MATH53 discussion: Parametric curves and their calculus

August 29 2022

We learned about parametric curves, defined by a pair of functions $x(t), y(t)$. This worksheet is about understanding the curves traced out by these functions, and calculus on these curves. Some useful formulae:

1. The slope of a tangent line to a parametric curve at a point $(x(t), y(t))$ is
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}$$
2. The arc-length of a parametric curve, paramertized on $\alpha < t < \beta$, is
$$\int_\alpha^\beta \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

1 Lissajous curves

1. Consider the circle $C = \{x, y \in |x^2 + y^2 = 1\}$
   (a) Is C the graph of a function? Why or why not?
   (b) Find a parameterization of C (Hint: $\sin^2(\theta) + \cos^2(\theta) = 1$
   (c) What is the slope of the tangent line to the circle at the point $(\sqrt{2}/2, \sqrt{2}/2)$? Compute this geometrically and using the parameterization
   (d) What is the length of the circle? Compute this geometrically, and using the arclength formula for parametric curves.
2. Consider the curve $x(t) = a\sin(t)$, $y(t) = b\cos(t), t \in [0, 2\pi]$
   (a) Sketch this curve when $a = 1, b = 2$ and when $b = 1, a = 2$
   (b) eliminate the parameter, to construct a single defining equation in x, y, a, and b. (Hint: look at the question above). What curve does this equation describe?
   (c) Find the slope of the tangent line to this curve at $t = 0, t = \pi/4, t = \pi/2$
3. How about the curve $x(t) = \sin(2 * t)$, $y(t) = \cos(t), t \in [0, 2\pi]$
   (a) Sketch this curve by plotting values of x, y for certain t. Try the values of t where you know the values of sin and cos, like multiples of $\pi/2$ and $\pi/3$.
4. Match up the following parametric equations with their curves:

2 The Cycloid

A cycloid describes the path of a point on the outside of a circle as it rolls along the ground. Imagine an ant, hitching a ride on the rim of your bicycle wheel. Say your bicycle has wheels of radius r. Since it rolls without slipping, as the wheel rotates θ radians, it moves to the right $r\theta$ radians.

1. Find a parametric equation describing the path of the ant.
2. After getting on the wheel, how far will the wheel move before the ant is moving parallel to the ground again? In other words, when $y'(t) = 0$, find $x(t)$.
3. How far does the ant travel before it can get off your bicycle wheel again? In other words, how long is one lobe of the cycloid? Use the arclength formula (Hint: $\sin^2(\theta/2) = (1 - \cos(\theta))/2$

(1,1) $x(t) = \cos(t)$, $y(t) = \sin(t)$

(2,1) $x(t) = \cos(2t)$, $y(t) = \sin(t)$

(1,2) $x(t) = \cos(2t)$, $y(t) = \sin(2t)$

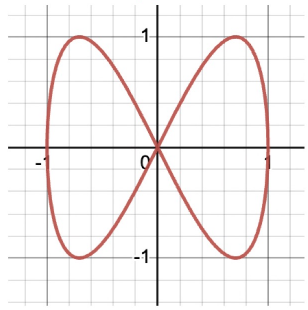
(3,1) $x(t) = \cos(3t)$, $y(t) = \sin(t)$

(1,5) $x(t) = \cos(t)$, $y(t) = \sin(5t)$

(2,3) $x(t) = \cos(2t)$, $y(t) = \sin(3t)$

(3,4) $x(t) = \cos(3t)$, $y(t) = \sin(4t)$

A



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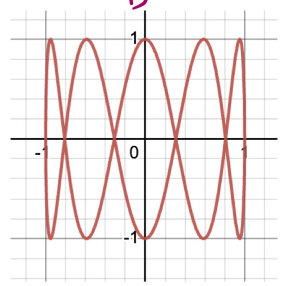

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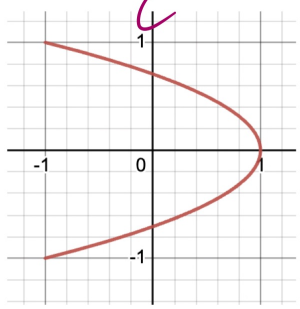

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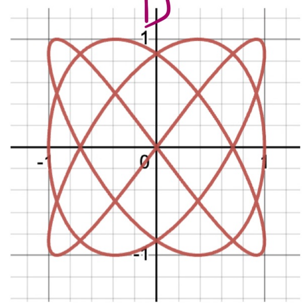

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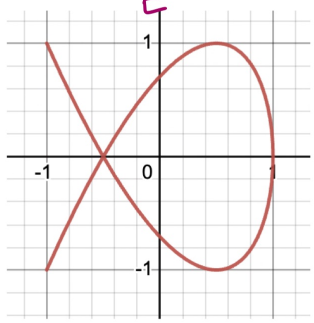

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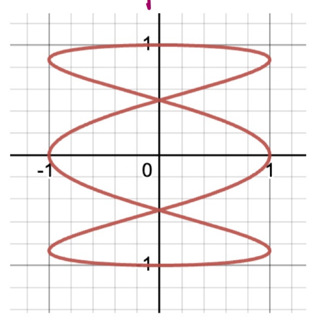

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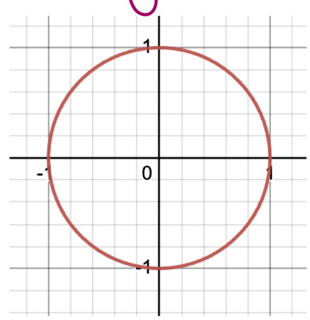

3 Parameterizing the world

With your group, look around the building for an interesting looking curve (A crooked treebranch, the metal tube in a bike rack, anything you can find). Take a picture of it. Then, try and come up with a parametric curve which matches the curve in the picture. You can check how well your curve matches your picture using https://www.desmos.com/calculator. Try for the most interesting picture/curve combo.