Quiz 1: Parametric equations, polar coordinates, vector operations

Name: Answer key

1. Consider the vectors $\vec{u} = \langle 0, 1, 1\rangle$, $\vec{v} = \langle 2, 1, -1\rangle$.

   (a) (2 points) What is the angle between $\vec{u}$ and $\vec{v}$?

$\vec{u} \cdot \vec{v} = 0 \cdot 2 + 1 \cdot 1 + 1 \cdot (-1) = 0 + 1 + -1 = 0$

$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|u||v|} = 0 \Rightarrow \cancel{\theta = 2\pi}$ $\boxed{\theta = \pi/2}$ the vectors are orthogonal

   (b) (3 points) Construct a vector perpendicular to both $\vec{u}$ and $\vec{v}$.

$u \times v$: $\begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = \left(\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}, -\begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} \right)$

$= \left(1(-1) - 1 \cdot 1, -(0(-1) - 1 \cdot 2), (0 \cdot 1 - 1 \cdot 2) \right)$

$\boxed{= (-2, 2, -2)}$

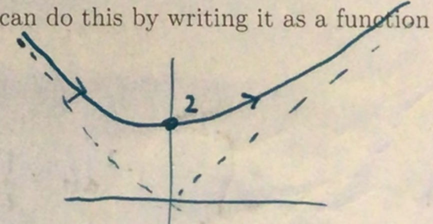
(optional) check:

$(u \times v) \cdot u = -2 \cdot 0 + 2 \cdot 1 + -2 \cdot 1 = 0 + 2 - 2 = 0$

$(u \times v) \cdot v = -2 \cdot 2 + 2 \cdot 1 + (-2) \cdot (-1)$
$= -4 + 2 + 2 = 0$

2. Consider the curve given by parametric equations $x(t) = t - 1/t$, $y(t) = t + 1/t$

   (a) (2 points) Sketch the curve traced by this parametric equation (You can do this by writing it as a function $y(x)$, plotting points, or any other means)



$x(t)^2 = t^2 - 2 + (\frac{1}{t}) + (\frac{1}{t})^2 = t^2 + 1/t^2 - 2$

$y(t)^2 = t^2 + 2 + (\frac{1}{t}) + (\frac{1}{t})^2 = t^2 + 1/t^2 + 2$

$x(t)^2 - y(t)^2 = -4$, so this traces a subset of the curve $y = \sqrt{x^2+4}$

this is a hyperbola, with asymptotes $y = \pm x$ & y-intercept 2

as $t \to 0$, $x \to -\infty$. as $t \to \infty$, $x \to \infty$. therefore,

~~as t increases, x inc~~ the entire ~~para~~ hyperbola is encompassed in this curve. for $-\infty < t < 0$, get lower half of hyperbola

(b) (3 points) Find the equation for the tangent line to this curve at $t = 1$.

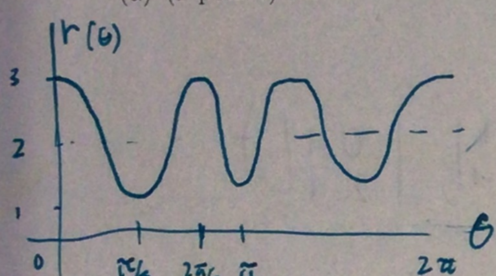
$x(1) = 1 - 1/1 = 0 \qquad \frac{dx}{dt} = \dot{x} = 1 + 1/t^2 \qquad \dot{x}(1) = 1 + 1/1 = 2$

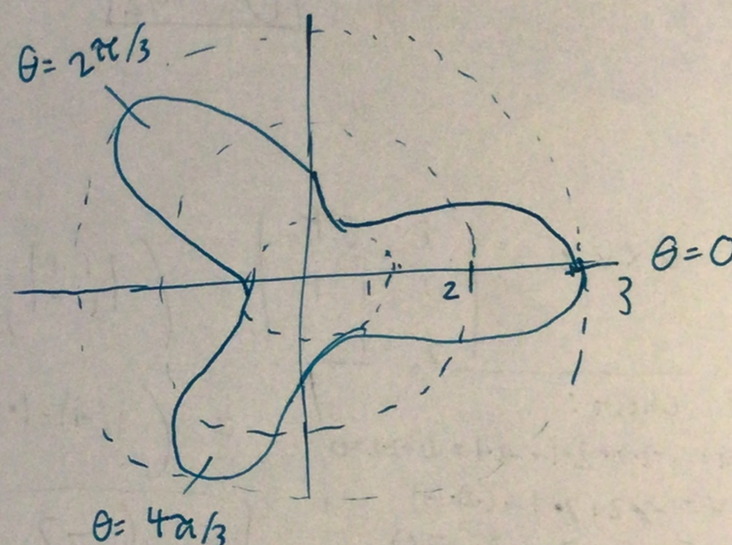
$y(1) = 1 + 1/1 = 2 \qquad \frac{dy}{dt} = \dot{y} = 1 - 1/t^2 \qquad \dot{y}(1) = 1 - 1/1 = 0$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{0}{2} = 0$, so the tangent line is horizontal w/ y-intercept 2, thus, it's equation is $\boxed{y = 2}$

3. Consider the polar curve $r(\theta) = 2 + \cos(3\theta)$

(a) (2 points) sketch the curve





(b) (3 points) Find the area inside the curve. (you may use that $\int_0^{2\pi} \cos^2(x)dx = \pi$.)

$u = 3\theta$

$du = 3d\theta$

$$\text{area} = \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \frac{1}{2}\int_0^{2\pi} 4 + 4\cos(3\theta) + \cos^2(3\theta)\, d\theta$$

$$= \frac{1}{2}\left(4 \cdot 2\pi + 4 \cdot \int_{u=0}^{u=6\pi} \cos(u) \frac{du}{3} + \int_0^{6\pi} \cos^2(u) \frac{du}{3}\right)$$

$$= \frac{1}{2}\left(8\pi + \underbrace{\frac{4}{3} \sin u \Big|_0^{6\pi}}_{0} + \frac{3}{3} \int_0^{2\pi} \cos^2(u) du\right) = \frac{1}{2}(8\pi + \pi)$$

$$\boxed{\text{area} = \frac{9\pi}{2}}$$