Quiz 2: Multivariable functions and limits

Name: Answer Key Section:

1. Consider the function

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$$

(a) (5 points) Compute the partial derivative of $f(x, y)$ with respect to x

$$\frac{\partial}{\partial x}\frac{x^2-y^2}{x^2+y^2} = \frac{(x^2+y^2)\,\partial_x(x^2-y^2) - (x^2-y^2)\,\partial_x(x^2+y^2)}{(x^2+y^2)^2}$$

quotient rule

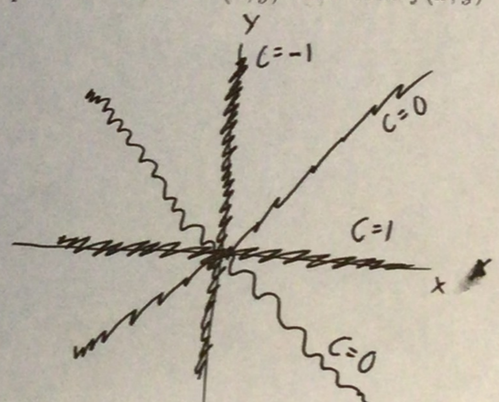
$$= \frac{2x(\cancel{x^2}+y^2) - 2x(\cancel{x^2}-y^2)}{(x^2+y^2)^2} = \boxed{\frac{4xy^2}{(x^2+y^2)^2}}$$

(b) (5 points) sketch the level curves of $f(x, y)$. That is, plot the values of (x, y) such that $f(x, y) = c$ for $c = -1, 0, 1$.

$$\frac{x^2-y^2}{x^2+y^2} = 0 \Downarrow x^2 - y^2 = 0 \Downarrow x^2 = y^2 \Downarrow y = \pm x$$

$$\frac{x^2-y^2}{x^2+y^2} = 1 \Rightarrow x^2 - y^2 = x^2 + y^2 \Rightarrow 2y^2 = 0 \Rightarrow y = 0$$

$$\frac{x^2-y^2}{x^2+y^2} = -1 \Rightarrow x^2 - y^2 = -x^2 - y^2 \Rightarrow 2x^2 = 0 \Rightarrow x = 0$$



(c) (5 points) Does the limit as $(x, y) \to (0, 0)$ exist? (Hint: Consider the limit along the path $(x(t), y(t)) = (t, mt)$ for different m).

$m=0$: $f(x(t), y(t)) = f(t, 0) = t^2/t^2 = 1$

$$\lim_{t\to 0} f(x(t), y(t)) = \lim_{t \to 0} 1 = 1$$

$m=1$: $f(x(t), y(t)) = f(t, t) = \frac{t^2 - t^2}{t^2 + t^2} = \frac{0}{2t^2} = 0$

$$\lim_{t\to 0} f(x(t), y(t)) = \lim_{t\to 0} 0 = 0$$

The paths $(t, 0)$ & (t, t) both approach $(0,0)$ as $t \to 0$

~~these paths~~

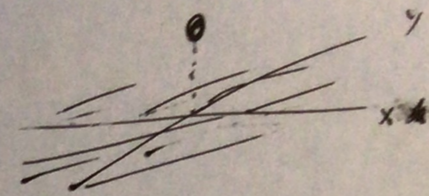
$f(x,y)$ limits to a different value along these paths

therefore, $\boxed{\lim_{(x,y)\to(0,0)} f(x,y) \text{ DNE}}$

2. (5 points) Recall that a multivariate function is *Continuous* at a point $\vec{a}$ if $\lim_{(x,y)\to\vec{a}} f(x,y)$ exists, and its value equals $f(\vec{a})$. Give an example of a function where $\lim_{(x,y)\to\vec{0}} f(x,y)$ exists, but $f(x,y)$ is not continious at zero.

Define

sketch of $f(x,y)$



$$f(x,y) = \begin{cases} 1 & x=y=0 \\ 0 & \text{else} \end{cases}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

~~But f(x,y)~~

But, $f(0,0) = 1 \neq \lim_{(x,y)\to(0,0)} f$

So $f(x,y)$ is not continous at $(0,0)$