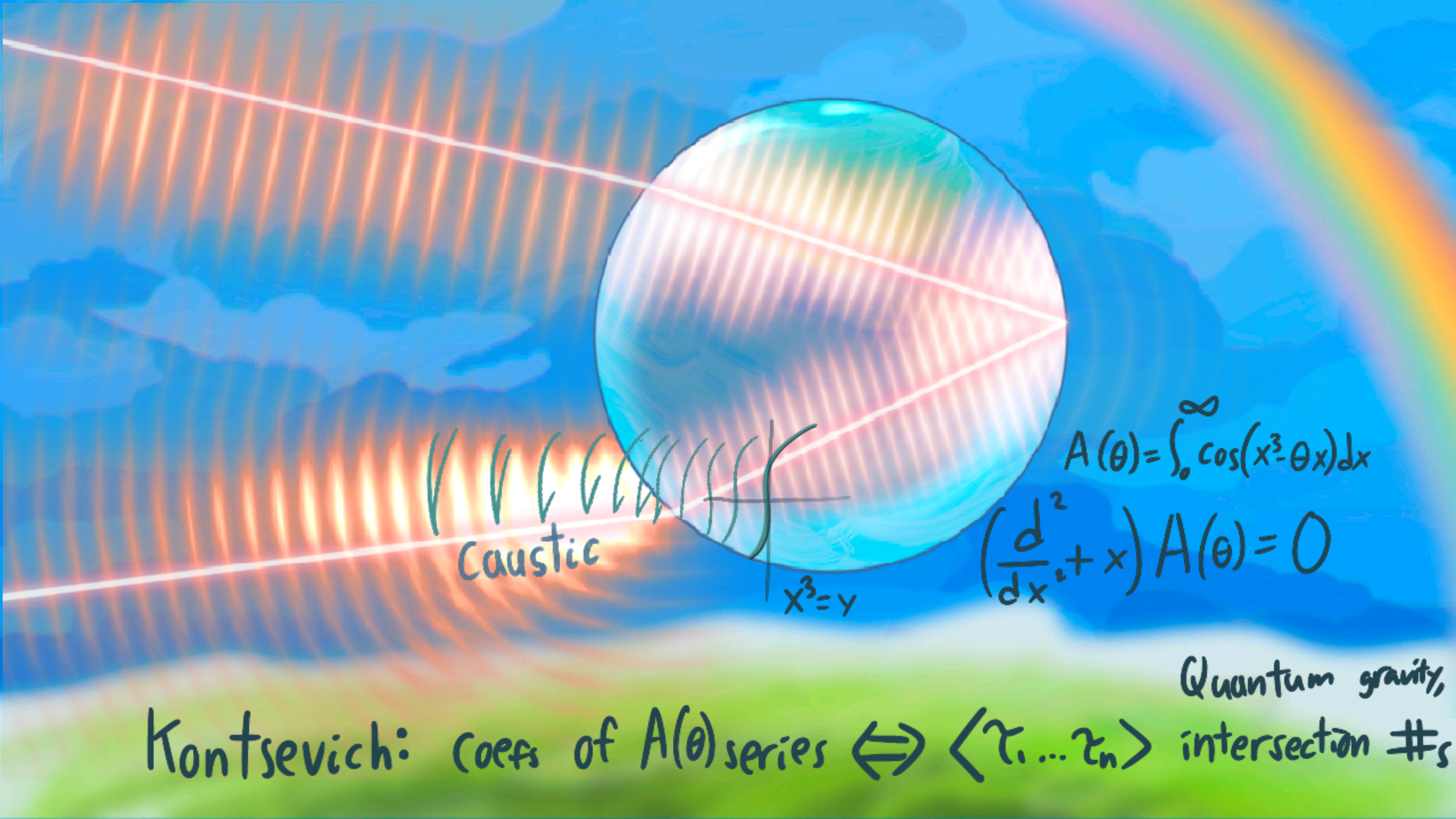
Topological Recursion

~For fun & Profit~


R
O Y
G B
V G?
V? G?
V?


$A(\theta) = \int_0^\infty \cos(x^3 - \theta x)\,dx$
$\left(\frac{d^2}{dx^2} + x\right) A(\theta) = 0$
caustic
$x^3 = y$
Quantum gravity,
Kontsevich: coefs of $A(\theta)$ series $\Leftrightarrow$ $\langle \tau_1 \dots \tau_n \rangle$ intersection #s

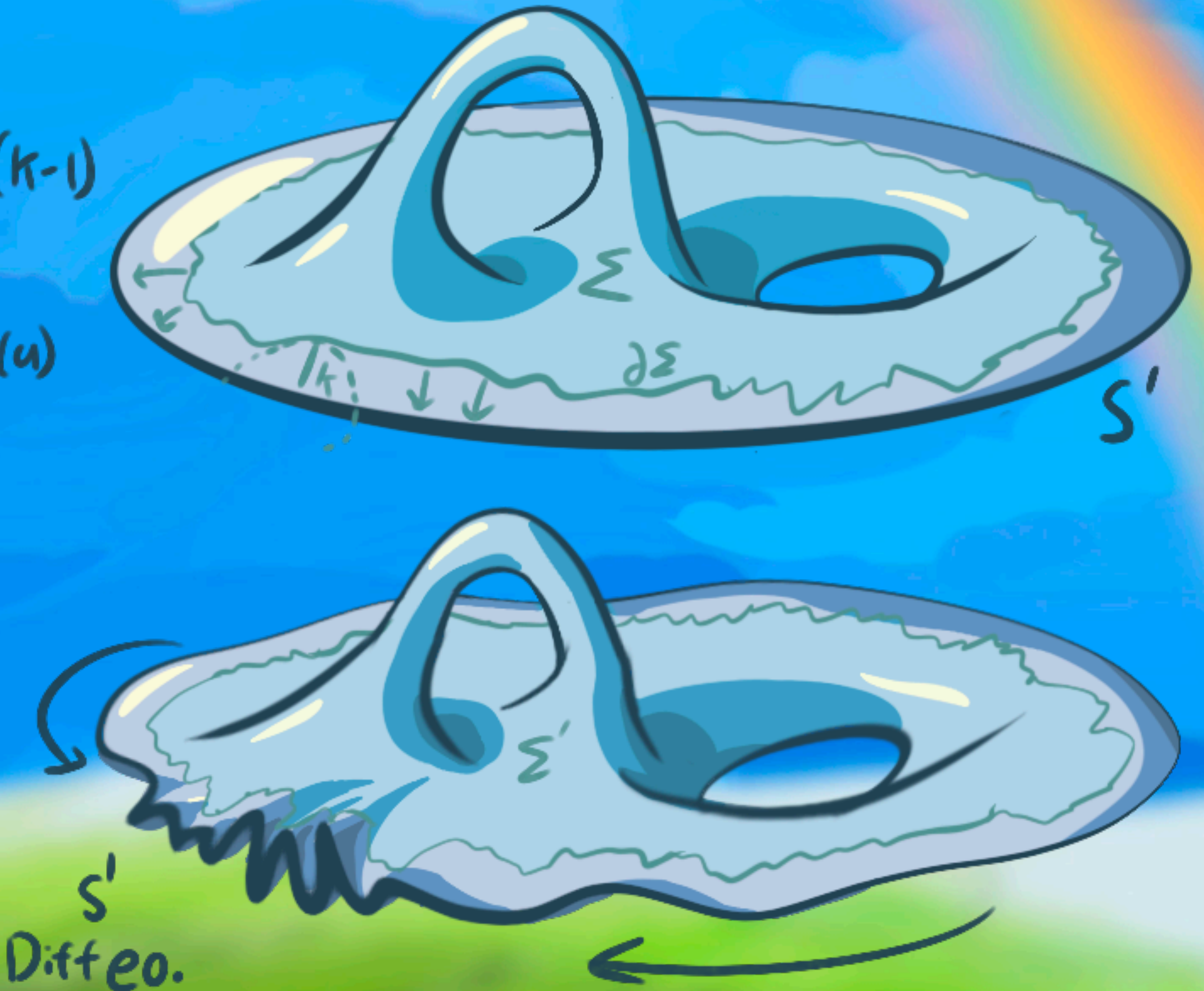

Topological
gravity

JT gravity: $S_{bulk} = \int_\Sigma \phi(R+2)$

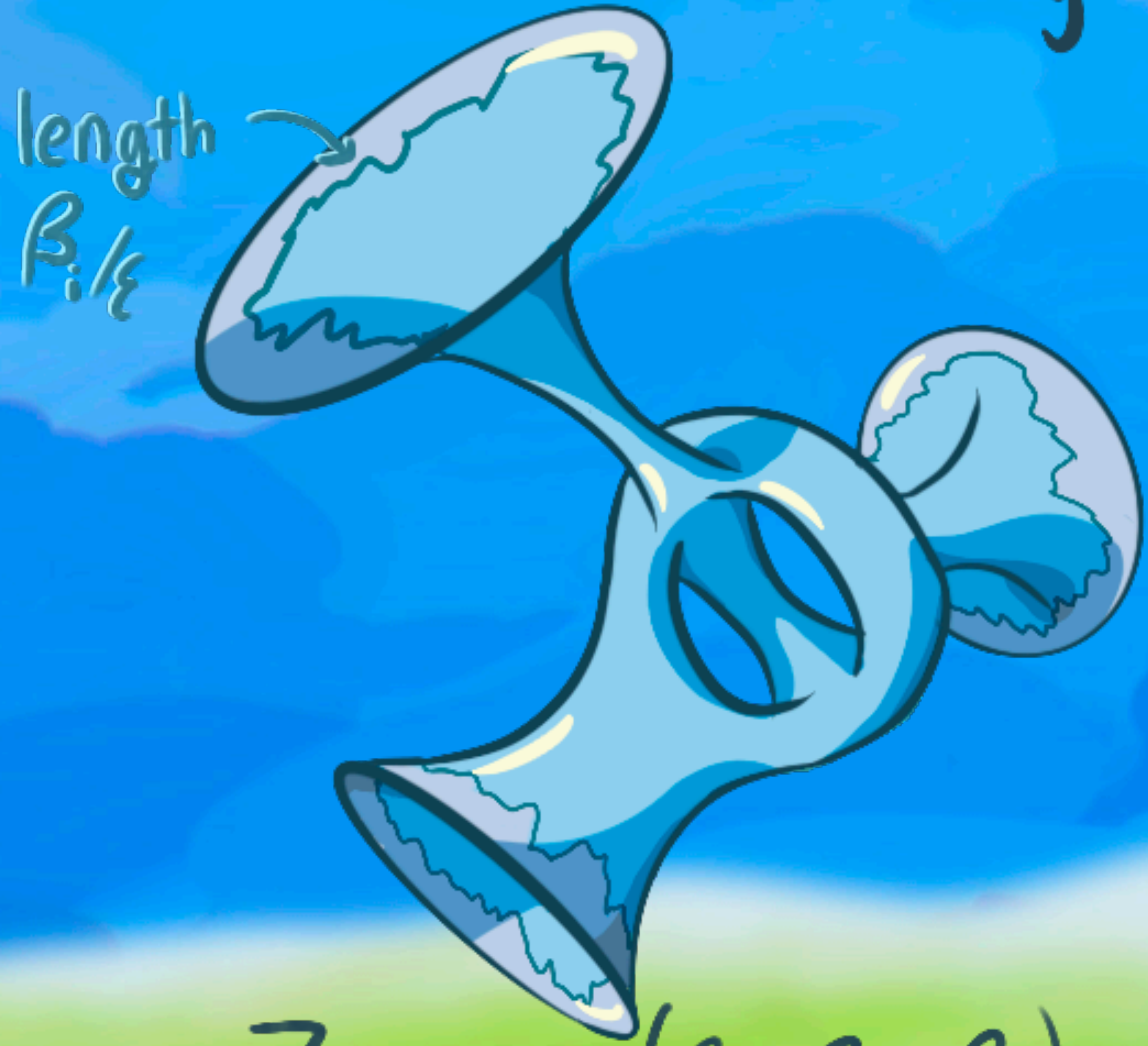
$$S = S_0\underset{\chi(\Sigma)}{\underset{\shortparallel}{\left(\int_\Sigma R + \int_{\partial\Sigma} K\right)}} + \underset{R=-2}{\underset{\Downarrow}{\int_\Sigma \phi(R+2)}} + \underset{\int_{S^1}\mathrm{Sch}(u)}{\underset{\Downarrow}{\int_{\partial\Sigma}\phi(K-1)}}$$

$$Z = \int_{\frac{\mathrm{Diff}(S^1)}{PSL(2,\mathbb{R})}} du \; e^{\chi + \int_{S^1}\mathrm{Sch}(u)}$$

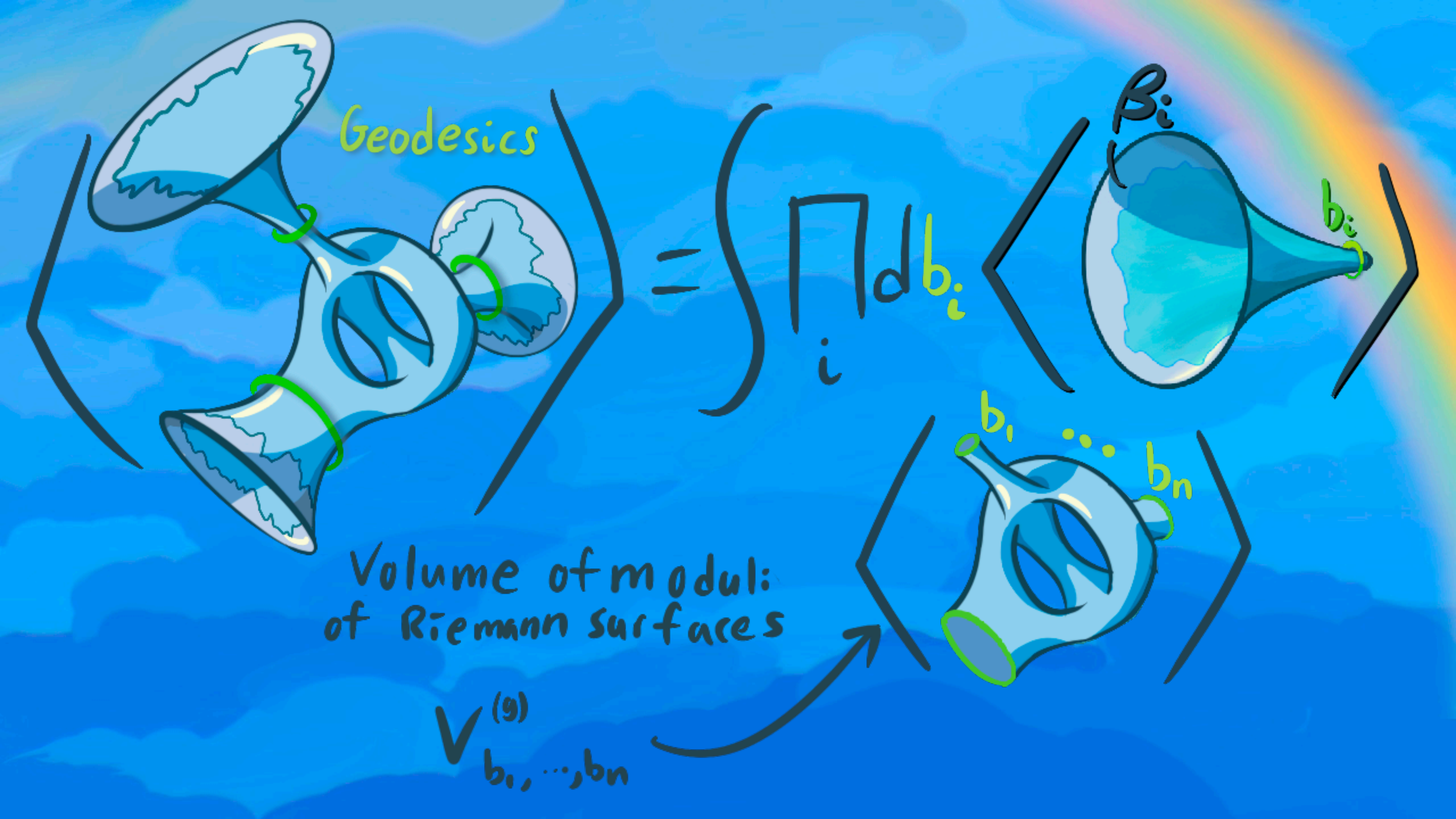
"Universal Teichmüller space"

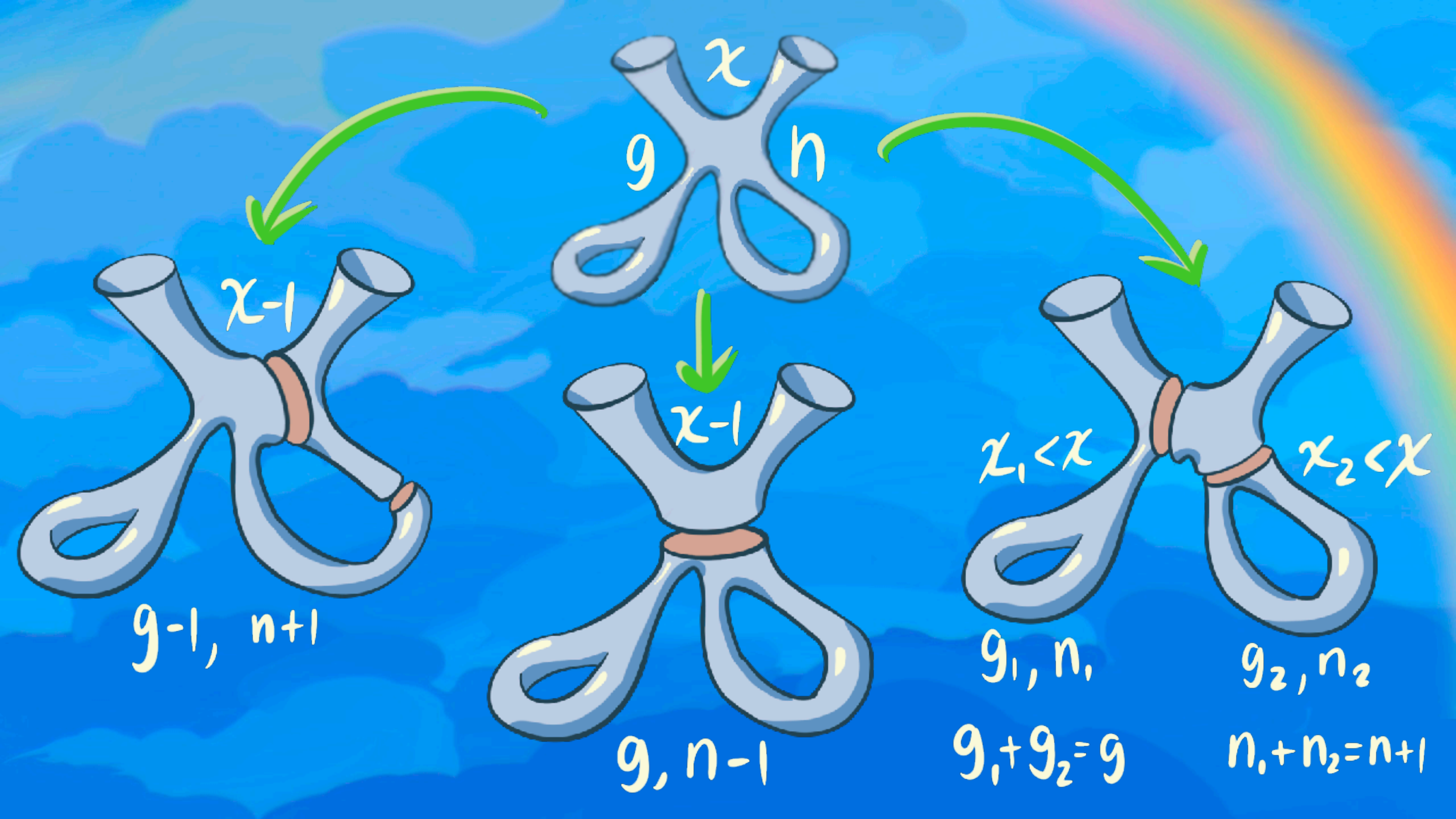


$$\langle Z(\beta_1) \cdots Z(\beta_n) \rangle = \sum_g \frac{\langle \cdots \rangle_g}{\hbar^{2g-2+n}}$$

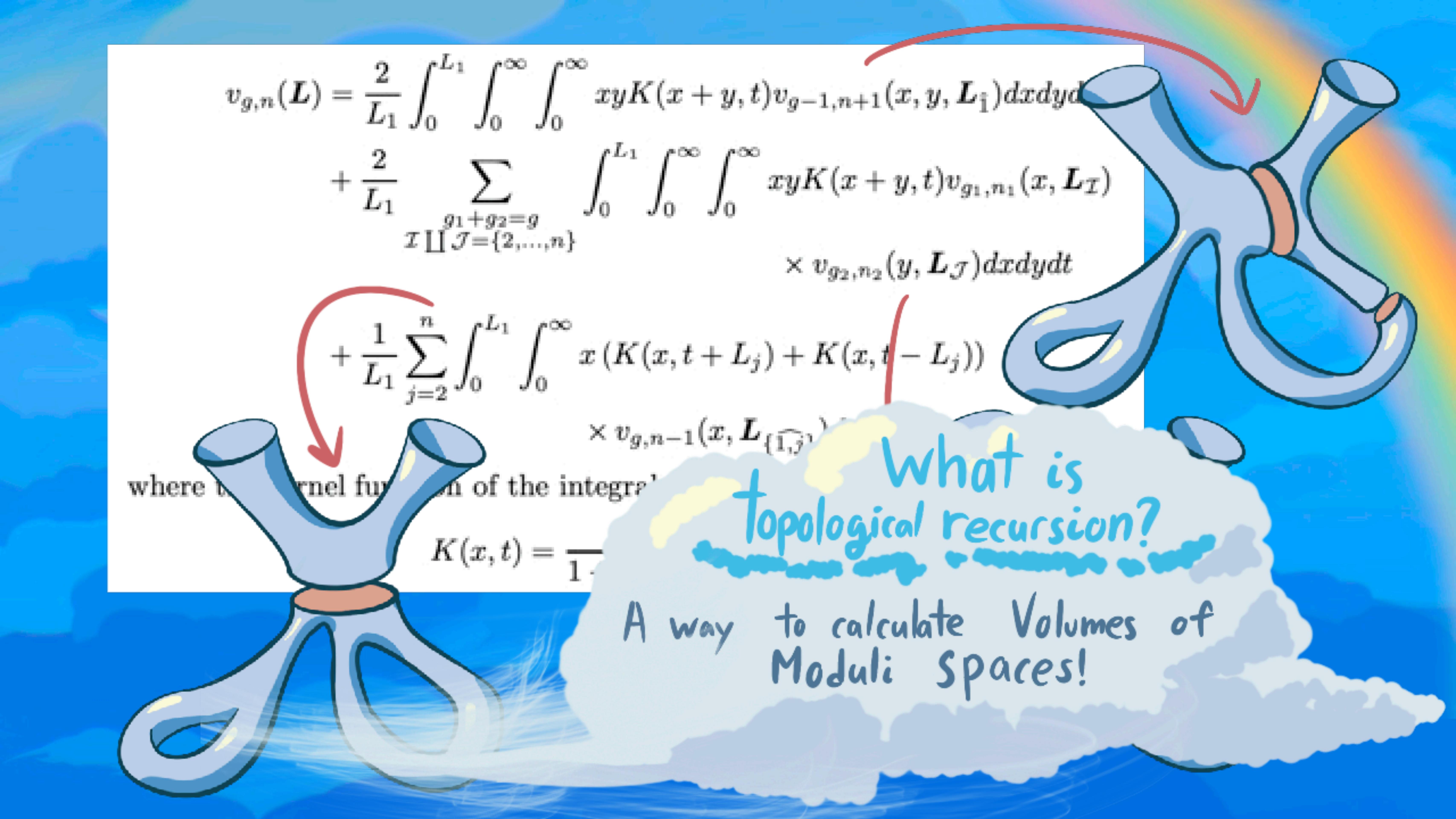


$$Z_{g=2,\, n=3}(\beta_1, \beta_2, \beta_3)$$


Geodesics
$= \int \prod_i d b_i$
β_i
b_i
$b_1 \dots b_n$
Volume of moduli of Riemann surfaces
$V^{(g)}_{b_1, \dots, b_n}$



χ
g
n
$\chi - 1$
$g-1, \; n+1$
$\chi - 1$
$g, \; n-1$
$\chi_1 < \chi$
$\chi_2 < \chi$
$g_1, \; n_1$
$g_2, \; n_2$
$g_1 + g_2 = g$
$n_1 + n_2 = n+1$



$v_{g,n}(\boldsymbol{L}) = \frac{2}{L_1}\int_0^{L_1}\int_0^{\infty}\int_0^{\infty} xyK(x+y,t)v_{g-1,n+1}(x,y,\boldsymbol{L}_{\hat{1}})dxdyd$
$+ \frac{2}{L_1}\sum_{\substack{g_1+g_2=g \\ \mathcal{I}\coprod\mathcal{J}=\{2,\dots,n\}}}\int_0^{L_1}\int_0^{\infty}\int_0^{\infty} xyK(x+y,t)v_{g_1,n_1}(x,\boldsymbol{L}_{\mathcal{I}})$
$\times v_{g_2,n_2}(y,\boldsymbol{L}_{\mathcal{J}})dxdydt$
$+ \frac{1}{L_1}\sum_{j=2}^{n}\int_0^{L_1}\int_0^{\infty} x\,(K(x,t+L_j)+K(x,t-L_j))$
$\times v_{g,n-1}(x,\boldsymbol{L}_{\{\widehat{1,j}\}})$
What is topological recursion?
A way to calculate Volumes of Moduli Spaces!


Random
Matrix
Theory

$$\int e^{N \operatorname{Tr} V(M)} = \int e^{N \operatorname{Tr} M^2 + g_3 M^3 + \dots}$$

$$N g_4 \delta_{ij} \delta_{kl} \delta_{mn} \delta_{pq}$$

$$\sum \delta_{ij} \delta_{jk} \delta_{kl} \delta_{li} = N$$

$$\langle M_{ij} M_{kl} \rangle = \frac{1}{N} \delta_{ij} \delta_{kl}$$

Ribbon graph

N^2

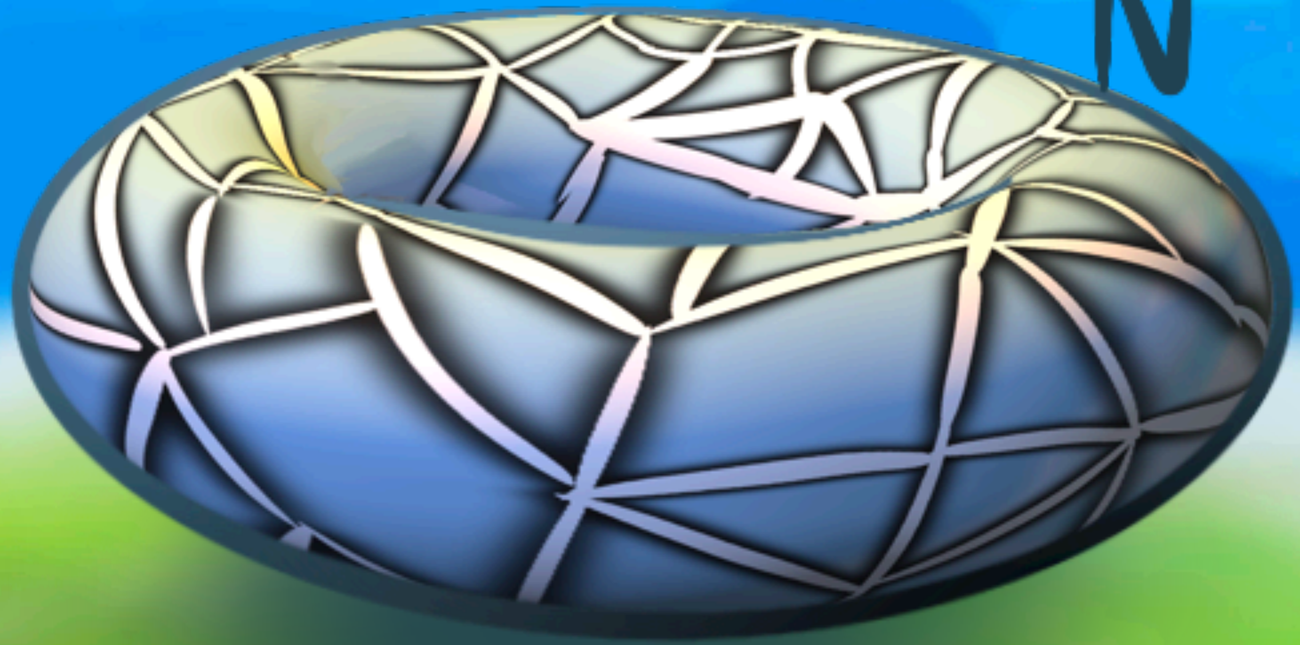
N^0

$\#N = \#\text{vertex} - \#\text{edge} + \#\text{face} = \chi$

N - expansion = genus expansion

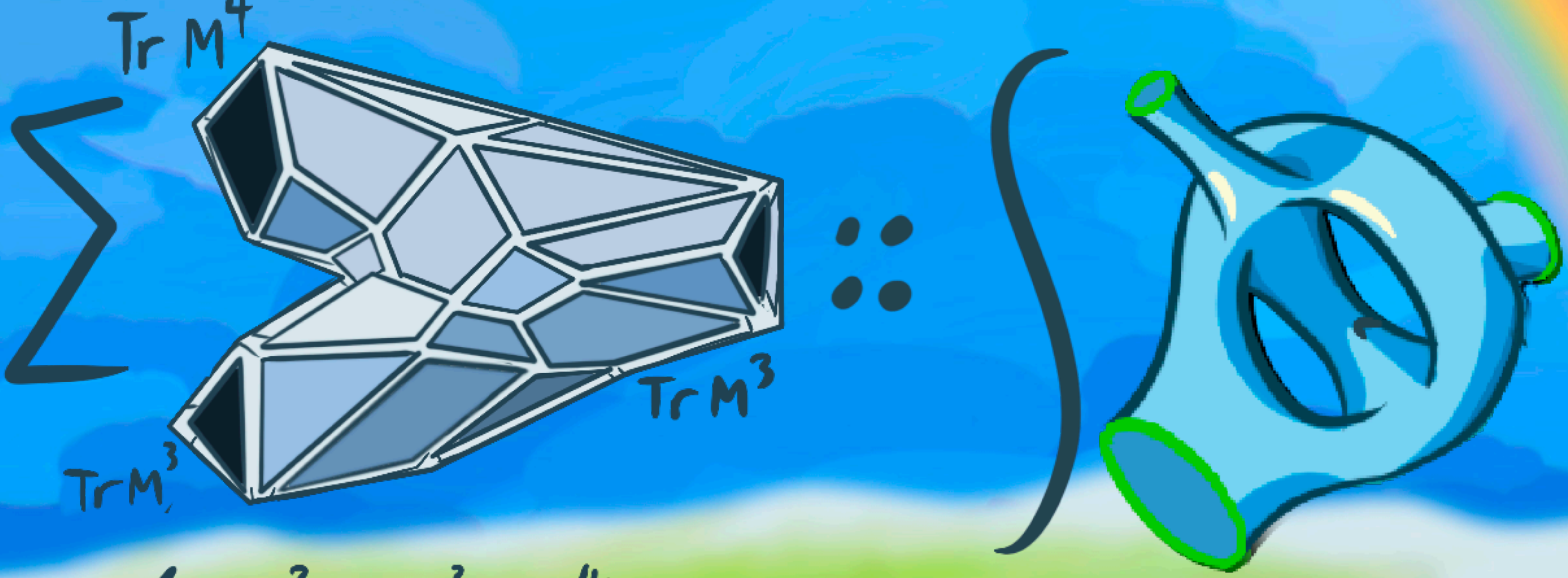
Combinatorial Riemann Surface!

N^2



N^0

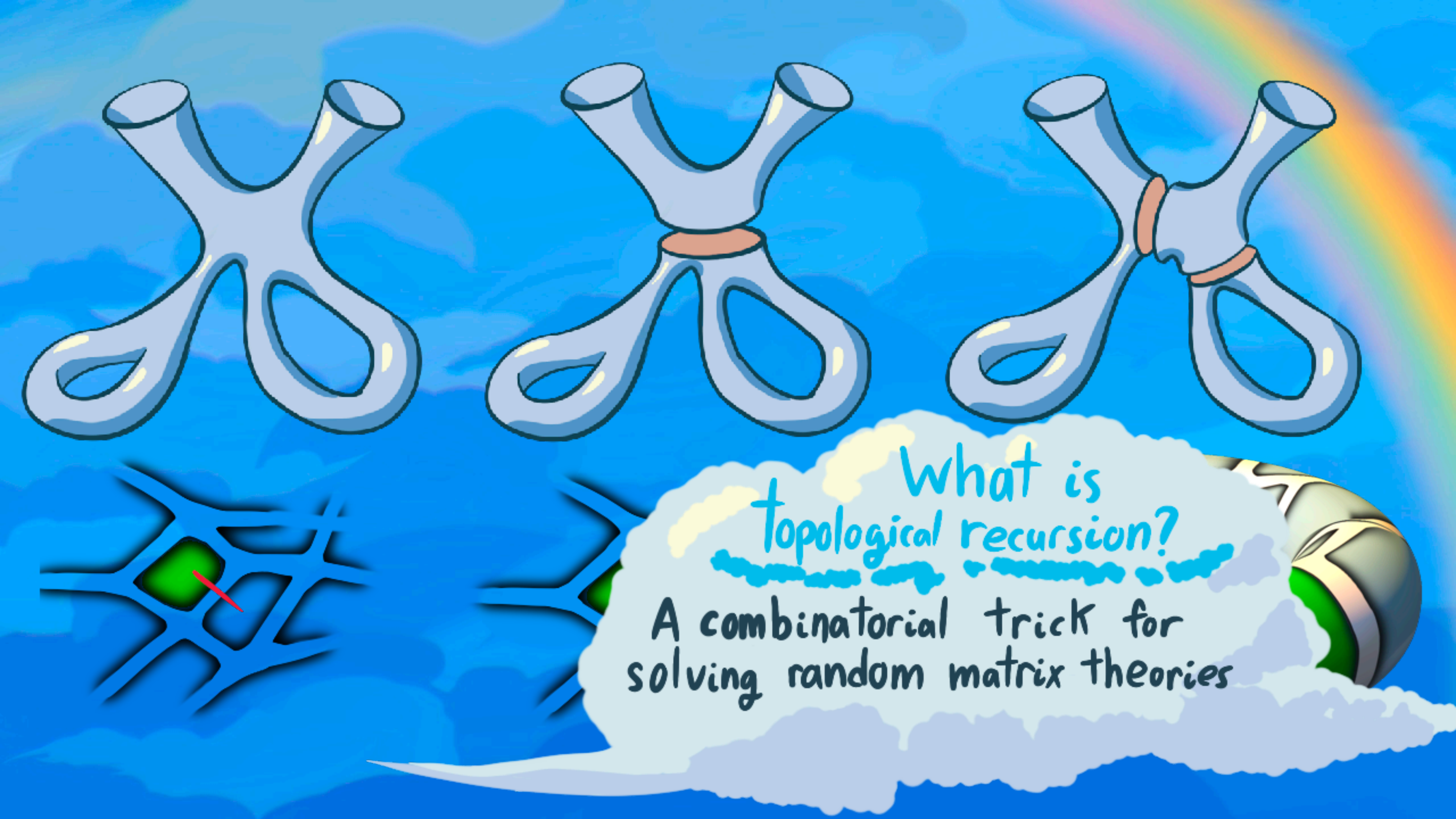
$$\langle \mathrm{Tr}\, M^3 \rangle = \langle M^i{}_j M^j{}_k M^k{}_i \rangle =$$



$$\langle \mathrm{Tr} M^3\, \mathrm{Tr} M^3\, \mathrm{Tr} M^4 \rangle$$

$$0 = d\langle \mathrm{Tr}M^{l_1} \cdots \mathrm{Tr}M^{l_2}\rangle = \sum_{i,j}\int \frac{\partial}{\partial M_{ij}} \underbrace{\left(M^{a_1}_{b_1} \cdots M^{a_{l_1}}_{b_{l_1}} \right)}_{\mathrm{Tr}M^{l_1}} \cdot \ldots \cdot e^{N\,\mathrm{Tr}\,V(M)}$$

$$N\Big\langle \mathrm{Tr}\,M^{\mu_1+1}\,\mathrm{Tr}\,M^{\mu_2}\cdots\mathrm{Tr}\,M^{\mu_n}\Big\rangle = \underbrace{\sum_{j=0}^{\mu_1-1}\Big\langle \mathrm{Tr}\,M^{j}\,\mathrm{Tr}\,M^{\mu_1-j-1}\,\mathrm{Tr}\,M^{\mu_2}\cdots\mathrm{Tr}\,M^{\mu_n}\Big\rangle}_{1}$$
$$+\underbrace{\sum_{i=2}^{n}\mu_i\Big\langle \mathrm{Tr}\,M^{\mu_1+\mu_i-1}\prod_{\substack{j=2\\ j\neq i}}^{n}\mathrm{Tr}\,M^{\mu_j}\Big\rangle}_{2} + \underbrace{\sum_{k=3}^{\infty}t_k\Big\langle \mathrm{Tr}\,M^{\mu_1+k-1}\,\mathrm{Tr}\,M^{\mu_2}\cdots\mathrm{Tr}\,M^{\mu_n}\Big\rangle}_{3}.$$


What is topological recursion?
A combinatorial trick for solving random matrix theories

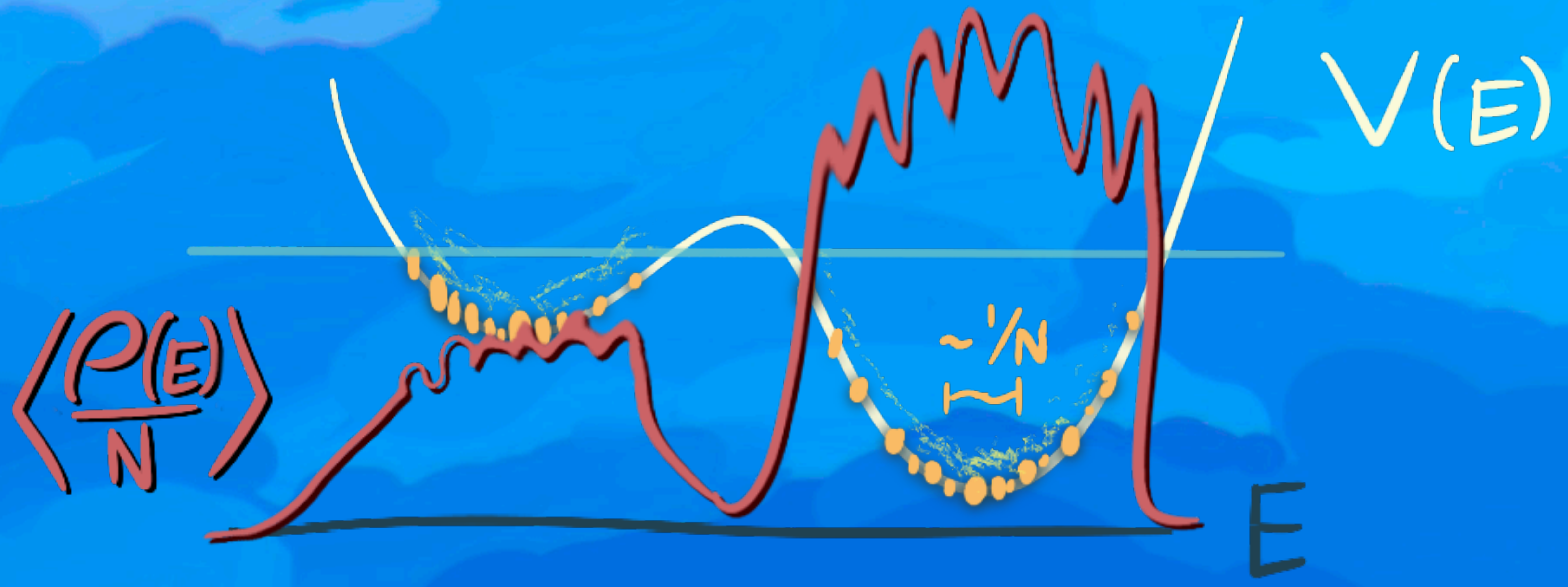

WKB

$$\int_{\underline{\mathcal{U}}(N)} dM\, e^{N \operatorname{Tr} V(M)} = \int \prod d\lambda_i \prod (\lambda_i - \lambda_j)^2\, e^{N \sum V(\lambda_i)}$$

$$= \int \prod d\lambda_i\, e^{N \sum_{i,j} V(\lambda_i) + 2 \ln(\lambda_i - \lambda_j)}$$

large N limit

$\Rightarrow$ saddle point approx.



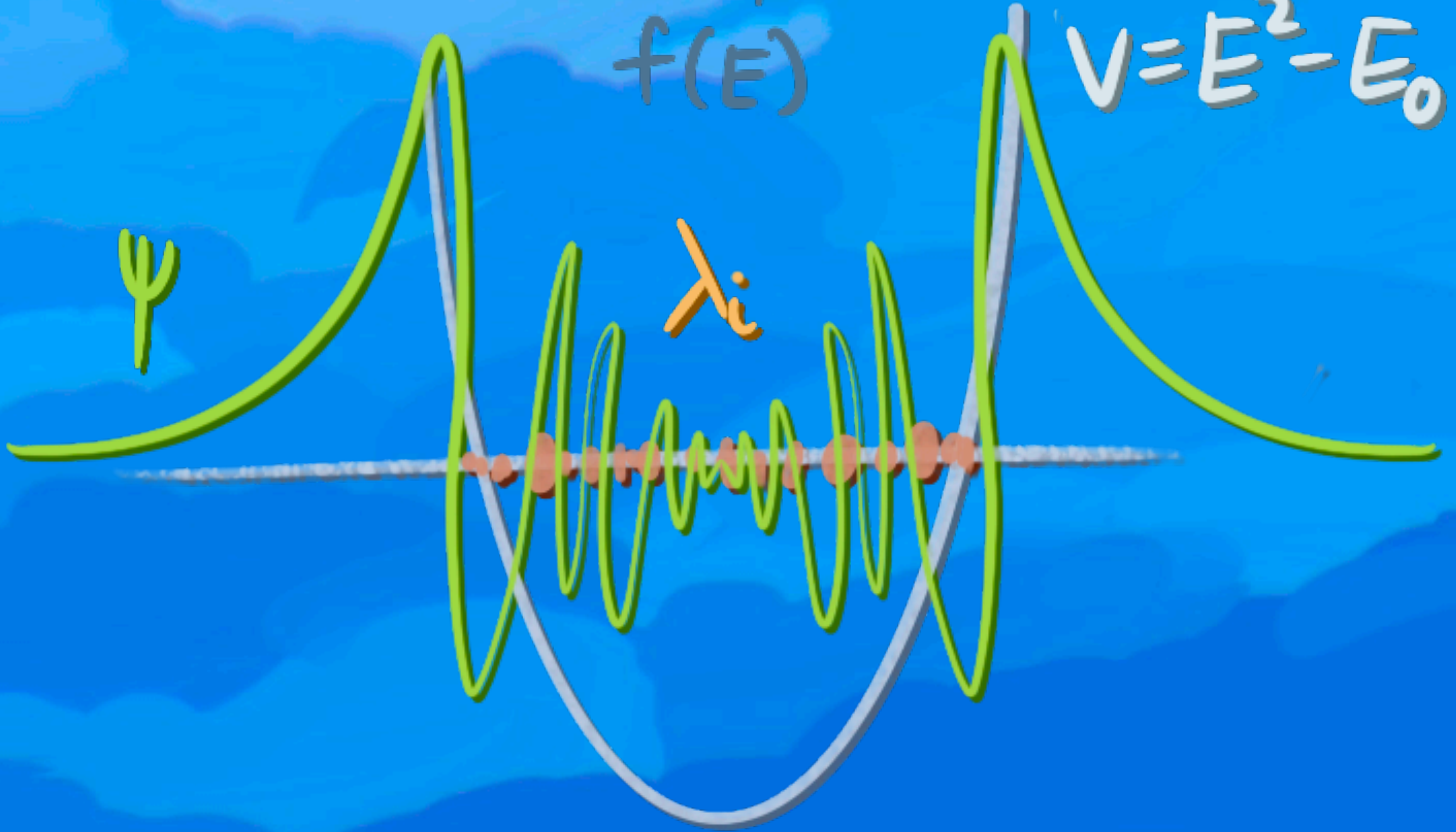
$\Psi(E) = e^{-V/2}\langle \det(M-E)\rangle$ charecteristic polynomial!

gauge transform

$$\left[\frac{1}{N^2}\frac{d^2}{dE^2} - \left(\frac{1}{4}V'^2 + \frac{1}{2}V'' + P(E)\right)\right]\Psi = 0, \qquad P(E) = \left\langle \frac{1}{N}\sum \frac{V'(E) - V'(\lambda_i)}{E-\lambda_i}\right\rangle$$

rational in E

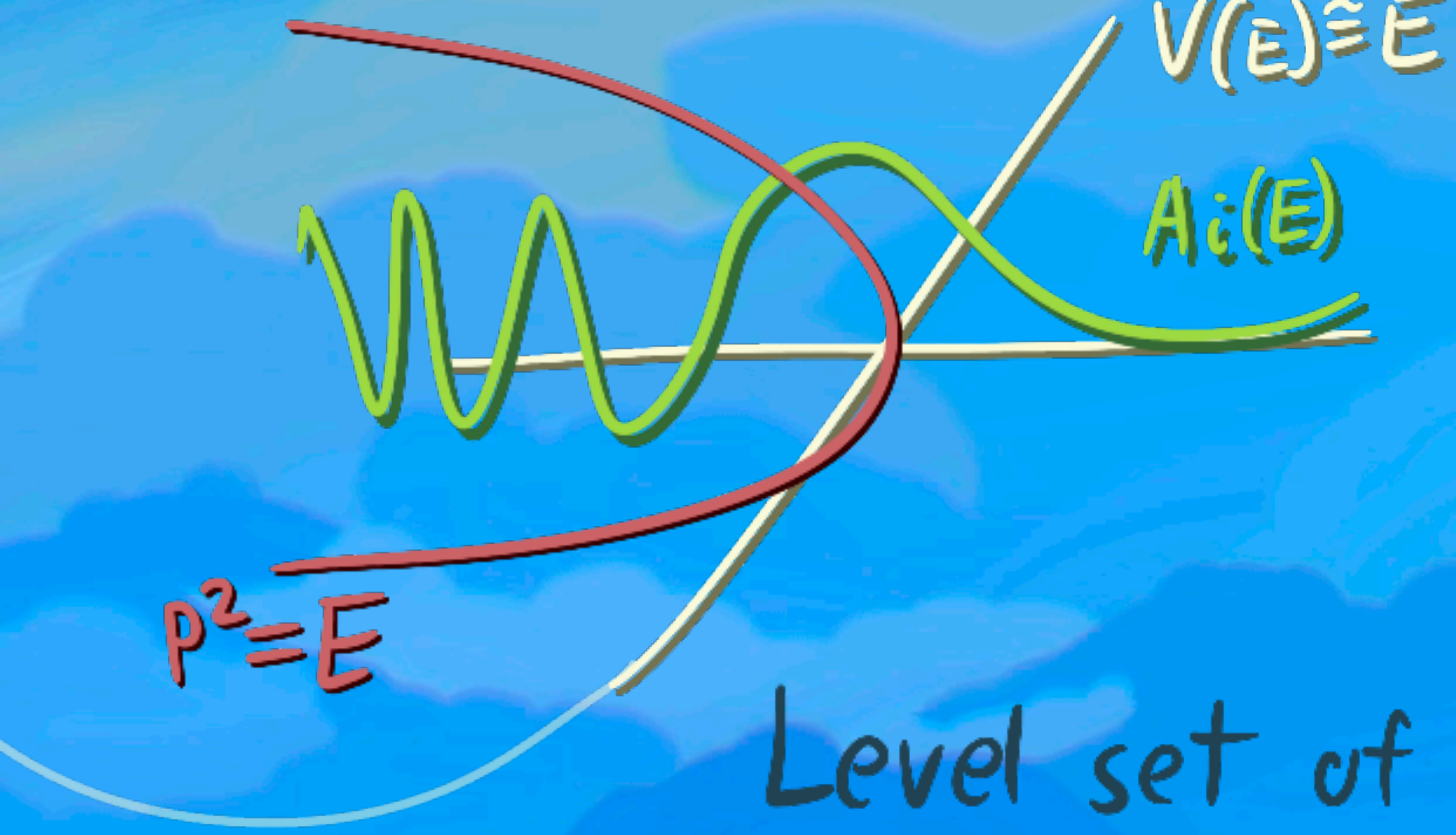
$f(E)$



$$\frac{1}{N^2}\Psi'' + x^2\Psi - E\Psi = 0$$

WKB:

$$\Psi = e^{\frac{i}{N}\sum \frac{1}{N^m} S_m}$$

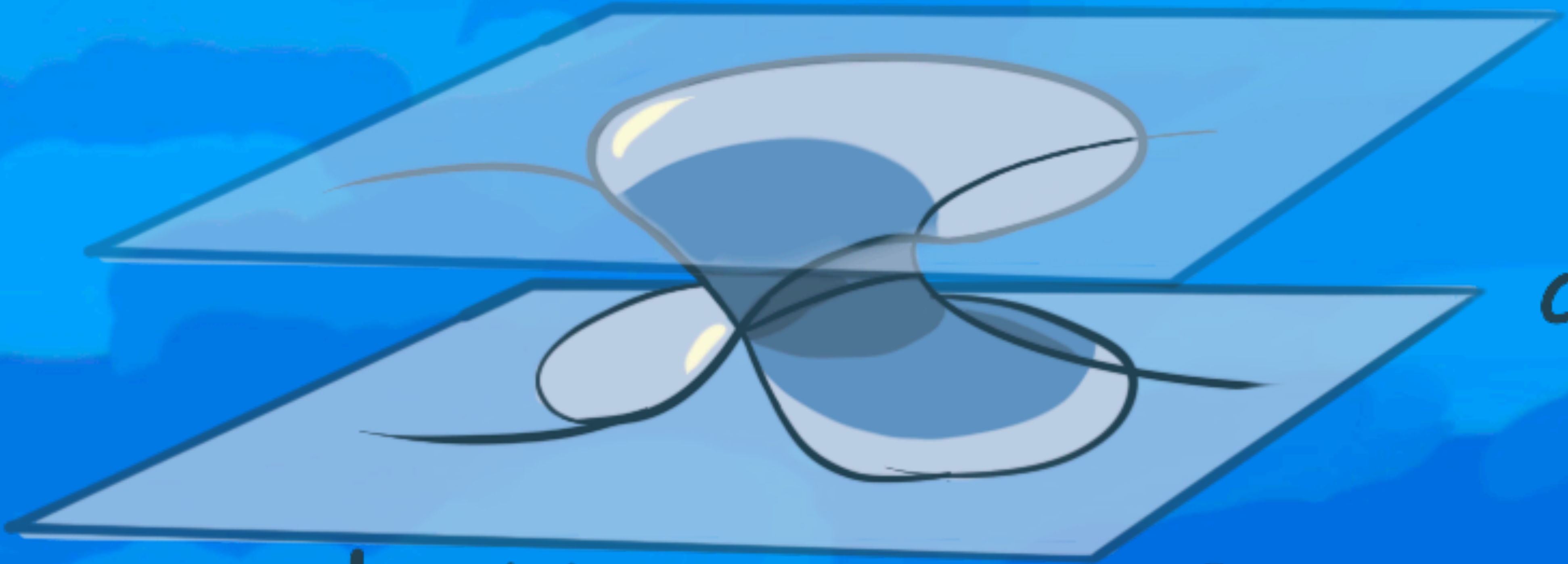


Complexify! $E \in \mathbb{C}$

$\frac{d}{dE}$ acts on complex $\Psi(E)$

$H = p^2/2m + V(E) : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$

Level set of H:



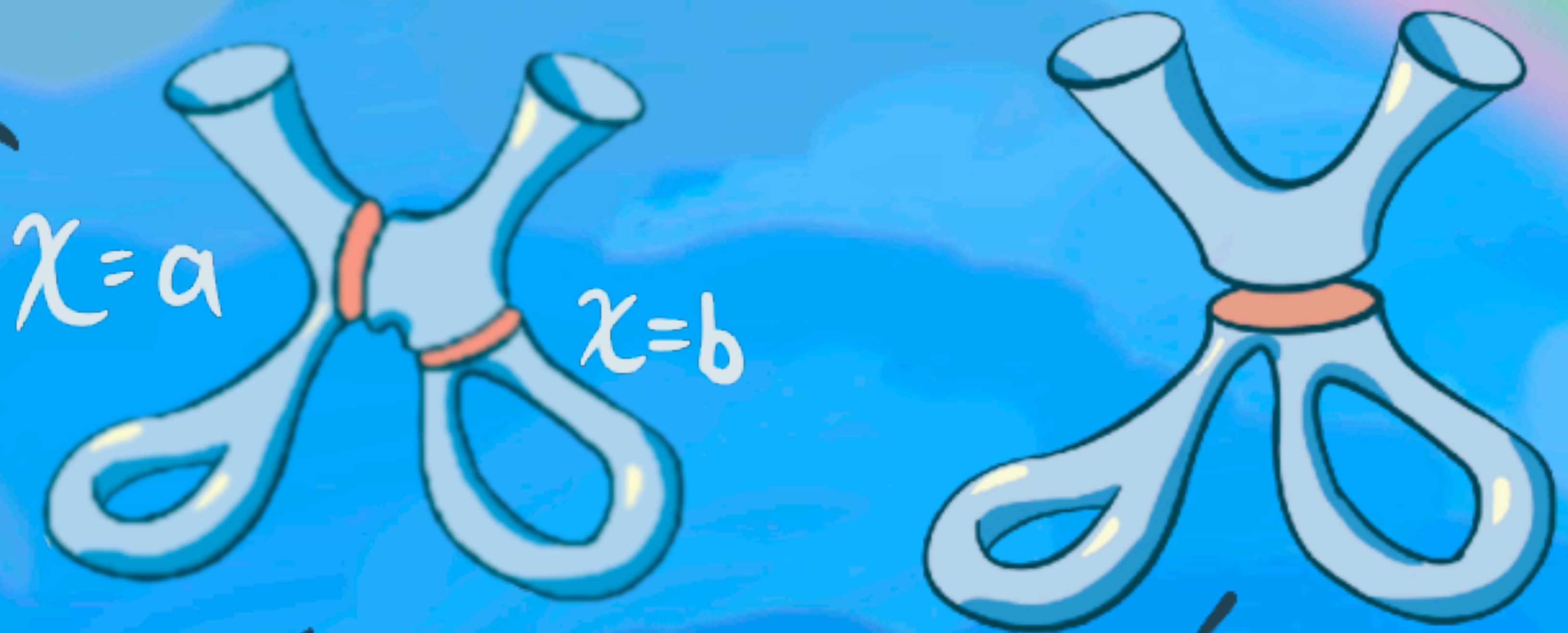
double cover of $\mathbb{C}$

$\Longrightarrow$ compact

Spectral curve!!

double cover of $\mathbb{CP}^1$

higher order WKB:



$$\frac{1}{N^{m+1}} : S_m'' + \sum_{a+b=m+1} S_a' S_b' + f(E)\, S_{m+1}' = 0$$

What is topological recursion?

Gives a semiclassical expansion for a diff e.q.


Topological
Topological
Topological
Topological
Topological
Rercursion
Rercursion
Rercursion
Rercursion
Rercursion

Eynard-Orantin topological recursion

spectral curve (S, x, y)

$\Rightarrow$ multidifferentials W^g_n

$$W^0_1 = y\,dx \qquad W^0_2 = B(z,z')$$

$$\int B(z,z') f(z') = f(z)$$

"Bergman Kernel"

$$W^g_{n+1}(z_0, J) = \sum_{\substack{a \\ \text{branch}}} \frac{\int_z^{\bar z} B(z_0,z')}{(y(z)-y(\bar z))\,dx}\left[W^{g-1}_{n+2}(z,\bar z,J) + \sum_{h=0}^{g}\sum_{I\subset J} W^h_{1+|I|}(z,I)\, W^{g-h}_{1+n-|I|}(\bar z, J\setminus I)\right]$$

Topological recursion: Algebraic invariants of spectral curve

topological gravity

A way to calculate Volumes of Moduli Spaces!

Random Matrix theory

A combinatorial trick for solving random matrix theories

WKB approximation

Gives a semiclassical expansion for a diff e.q.

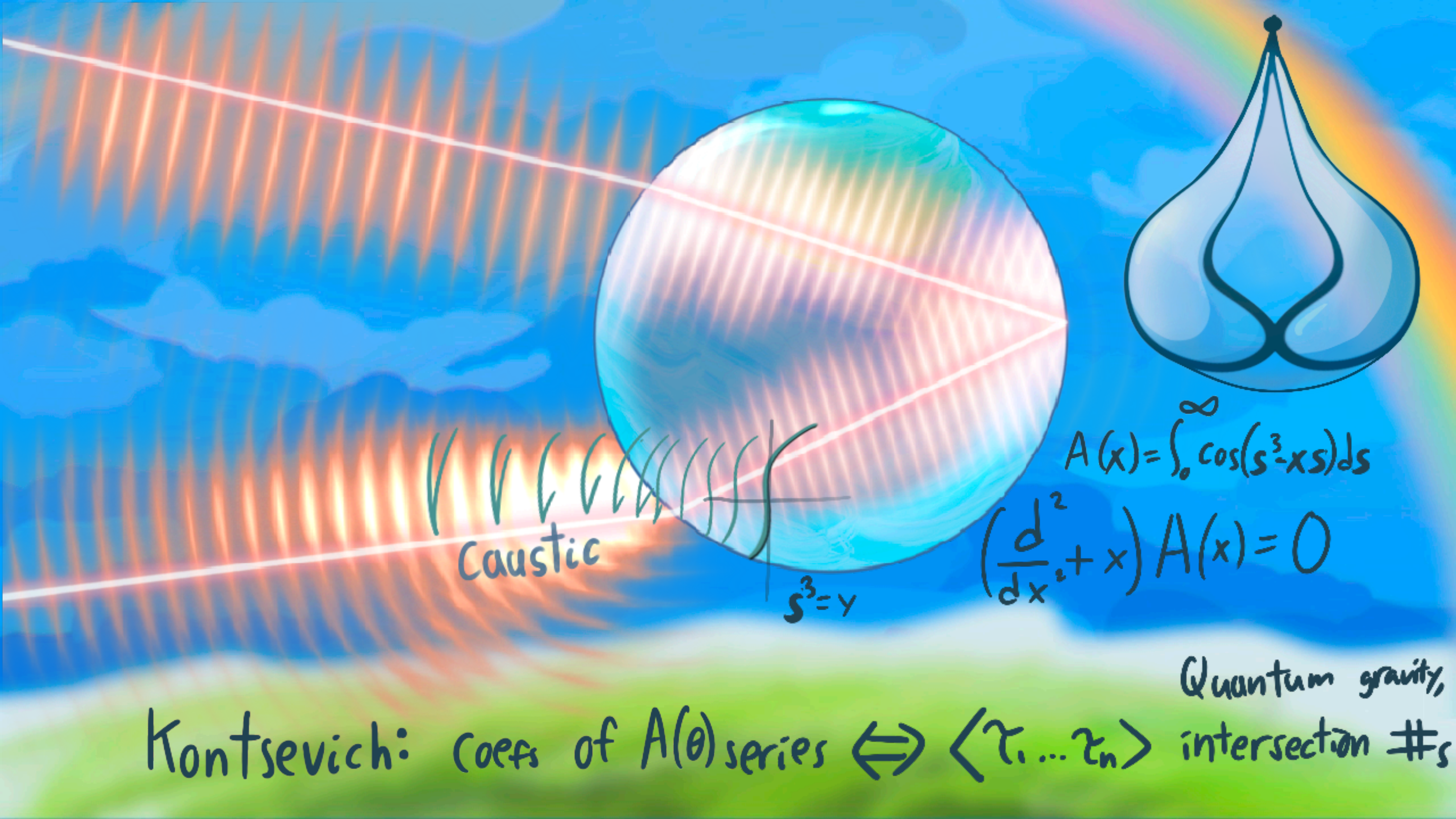
$$V_n^g(\beta_1,\dots,\beta_n) = \int dz_i\, e^{-\sum z_i\beta_i}\, W_n^g(z_1,\dots,z_n)$$

laplace transform

$$\langle \mathrm{Tr}\, M^n \rangle = \sum \frac{V^g{}_n}{N^{2g-2+n}}$$

$$S_{m+1} = \sum_{2g-2+n=m} \frac{1}{n!} \underbrace{\int\cdots\int}_{n} \Pi dz\, W_n^g(z_1,\dots,z_n)$$

A-model moduli-space	B-model Spectral curve
Kontsevich-Witten intersection numbers $W_{g,n} = \sum_{\mathrm{d}} < \tau_{d_1} \dots \tau_{d_n} >_g \prod_{i=1}^{n} \frac{(2d_i-1)!!}{z_i^{2d_i+2}} \, dz_i$	$y^2 = x$
Weil-Petersson volumes $\mathcal{V}_{g,n}(L_1, \dots, L_n)$	$y = \frac{\sin(2\pi\sqrt{x})}{4\pi}$
Hurwitz numbers $H_g(\mu)$	$y\,\mathrm{e}^{-y} = \mathrm{e}^x$
Random Matrix: asymptotic expansion of correlation functions $\mathbb{E}(\prod_i \operatorname{Tr}(x_i - M)^{-1})$	$y = 2\mathrm{i}\pi\,\rho_{\mathrm{eq}}(x)$
Toric Calabi-Yau Gromov-Witten invariants	mirror curve $H(\mathrm{e}^x, \mathrm{e}^y) = 0$
Knot theory Jones polynomial	A-polynomial $A(\mathrm{e}^x, \mathrm{e}^y) = 0$ character variety



Caustic
$s^3 = y$
$A(x) = \int_0^\infty \cos(s^3 - xs)\,ds$
$\left(\frac{d^2}{dx^2} + x\right) A(x) = 0$
Quantum gravity,
Kontsevich: coefs of $A(\theta)$ series $\Leftrightarrow$ $\langle \tau_1 \ldots \tau_n \rangle$ intersection #s

resolvent $R(E) = \mathrm{Tr}\,\frac{1}{E-H} = \sum \frac{1}{E-\lambda_i}$

$\oint_{\mathcal{C}} R(E)$ counts # λ_i in $\mathcal{C}$

series expansion $\Rightarrow$ $R^2 + \frac{1}{N} R' = V' R - P$ Riccati eq.

log det = Tr log $\Rightarrow$ $R = \frac{1}{N} \left(\log \det (E-H) \right)$

$\psi = e^{-V/2} \det(E-H)$ $\Rightarrow$ $\frac{1}{N^2} \psi'' + (V'^2 - P)\, \psi = 0$ schrödinger eq

$\mathcal{M}_{g,n}$

$\mathcal{L}_i = K_\Sigma \big|_{p_i}$

Chern class ψ_i 2-form — line bundle on $\mathcal{M}_{g,n}$

observable

$\pi : \overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$ $\quad \tau_2 = \psi_1^2$ is 4-form on $\overline{\mathcal{M}}_{g,1}$

$\kappa = \pi_*(\tau_2)$ 2-form on $\overline{\mathcal{M}}_g$

$\kappa \simeq \omega_{wp}$

$V_g = \int \kappa^{3g-3} \neq \langle \tau^{3g-3} \rangle$

hyperbolic lengths

$V_{g,b} \propto \int_{\overline{\mathcal{M}}_{g,n}} \exp\left(\omega + \frac{1}{2} \sum^{n} b_i^2 \psi_i\right) \Rightarrow$ power series in τ_i

$\dots \Rightarrow y^2 = \sin x$