MATH53 discussion: Parametric curves and their calculus

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We learned about parametric curves, defined by a pair of functions $x(t), y(t)$. This worksheet is about understanding the curves traced out by these functions, and calculus on these curves. Some useful formulae:

1. The slope of a tangent line to a parametric curve at a point $(x(t), y(t))$ is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}$$

2. The arc-length of a parametric curve, paramertized on $\alpha < t < \beta$, is

$$\int_{\alpha}^{\beta} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \mathrm{d}t$$

1 Lissajous curves

1. Consider the circle $C = \{x, y \in \mathbb{R} | x^2 + y^2 = 1\}$

   (a) Is C the graph of a function? Why or why not?
   No, as it fails the vertical line test

   (b) Find a parameterization of C (Hint: $\sin^2(\theta) + \cos^2(\theta) = 1$ Use $x(\theta) = \cos(\theta), y(\theta) = \sin\theta$. This corresponds to a point rotating at a constant rate around the circle. To check this works, note $x(\theta)^2 + y(\theta)^2 = \cos(\theta)^2 + \sin(\theta)^2 = 1$

   (c) What is the slope of the tangent line to the circle at the point $(\sqrt{2}/2, \sqrt{2}/2)$? Compute this geometrically and using the parameterization Geometrically, this point is at 45 degrees along the unit circle. The tangent line is perpendicular to the radial line of that point, which is $y = x$ and has slope $s = 1$. The perpendicular line has slope $-1/s = \boxed{-1}$.
   Parametrically, we calculate

$$x'(\theta) = -\sin(\theta) \tag{1}$$
$$y'(\theta) = \cos(\theta) \tag{2}$$

   Thus, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y'}{x'} = -\frac{\sin(\theta)}{\cos(\theta)} = -\tan(\theta)$$

   The point $(\sqrt{2}/2, \sqrt{2}/2)$ corresponds to $\theta = \pi/4$, so the slope is $-\tan(\pi/4) = \boxed{-1}$.

   (d) What is the length of the circle? Compute this geometrically, and using the arclength formula for parametric curves.
   Geometrically, the length of the circle is simply the perimeter of a circle with radius 1. Thus, the length l is $2\pi r$ for $r = 1$, giving $\boxed{2\pi}$

Parametrically, we use the arclength formula. The path from $\theta = 0$ to $\theta = 2\pi$ goes over the whole circle exactly once, so these are the endpoints for our integral. The length ℓ is

$$\begin{aligned}
\ell &= \int_0^{2\pi} \sqrt{x'(\theta)^2 + y'(\theta)^2} \mathrm{d}\theta \\
&= \int_0^{2\pi} \sqrt{(-\sin(\theta))^2 + (\cos(\theta))^2} \mathrm{d}\theta \\
&= \int_0^{2\pi} 1 \mathrm{d}\theta \\
&= \theta|_0^{2\pi} = \boxed{2\pi}
\end{aligned}$$

2. Consider the curve $x(t) = a\sin(t)$, $y(t) = b\cos(t), t \in [0, 2\pi]$

   (a) Sketch this curve when $a = 1, b = 2$ and when $b = 1, a = 2$ Choosing $a = 1, b = 2$ is stretching all y-values by a factor of 2. This means the circle gets streched into a vertically oriented ellipse. Similarly, $a = 2, b = 1$ gives stretches the circle into a horizontally oriented ellipse.

   (b) eliminate the parameter, to construct a single defining equation in x, y, a, and b. (Hint: look at the question above). What curve does this equation describe?

   The equation is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

   this is the equation for an ellipse, with vertical height $2b$ and horizontal length $2a$. To see this works, plug in the parameterized $x(t), y(t)$ into the equation.

   (c) Find the slope of the tangent line to this curve at $t = 0, t = \pi/4, t = \pi/2$

$$x'(\theta) = a\cos(\theta) \tag{3}$$
$$y'(\theta) = -b\sin(\theta) \tag{4}$$

   Thus, we have

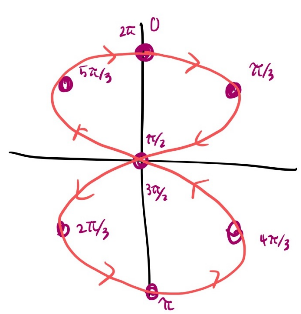
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y'}{x'} = \frac{-b\sin(\theta)}{a\cos(\theta)} = -\frac{b}{a}\tan(\theta)$$

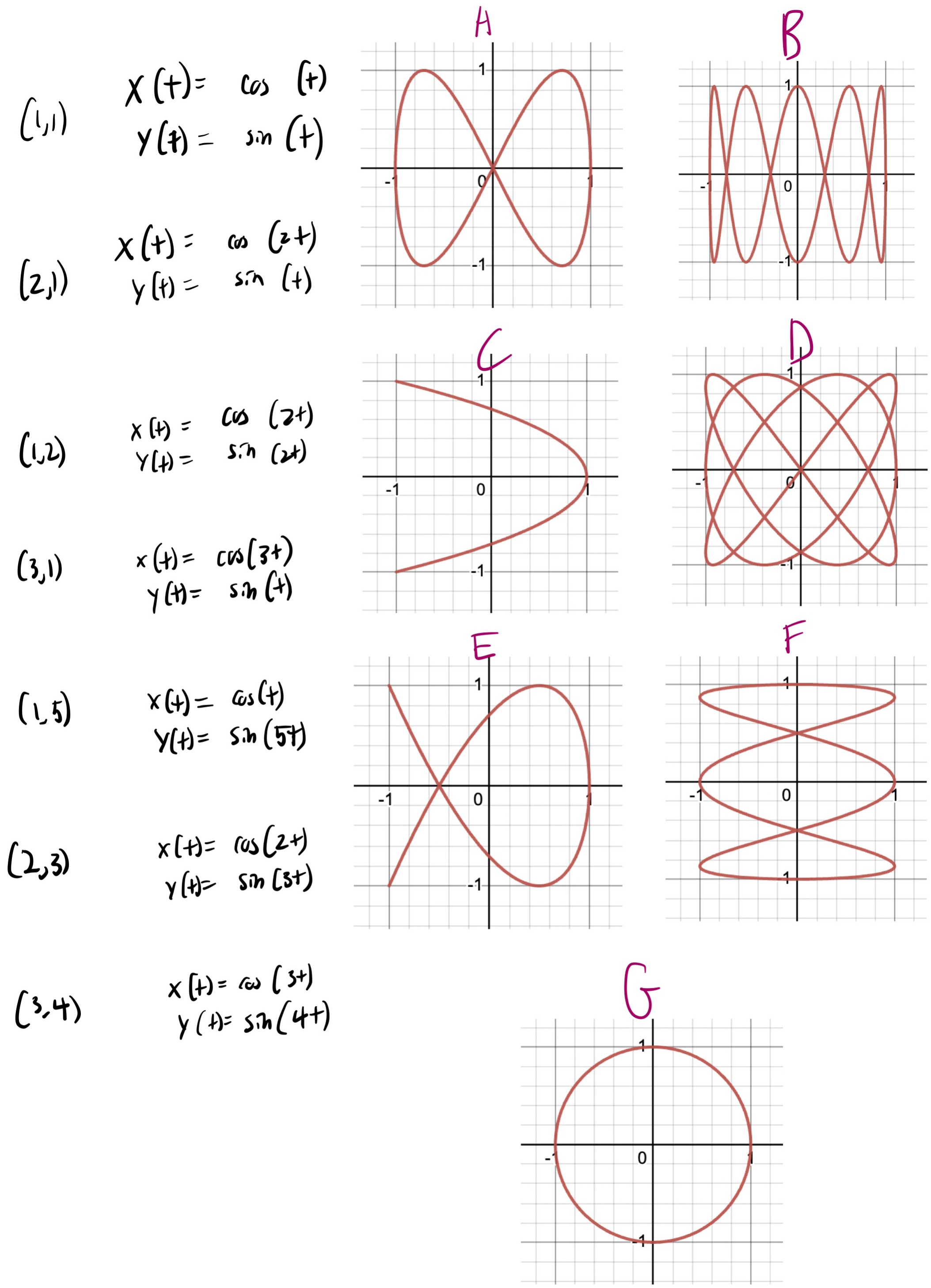
   Plugging in, we see the slope at $\theta = 0$ is 0, the slope at $\theta = \pi/2$ is infinite, and the slope at $\theta = \pi/4$ is $-b/a$.

3. How about the curve $x(t) = \sin(2 * t)$, $y(t) = \cos(t), t \in [0, 2\pi]$

   (a) Sketch this curve by plotting values of x, y for certain t. Try the values of t where you know the values of sin and cos, like multiples of $\pi/2$ and $\pi/3$.

t	$x = \sin 2t$	$y = \cos t$
0	0	1
$\pi/3$	$\sqrt{3}/2$	$1/2$
$\pi/2$	0	0
$2\pi/3$	$-\sqrt{3}/2$	$-1/2$
π	0	-1
$4\pi/3$	$\sqrt{3}/2$	$-1/2$
$3\pi/2$	0	0
$5\pi/3$	$-\sqrt{3}/2$	$1/2$
2π	0	1





4. Match up the following parametric equations with their curves: (on back)

Note that this problem has a typo: $(1, 2)$ should read $x(t) = \cos(t), y(t) = \sin(2t)$.

If $x(t) = \sin(at), y(t) = \cos(bt)$ for a, b integers,then it means x oscillates a times in the time it takes for y to oscillates b times. The number of times it oscillates is equivalently the number of times it hits one extreme. For example, in graph A, it hits the y extreme $y = 1$ twice, while only hits the x extreme $x = 1$ once. So, it has to have b be twice a. There's a further subtlty: If $a = ka'$, $b = kb'$ for some integers k, a', b', then it traces the same curve as that for a', b', just k times as fact. This is not the case for any of the options on the left, so we don't

(a,b)	graph letter
$(1,1)$	G
$(2,1)$	C
$(1,2)$	A
$(3,1)$	F
$(1,5)$	B
$(2,3)$	E
$(3,4)$	D

have to worry about it, and the number of extremes match the numbers a, b. So, we have:

There is one further subtlty: In graph E and C, on the right, both x- extremes lie at the same point, so it only looks like it hits once. On the left though, you can see it hit both times to get the proper count.

You aren't expected to come up with this particular solution to succeed in this class. There are many ways to go about this problem, and anything that works, works. This is just supposed to show you one way to identify parametric curves from their equations.

2 The Cycloid

A cycloid describes the path of a point on the outside of a circle as it rolls along the ground. Imagine an ant, hitching a ride on the rim of your bicycle wheel. Say your bicycle has wheels of radius r. Since it rolls without slipping, as the wheel rotates θ radians, it moves to the right $r\theta$ radians.

1. Find a parametric equation describing the path of the ant. You are adding together two parametric equations. First, rotation of the circle, with angle measured from vertical:

$$x_1(t) = r\sin(t), \qquad y_1(t) = r(1 - cos(t))$$

and second, translational motion from rolling:

$$x_2(t) = rt, \qquad y_2(t) = 0$$

Together these give

$$\boxed{x(t) = x_1(t) + x_2(t) = r(1 + \sin(t))} \qquad \boxed{y(t) = y_1(t) + y_2(t) = r(1 - \cos(t))}$$

2. After getting on the wheel, how far will the wheel move before the ant is moving parallel to the ground again? In other words, when $y'(t) = 0$, find $x(t)$. We compute $y' = r\sin(t)$. This is zero when t equals $0, \pi, \pi....$ $t = 0$ is when the ant first gets on the wheel, so the second time is $t = \pi$. Plugging this in to x, we see $x(\pi) = r(\pi + \sin(\pi)) = \boxed{\pi r}$.

3. How far does the ant travel before it can get off your bicycle wheel again? In other words, how long is one lobe of the cycloid? Use the arclength formula. (Hint: $\sin^2(\theta/2) = (1 - \cos(\theta))/2$ Note that

$$x'(t) = r(1 + \cos(t))y'(t) = r\sin(t)$$

We want to integrate the path length between the two times where $y = 0$, so from $t = 0$ to $t = 2\pi$. Plugging in to the path length formula, the length L is:

$$\begin{aligned} L &= \int_0^{2\pi} \mathrm{d}t \sqrt{(r(1 + \cos(t)))^2 + (r\sin(t))^2} \\ &= \int_0^{2\pi} \mathrm{d}t \sqrt{r^2(1 + 2\cos(t) + \cos^2(t) + \sin^2(t))} \\ &= r\int_0^{2\pi} \mathrm{d}t \sqrt{1 + 2\cos(t) + 1)} \\ &= r\sqrt{2}\int_0^{2\pi} \mathrm{d}t \sqrt{1 + \cos(t)} \end{aligned}$$

Using the double angle formula, $1 + \cos(t) = 2\sin^2(t/2)$. Plugging this in,

$$\begin{aligned} L &= r\sqrt{2}\int_0^{2\pi} \mathrm{d}t \sqrt{2\sin^2(t/2)} \\ &= 2r\int_0^{2\pi} \mathrm{d}t \sin(t/2) \end{aligned}$$

use a u-sub $u = t/2$, $\mathrm{d}u = \mathrm{d}t/2$:

$$\begin{aligned} L &= 2r\int_0^{\pi} 2\mathrm{d}u \sin(u) \\ &= -4r\cos(u)|_0^{\pi} \\ &= -4r(-1-1) \\ &\boxed{= 8r} \end{aligned}$$

3 Parameterizing the world

With your group, look around the building for an interesting looking curve (A crooked treebranch, the metal tube in a bike rack, anything you can find). Take a picture of it. Then, try and come up with a parametric curve which matches the curve in the picture. You can check how well your curve matches your picture using https://www.desmos.com/calculator. Try for the most interesting picture/curve combo.

There is no right answer! anything you can think of is A-okay.