MATH53 discussion: Polar coordinates

September 2, 2022

We learned about polar functions, which are a special case of parametric curves where the parameter is the angle from the origin, and the curve is specified by the radius as a function of that angle:

$$x(\theta) = r(\theta)\cos(\theta)$$

$$y(\theta) = r(\theta)\sin(\theta)$$

1. The arc length of a polar curve is

$$\int_a^b \sqrt{r(\theta)^2 + r'(\theta)^2}\,\mathrm{d}\theta$$

2. The area of a polar curve is

$$\int_a^b \frac{1}{2} r^2 \mathrm{d}\theta$$

1 The Cardiod

The cardioid is a shape similar to a cycloid. Instead of tracing a circle rolling around the ground, its formed by a circle rolling around another circle. For two circles of radius a, this equation is parametrically

$$x(\theta) = 2a(1 - \cos(\theta))\cos(\theta) \qquad y(\theta) = 2a(1 - \cos(\theta))\sin(\theta)$$

1. Write this in terms of polar coordinates: what is $r(\theta)$?
2. Sketch a graph of this shape. Why do you think it's called a cardioid? (hint- think about cardiac arrest)
3. What is the length of a cardioid?
4. What is the area inside of a cardioid?

If you finish above: Consider instead the polar equation $r(\theta) = a(1 - c\cos(\theta))$. Sketch this for some different values of c. How does the shape vary?

Cardiods are interesting because they show up as *Caustics*, the envelopes of bright light that can show up when light bounces or refracts off a curved surface. For parallel light rays bouncing off a circle, they fold over themselves in a cardioid. You can see this in the bottom of your coffee cup: Try holding a mug sideways in the sunlight, and looking at the bottom. If you instead make the light rays come from a point close by, like from your phone flashlight, then the distortion of the rays distorts the caustics from a perfect cardioid. This distortion is like changing the value of c in the equation above. For the perfect light position, you get $c = 1$, which is a cardioid!

Figure 1: Cardioid in a coffee cup