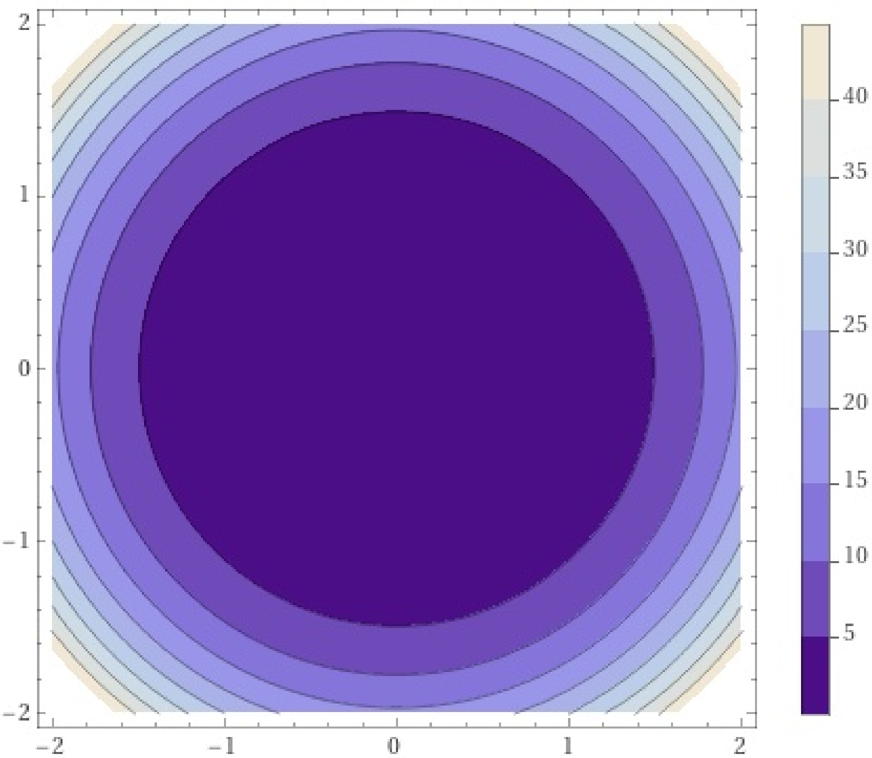
Quiz 3: Gradients, Tangent Planes, chain Rule

Name: Section:

All questions on this quiz deals with the function

$$f(x, y) = (x^2 + y^2)^2$$

To save time, here is a contour plot of this function (thanks to wolfram alpha):



1. (5 points) Compute the gradient of f at the point $(1, 0)$. Draw the resulting vector on the contour plot, starting from $(1, 0)$

$$\begin{aligned}\nabla f &= (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \\ &= (\partial_x(x^2 + y^2) \times 2(x^2 + y^2), \partial_y(x^2 + y^2) \times 2(x^2 + y^2)) \\ &= (4x(x^2 + y^2), 4y(x^2 + y^2))\end{aligned}$$

Plugging in $x = 1, y = 0$, we get

$$\nabla f(1, 0) = \boxed{\langle 4, 0\rangle}$$

We plot this on the graph as a vector, starting at the point $1, 0$, and pointing to the right. The scale itself isn't too important, as long as the direction is right and the vector is labeled.

2. (5 points) Find the equation for the tangent plane to the graph $z = f(x,y)$ at the point $(1, 0, f(1,0))$.

The tangent plane is tangent to the level set of the function $F(x,y,z) = z - f(x,y)$. The normal vector to the tangent curve is the gradient of this function, which is

$$\nabla F = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)$$

This gives the coefficients of the plane's equation. At the point $(1, 0, f(1,0)) = (1,0,1)$, the normal vector is $(-4, 0, 1)$. Since this plane passes thru the point $(1,0,1)$, its formula is

$$-4(x-1) + 0y + 1(z-1) = 0$$

Simplified, our final answer is

$$\boxed{-4(x-1) + z - 1 = 0}$$

3. (5 points) Consider the path

$$\gamma(t) = (x(t), y(t)) = (\cos(t), \sin(t))$$

Define the function $g(t) = f \circ \gamma = f(x(t), y(t))$. Compute $\dot{g}(0)$, the first derivative of g with respect to t at $t = 0$. (Hint: Try to directly substitute γ into f)

$$g(t) = f(x(t), y(t)) = \left(\sin^2(t) + \cos^2(t)\right)^2 = 1^1 = 1$$
$$\dot{g}(0) = \frac{\mathrm{d}}{\mathrm{d}t}1 = \boxed{0}$$

4. (5 points) Now consider the path

$$\rho(t) = (x(t), y(t)) = (1 + \cos(t), -1 + \sin(t))$$

Define the function $h(t) = f \circ \rho = f(x(t), y(t))$. Compute $\dot{h}(\pi/2)$, the first derivative of h with respect to t at $t = \pi/2$. (Hint: Direct substitution will make you sad. Try and use the chain rule)

The chain rule states

$$\frac{\mathrm{d}}{\mathrm{d}t} f(x(t), y(t)) = \frac{\partial f}{\partial x}(x(t), y(t))\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}(x(t), y(t))\frac{\mathrm{d}y}{\mathrm{d}t}$$

Evaluating this at $t = \pi/2$, we get

$$x(\pi/2) = 1 \qquad y(\pi/2) = 0$$

So, the partial derivatives of f are exactly those we calculated in the first problem of the quiz: $\partial_x f = 4$, $\partial_y f = 0$. We also compute the derivatives of x, y:

$$\frac{\mathrm{d}x}{\mathrm{d}t}(\pi/2) = -\sin(\pi/2) = -1$$
$$\frac{\mathrm{d}y}{\mathrm{d}t}(\pi/2) = \cos(\pi/2) = 0$$

Putting it together,

$$\frac{\mathrm{d}}{\mathrm{d}t} f(x(t), y(t))|_{t=\pi/2} = 4 \cdot (-1) + 0 \cdot 0 = \boxed{-4}$$