - \mathbf{a}) \cdot (1, 1, -1) = 0$ is the equation of the plane.
   (b) Explain why $\mathbf{i} + \mathbf{j} - \mathbf{k}$ is a normal vector to the plane.
   (c) Show that if $ax + by + cz = d$ is the equation of a plane where $a$, $b$, $c$, and $d$ are constants, then $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a normal vector.
3. Here we find the equation of a plane containing the points $\mathbf{a} = (0, 0, 1)$, $\mathbf{b} = (0, 1, 2)$ and $\mathbf{c} = (1, 2, 3)$.
   (a) Let $\mathbf{u}$ and $\mathbf{v}$ be vectors connecting $\mathbf{a}$ to $\mathbf{b}$ and $\mathbf{a}$ to $\mathbf{c}$. Compute $\mathbf{u}$ and $\mathbf{v}$.
   (b) Find a vector perpendicular to the plane.
   (c) Use the normal vector to find the equation of the plane.
   (**Hint:** First write the equation in the form given in Problem 2(a).)

# Additional Problems

1. Suppose that you are looking to the side as you walk on a windless, rainy day. Now you stop walking.[1]
   (a) How does the apparent direction of the falling rain change?
   (b) Explain this observation in terms of vectors.
   (c) Suppose you know your walking speed. How could you determine the speed at which the rain is falling?

[1] From *Basic Multivariable Calculus* by Marsden, Tromba, and Weinstein.