

The Riemann Hypothesis

& Hamiltonian dynamics

Part 0: the Riemann hypothesis

Define the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

some facts about our soup de jour:

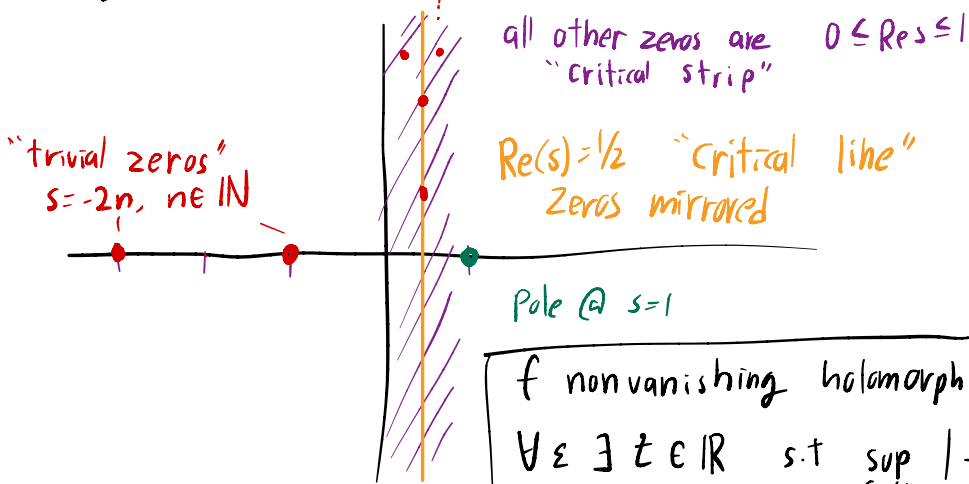
- $\zeta(s)$ converges for s real, $s \geq 1$. Here, ζ is analytic.

↳ $\zeta(s)$ for $s \in \mathbb{C}$ is the unique analytic continuation of $\sum \frac{1}{n^s}$

- $\zeta(s)$ satisfies a functional equation $\zeta(s) = \frac{(2\pi)^s}{\pi} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$

so we know how $\zeta(s)$ looks reflected around $\text{Re}(s) = \frac{1}{2}$ *things we understand*

- using the functional equation, we can deduce the following structure of zeros & poles



My favorite ζ fact:

ζ -fn universality

for U simply connected w/

$$U \subset \{z \mid \frac{1}{2} < \text{Re}(z) < 1\}$$

f nonvanishing holomorphic function on U ,

$$\forall \varepsilon \exists t \in \mathbb{R} \text{ s.t. } \sup_{s \in U} |f(s) - \zeta(s+it)| < \varepsilon$$

"You can approximate any holo fn. by scrolling up on the critical strip"

Birkhoff: this property is generic among holo fns.

But, our only explicit examples are ζ -like functions.



Riemann hypothesis: all zeros of $\zeta(s)$ w/ $0 \leq \text{Re}(z) \leq 1$ have $\text{Re}(z) = \frac{1}{2}$

"Hilbert-Polya conjecture". one potential approach is to find a self-adjoint operator $\hat{H}$ s.t. the e.vals of $\frac{1}{2} + i\hat{H}$ are the zeros of $\zeta(s)$.

$\hat{H}$ self-adjoint $\Rightarrow$ e.vals of $\hat{H}$ are real $\Rightarrow$ zeros of $\zeta(s)$ are $\frac{1}{2} + it, t \in \mathbb{R}$ ■

In this talk, I will discuss some evidence that this magic operator $\hat{H}$ is more than a pipe dream, & some attempts to construct $\hat{H}$.

Part 1: RH & distribution of primes

People always talk about the RH in relation to prime numbers, which I always found mysterious. As my civic duty, I will explain what RH says about primes and use this to motivate our operator $\hat{H}$.

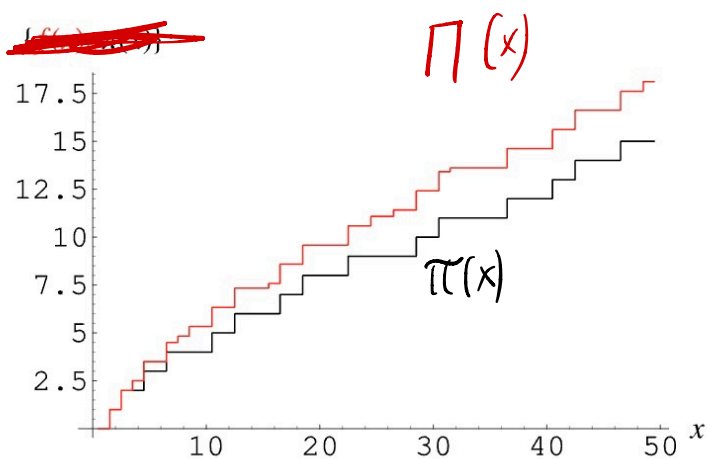
We capture the distribution of primes w/ the prime counting function

$$\pi(x) = \{0 \leq p \leq x \mid p \text{ prime}\} \quad \text{or} \quad \Pi(x) = \sum_{q=1}^x \begin{cases} 1/n & q=p^n, p \text{ prime} \\ 0 & \text{else} \end{cases}$$

"# of primes $\leq x$ "

"weighted count of prime powers $\leq x$ "

π & Π can be reconstructed from one another



The prime number theorem states

$$\pi(x) \sim \frac{x}{\log(x)}$$

$\Rightarrow$ density of primes $\sim \frac{1}{\log(x)}$

average spacing of primes $\sim \log(x)$

the n th prime $p \sim n \log(x)$

We are interested in the fluctuation of $\pi(x)$ around its avg value $\frac{x}{\log(x)}$

the prime # thm is proved using the ζ function.

Euler product form: $\zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$

$$\sum \frac{1}{n^s} = 1^{-s} + 2^{-s} + 3^{-s} + 4^{-s} + 5^{-s} + 6^{-s} + \dots$$

$$= 1 + 2^{-s} + 3^{-s} + (2 \cdot 2)^{-s} + 5^{-s} + (2 \cdot 3)^{-s} + \dots$$

split n into prime factors

$$= (1 + 2^{-s} + (2 \cdot 2)^{-s} + \dots) (1 + 3^{-s} + (3 \cdot 3)^{-s} + \dots)$$

factor sum by prime factors

$$= \left(\sum_{n=0}^{\infty} (2^{-s})^n \right) \left(\sum_{n=0}^{\infty} (3^{-s})^n \right) \dots$$

$$= \frac{1}{1 - 2^{-s}} \frac{1}{1 - 3^{-s}} \dots$$

geometric series formula

Fundamental analogy:

	number theory	dynamics
	primes	simple closed orbits
we hope to study ζ using a dynamical system.	integers	orbit sets (multisets of closed orbits)

Dynamical zeta function (for a contact manifold (Y, λ) w/ Reeb field R)

$$\zeta_R(s) = \prod_{\gamma \text{ simple closed orbit}} \frac{1}{1 - e^{-s \cdot T_\gamma}} = \sum_{\text{orbit set } \alpha} e^{-s \cdot A(\alpha)}$$

where T_γ is the period of γ , equal to the action $\int_\gamma \lambda = A(\gamma)$

We will later try to construct a contact manifold whose dynamical zeta function equals the Riemann ζ -function. This will let us attack the Riemann hypothesis.

Relating $\zeta(s)$ to prime distributions

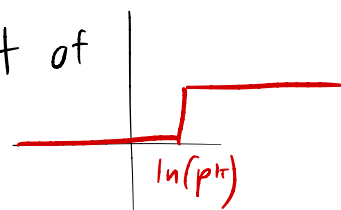
$$\log \zeta(s) = \log \prod \frac{1}{1-p^{-s}} = \sum -\log(1-p^{-s}) = \sum_{p \text{ prime}} \text{power series of } -\log(1-x)$$

$$p^{-s} + \frac{1}{2} p^{-2s} + \frac{1}{3} p^{-3s} + \dots$$

now represent p^{-s} as a Fourier-Laplace transform:

$$\int_a^\infty e^{-sx} dx = \frac{1}{s} e^{-sx} \Big|_a^\infty = -\frac{e^{-sa}}{s}$$

Plugging in $a = \ln(p)$, $p^{-ks} = -s \int_{\ln(p^k)}^\infty e^{-sx} dx = -\mathcal{L}(1_{[\ln(p^k), \infty)})$ f.t of



$$\log \zeta(s) = \sum_{p \text{ prime}} \left(\sum_{k=1}^\infty \frac{1}{k} \cdot \left(-s \int_{\ln(p^k)}^\infty e^{-sx} dx \right) \right)$$

$$= -s \left(\sum_{p \text{ prime}} \sum_{k=1}^\infty \mathcal{L}(1_{[\ln(p^k), \infty)}) \right)$$

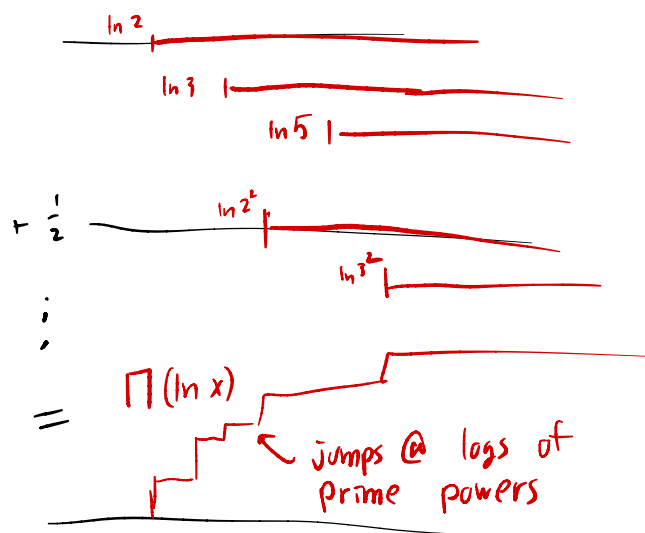
$$= -s \cdot \mathcal{L} \left(\sum_{p \text{ prime}} \sum_{k=1}^\infty \frac{1}{k} 1_{[\ln(p^k), \infty)} \right)$$

$$= -s \mathcal{L}(\Pi(\ln(x)))$$

by Fourier inversion formula,

$$\Pi(\ln(x)) = \mathcal{L} \left(\frac{\log \zeta(s)}{s} \right)$$

$$= \int_{\mathbb{C}} \frac{\log \zeta(s)}{s} e^{sx} ds$$



The Riemann zeta function gives the Fourier Transform of the prime power counting f_n !

From here, Riemann evaluates this Fourier transform. He does this by introducing an entire function $\zeta(s)$ w/ roots @ every nontrivial root of ζ :

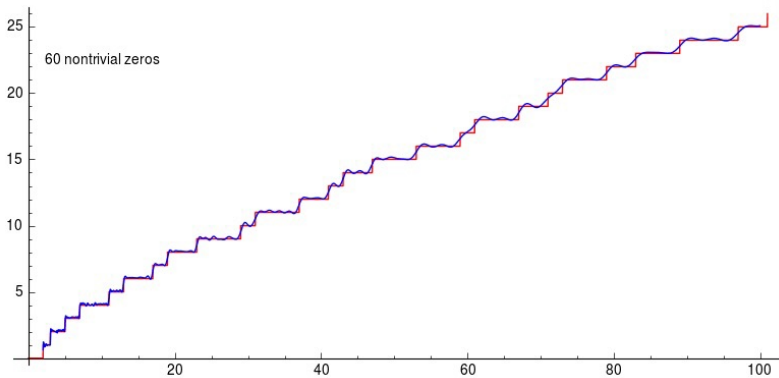
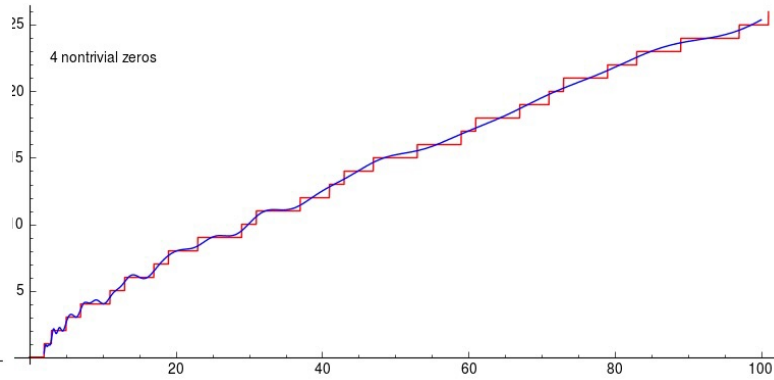
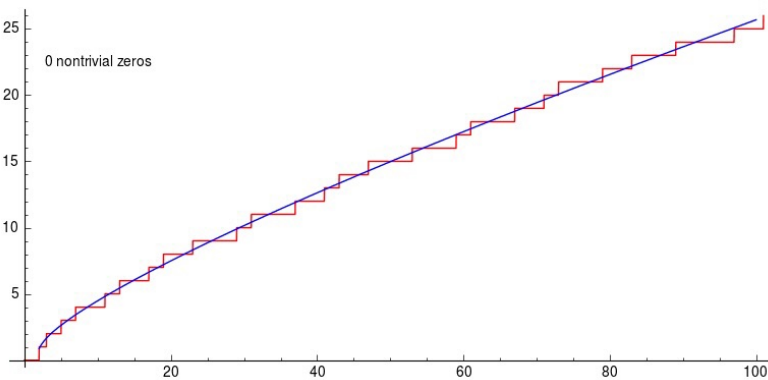
$$\zeta(s) = \frac{1}{s} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) \quad \rho \text{ nontrivial zero}$$

then proving a functional equation relating $\zeta(s)$ to $\zeta(\bar{s})$ using Γ .
Finally, he plugs all this in to $\log s$ & evaluates the integral:

$$\Pi(x) = \underbrace{\text{Li}(x)}_{\text{nontrivial } 0} - \sum_{\rho} \underbrace{\text{Li}(x^{\rho})}_{\text{oscillations in } \Pi(x) \text{ from avg, measured by roots of } \zeta(s)} + \underbrace{\log\left(\frac{1}{2}\right) + \int_x^{\infty} \frac{dt}{t + (t^2-1)\log t}}_{\text{Riemann's explicit formula, Chump change}}$$

$$\text{Li}(x) = \int_x^{\infty} \frac{1}{\log(t)} dt \sim \frac{x}{\log x}$$

first approx of $\Pi(x)$



Approximating $\Pi(x)$ using the explicit zeros gives Fourier expansion

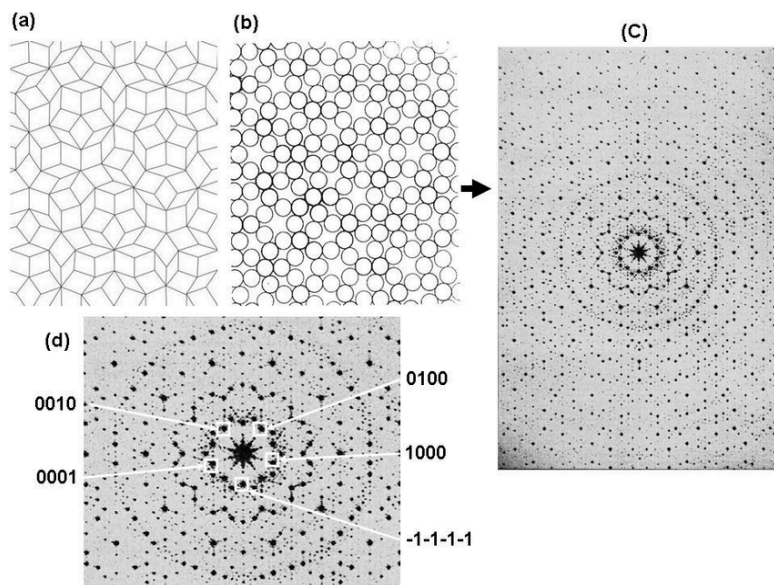
Here's a more workable modern form of the explicit formula by Weil:
let the nontrivial zeros $\rho_n = \frac{1}{2} + i\tau_n$. let φ be a test function. then

$$\underbrace{\sum \varphi(\tau_n)}_{\text{Sum over zeros}} - \underbrace{\left(\frac{1}{2\pi} \int \varphi(r) \frac{\Gamma'(\frac{1}{4} + \frac{i}{2}r)}{\Gamma(\frac{1}{4} + \frac{i}{2}r)} dr \right)}_{\text{governs prime \# thm } \pi(x) \sim \text{Li}(x)} + \underbrace{\varphi\left(\frac{i}{2}\right) + \varphi\left(-\frac{i}{2}\right)}_{\text{chump change}} = -2 \underbrace{\sum_{p \text{ prime}} \ln p \sum_n \frac{\hat{\varphi}(n \ln p)}{e^{n \ln p/2}}}_{\text{sum over prime powers } \ln p^n} - \underbrace{\hat{\varphi}(0) \ln \pi}_{\text{constant term}}$$

function φ measured on zeros of $\zeta \iff$ Fourier transform $\hat{\varphi}$ measured on $\ln(p^n)$

Roughly, sum of deltas supported at zeros of ζ is farrier dual to distribution supported on primes.

RH is true $\Leftrightarrow$ logs of prime powers forms a Quasicrystal

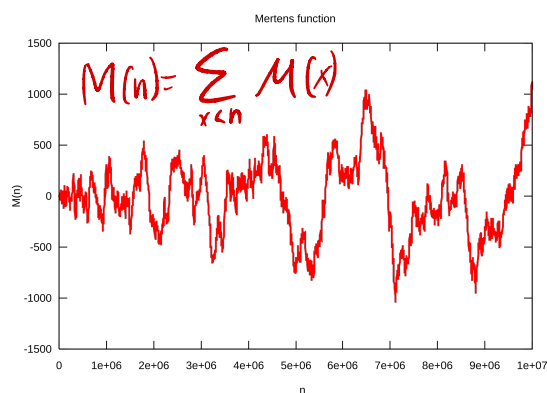


fourier transform of a discrete set is a discrete set

RH is true $\Rightarrow$ the primes have music in them

RH is true $\Rightarrow$ the primes are well distributed

define the mobius function
$$\mu(n) = \begin{cases} 1 & n \text{ has an even \# of prime factors} \\ -1 & n \text{ has an odd \# of prime factors} \\ 0 & n \text{ has a prime square} \end{cases}$$



$$RH \Rightarrow M(n) = O(\sqrt{n})$$

(what you would expect from a random walk, if $\mu(n)$ had random parity)

the Merten's conjecture, $M(n) \leq \sqrt{n}$, is

False. First counterexample c $10^{16} \leq c \leq 10^{1064}$

Part 2: trace formula

We want to linearize the RH, by finding an operator whose zeros give the zeros of $\zeta(s)$. To make this more plausible, let me describe a different zeta function whose zeros are the spectrum of an operator.

Analogy:	Number Theory	Riemannian geometry
	Primes	simple closed geodesics
	zeros of $\zeta(s)$	e.vals of laplacian
	Zeta function	Dynamical zeta function

classical periodic orbits
Quantum periodic orbits for schrodinger equation

Example: Poisson summation formula

consider riemannian manifold (S, g) w/ length 1

spectral theory	Geodesic flow
laplacian $-\partial_\theta^2$	single periodic orbit, length 1
e. functions $e^{2\pi i k \theta}$, $k \in \mathbb{Z}$	eval k
spectrum $k \in \mathbb{Z}$	length spectrum $= \{ \text{length } \gamma \mid \gamma \text{ closed geodesic} \}$ $= k \in \mathbb{Z}$

h test function, $\text{tr}_{\text{test fn}}(\varphi(\Delta)) = \sum_{k \in \mathbb{Z}} \varphi(k) = \sum_{k \in \mathbb{Z}} \hat{\varphi}(k)$

"Trace of function of laplace operator given by sum over periodic orbits"

Example: Selberg trace formula

Let (Σ, g) be a compact, hyperbolic surface.

denote the e.vals of Δ
by $0 < \lambda_1 \leq \dots \leq \lambda_n \leq \dots$

denote $\mathcal{P} = \{ \text{simple, closed geodesics} \}$
let l_p denote the length of $p \in \mathcal{P}$

$$\underbrace{\sum \varphi(\sqrt{\lambda_n - 1/4})}_{\text{Tr}(\varphi \sqrt{\Delta - 1/4}) \text{ spectral term}} - \underbrace{\frac{\text{Area}(\Sigma)}{2\pi} \int \varphi(r) \tanh(\pi r) dr}_{\text{expected distribution of eigenvalues}} = \underbrace{\sum_{\mathcal{P}} l_p \sum_{n \in \mathbb{N}} \frac{\hat{\varphi}(n l_p)}{e^{n l_p/2} - e^{-n l_p/2}}}_{\text{sum over closed geodesics } p^n}$$

moral: geodesics govern fluctuations of evalue distribution around expected distribution.

The selberg trace formula looks strikingly similar to the Weil explicit formula...

in fact, we can construct a ζ -fn whose zeros are $\sqrt{\lambda_n - \frac{1}{4}}$!

$$\zeta_\Sigma(s) = \prod_{\rho \in \mathcal{P}} \prod_{k \in \mathbb{N}} (1 - e^{-(s+k)\rho}) \quad \left(\text{I don't know why this isn't } \prod \frac{1}{1 - e^{-s\rho}} \dots \right)$$

This is the Selberg ζ -function.

- ρ is a zero of $\zeta_\Sigma \Leftrightarrow \rho(1-\rho)$ is an e.value of Δ (Zeros are spectral)

if $\rho_n = \frac{1}{2} + i r_n$, then $\lambda_n = \rho_n(1-\rho_n) = \frac{1}{4} + r_n^2 \Leftrightarrow r_n = \sqrt{\lambda_n - \frac{1}{4}}$ (this is why $\sqrt{\lambda_n - \frac{1}{4}}$ appears above)

$\Delta^{-\frac{1}{4}}$ is positive and self-adjoint, so r_n is always real. Thus, ρ_n lies on critical strip $\frac{1}{2} + i\mathbb{R}$.
So, the RH holds for the selberg ζ -fn!

Semiclassical trace formula:

now we generalize significantly. We wish for an operator $\hat{H}$ such that the eigenvalues give the zeros of ζ . Try as we might, no laplacians seem to work.

Guess: $\hat{H}$ is the Quantization of a classical Hamiltonian.

Quantization

(M, ω) symplectic $\longrightarrow$ Hilbert space $\mathcal{H}$
 $H: M \rightarrow \mathbb{R}$ Hamiltonian fn $\longrightarrow$ self-adjoint operator $\hat{H}: \mathcal{H} \rightarrow \mathcal{H}$

There are many schemes for doing this. Here's one which idk if it works. but, it's the most suggestive of symplectic homology

let (W, ω) be a Weinstein domain. W has a canonical complex structure.

$\partial W = Y$ is a contact manifold, $\bar{W}$ is W 's symplectic completion

let $\lambda^\#$ be the Liouville field, with $\lambda^\# = \partial u$. Then, $i\partial\bar{\partial}u = \omega$.

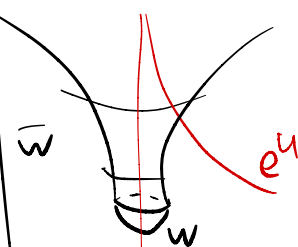
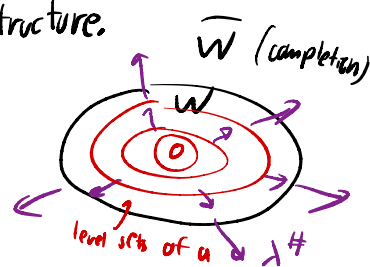
Define the Hilbert space

$$\mathcal{H} = H^2(\bar{W}, e^u) = \text{Holo. functions on } \bar{W} \cap \{f: \bar{W} \rightarrow \mathbb{C} \mid \|f\|_u^2 = \int \bar{f} f e^u \frac{\omega^n}{n!} < \infty\}$$

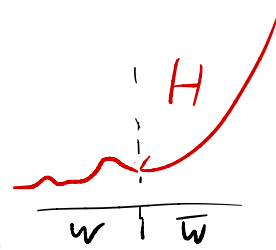
Holomorphic, L^2 functions wrt weight e^u

ω grows exponentially wrt symplectization coordinate

u grows linearly wrt symplectization coordinate



Let H be a function which is quadratic on $\bar{W} \setminus W$ (like symplectic homology)



Now we define $\hat{H}$:

define π by the orthogonal projection $\pi: L^2(W, e^u) \rightarrow H L^2(W, e^u) = \mathcal{H}$
 use $M_H: L^2(W, e^u) \rightarrow L^2(W, e^u)$ to denote multiplication by H , $M_H f = Hf$
 Define the Toeplitz operator $\hat{H} = \pi M_H \pi$, $\hat{H}: \mathcal{H} \rightarrow \mathcal{H}$

closest you can get to multiplying by H in $\mathcal{H}$

Let E_j denote the eigenvalues of $\hat{H}$ in order (we expect only accumulation pt to be ∞)

These energies satisfy a trace formula. (at least they ought to, I haven't found this particular setup written anywhere)

This one is asymptotic, probing the large-energies. let $\hbar \ll 0$ be the asymptotic parameter.

Semi-classical Trace formula:

fix a regular level E of H . Let $\Sigma_E = H^{-1}(E)$. Let $\mathcal{P}$ denote the simple closed orbits on Σ_E , & for $\gamma \in \mathcal{P}$, γ^n the n -fold orbit.

$$\underbrace{\sum \varphi\left(\frac{E_j - E}{\hbar}\right)}_{\text{Zoom in on energies near } E} \approx \hbar^{1-n} \text{Vol}(\Sigma_E) \hat{\varphi}(0) + \sum \overbrace{c_j(\hat{\varphi})}^{\text{lower order terms are differentiated operators of } \hat{\varphi}, \text{ evaluated at zero}}(0) \hbar^{1-n+j} + \sum_{\gamma \in \mathcal{P}} \frac{\overset{\text{Period}}{T_\gamma}}{2\pi} \sum_{n \in \mathbb{N}} \frac{e^{i A(\gamma^n)/\hbar} e^{\frac{i\pi}{4} \underbrace{CZ(\gamma^n)}_{\text{conley-Zhender index}}}}{\sqrt{\det |1 - P_\gamma|}} \hat{\varphi}(T_\gamma^n) + O(\hbar)$$

Linearized first return map

If we choose E so that H is linear already @ E , then we can describe this purely in terms of the contact manifold (Y, λ) & the reeb orbits.

$$\sum \varphi\left(\frac{E_j - E}{\hbar}\right) = \hbar^{1-n} \text{Vol}(Y) \hat{\varphi}(0) + \sum_{\gamma \in \mathcal{P}} \frac{A(\gamma)}{2\pi} \sum_n \frac{e^{in A(\gamma)/\hbar} e^{\frac{i\pi}{4} CZ(\gamma^n)}}{\sqrt{\det |1 - P_\gamma|}} \hat{\varphi}(n A(\gamma)) + \text{lower order terms}$$

first return map on contact structure

Selberg trace formula: application of this formula to $S^*\Sigma$, for Σ a hyperbolic surface. Selberg formula is exact, not requiring the lower order terms.

Trace formula Schema:

According to the path integral interpretation of Quantum mechanics, we expect the quantization to be given by an integral over the space of paths.

To take a trace, we integrate over the space of loops $\mathcal{LW}$

$$\sum \varphi\left(\frac{E_j - E}{\hbar}\right) \approx \int_{\mathcal{LW}} e^{i \int \left(\lambda + \varphi\left(\frac{H-E}{\hbar}\right) \right) / \hbar} \mathcal{D}\gamma$$

this is not well defined...

define $S(\gamma) = \int \lambda + \varphi\left(\frac{H-E}{\hbar}\right)$, the action. the integrand is $e^{iS(\gamma)/\hbar}$. we wish to compute this integral asymptotically as $\hbar \rightarrow 0$. to do this, we formally apply the stationary phase approximation.

Stationary phase in finite dimensions: if $\phi: \mathbb{R}^n \rightarrow \mathbb{R}$ is morse, then

$$\int e^{i\phi/\hbar} dx \sim (2\pi\hbar)^{n/2} \sum_{p \text{ crit pts } \phi} \frac{e^{i\pi\sigma_p/4}}{\sqrt{|\det \text{Hess}_p(\phi)|}} e^{i\phi(p)/\hbar} + O(\hbar^{n/2-1})$$

where σ_p is the signature of $\text{Hess}_p \phi$.

so the integral $\int_{\mathcal{LW}} e^{iS(\gamma)/\hbar} \mathcal{D}\gamma$ should concentrate around the critical points of $S(\gamma)$. These are closed hamiltonian orbits

Period 0 orbits: These are the constant loops, $W \in \mathcal{LW}$.

↳ contributes term $\hbar^{1-n} \text{vol}(\Sigma_E) \hat{\psi}(0) + O(\hbar^{2-n})$

Period $T \neq 0$ closed orbit γ :

↳ signature of Hessian = $CZ(\gamma)$

$$\hookrightarrow \frac{1}{\sqrt{\text{Hess}}} = \frac{T_\gamma}{\sqrt{|\det(I - P_\gamma)|}}$$

↳ phase $e^{iS(\gamma)/\hbar} = e^{i \int \lambda / \hbar}$

} Plugged in to stationary phase, these reproduce the semiclassical trace formula

Part 3: Proving (?) the RH

Suppose that the eigenvalues of $\hat{H}$ agreed with the heights of zeros of $\zeta(s)$. We need the explicit formula for ζ to agree w/ the trace formula.

This imposes strong restrictions on the dynamics of H . To prove RH:

- find H with the necessary "Riemann dynamics"
- find a sensible way to quantize H into $\hat{H}$
- Prove the e.vals of $\frac{1}{2} + i\hat{H}$ are exactly the zeros of $\zeta(s)$

Then, $\hat{H}$ is by construction self-adjoint. so, the RH is true.

Explicit formula

$$\sum \varphi(r_n) - \underbrace{\left(\frac{1}{2\pi} \int_0^\infty \varphi(r) \frac{\Gamma'(\frac{1}{4} + \frac{i}{2}r)}{\Gamma(\frac{1}{4} + \frac{i}{2}r)} dr + \varphi\left(\frac{i}{2}\right) + \varphi\left(-\frac{i}{2}\right) \right)}_{(4)} = -\hat{\varphi}(0) \ln \pi - 2 \sum_{p \text{ prime}} \ln p \sum_n \frac{\hat{\varphi}(n \ln p)}{e^{n \ln p/2}} \quad (2) \quad (1) \quad (3)$$

Semiclassical trace formula (4)

$$\sum \varphi(E_j - E) = \text{Vol}(\Sigma_E) \hat{\varphi}(0) + \sum_{\gamma \in \mathcal{P}} \frac{T_\gamma}{2\pi} \sum_{n \in \mathbb{N}} \underbrace{\left(\frac{e^{i\phi(\gamma^n)}}{\sqrt{\det |1 - P_\gamma|}} \right)}_{(3)} \underbrace{e^{\frac{i\pi}{4} L_2(\gamma^n)}}_{(2)} \hat{\varphi}(T_\gamma^n) + \dots \quad (5)$$

The "Riemann dynamics" must satisfy:

- (1) the periods of simple closed orbits must be $\ln p$, one orbit for each prime p . the dynamical zeta function is then

$$\zeta_{X_H}(s) = \prod_{p \text{ prime}} \frac{1}{1 - e^{-s \ln p}} = \prod_p \frac{1}{1 - p^{-s}} = \zeta(s)$$

So we are looking for a Hamiltonian / contact system whose dynamical zeta fn is the Riemann zeta fn. The trace formula suggests that, like the selberg zeta function, the quantization of the dynamics finds roots of the dynamical zeta function.

- (2) the Conley-Zehnder index of every (non) simple orbit is 4.

This already feels impossible. we might be saved by some crazy adelic spaces, see Connes.

- (3) $\det 1 - P_\gamma^n = e^{n \ln p} = e^{T_\gamma^n}$. Therefore, the first return map grows proportionally with the period for every orbit. Every orbit has Lyapunov exponent 1. The Riemann dynamics are uniformly hyperbolic.

- The Riemann dynamics are chaotic. The pair correlation function of the Riemann zeros exactly agrees w/ those of a random hermitian matrix. You expect this to be true for energies of chaotic systems

(4) controls $\text{Vol } H^{-1}(-\infty, E)$. This is hard to see in this form of the semiclassical trace formula. Instead, we compare:

$$N_Z(E) = \# \{ \text{zeros } \rho \mid \text{Im } \rho < E \}, \quad N_Z(E) \sim E \log E$$

$$N_H(E) = \# \{ \text{e.vals } \lambda \text{ of } \hat{H} \mid \lambda < E \}, \quad N_H(E) \sim \text{Vol } (H^{-1}(-\infty, E))$$

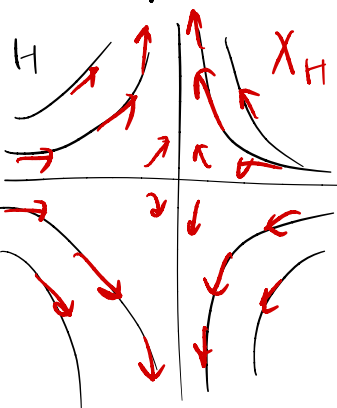
$$\text{So } \text{Vol } (H^{-1}(-\infty, E)) \sim E \log(E)$$

Weyl's law: # of e.vals $\leq E$
 $\sim$ volume of phase space $\leq E$

(5) Our Quantization scheme needs the trace formula to match the ζ -explicit formula. So the trace formula must be exact, like it is for the selberg trace formula

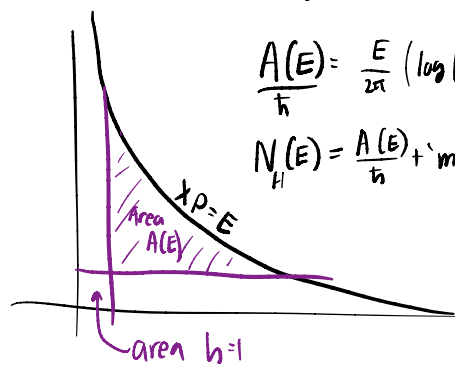
The xp hamiltonian: our best attempt at satisfying the conditions above

the simplest homogeneously hyperbolic hamiltonian is $H = xp$ on $T^*\mathbb{R} = \mathbb{R}^2$



Unfortunately, its sublevel sets are infinitely large so $N_H(E)$ is infinite.

Instead, we "regularize" chopping off the part near the bdy.



$$\frac{A(E)}{h} = \frac{E}{2\pi} \left(\log \left(\frac{E}{2\pi} \right) - 1 \right) + 1$$

$$N_H(E) = \frac{A(E)}{h} + \text{'maslov correction'} = \frac{E}{2\pi} \left(\log \left(\frac{E}{2\pi} \right) - 1 \right) + \frac{7}{8}$$

compare this to the smooth error term in the zero counting function:

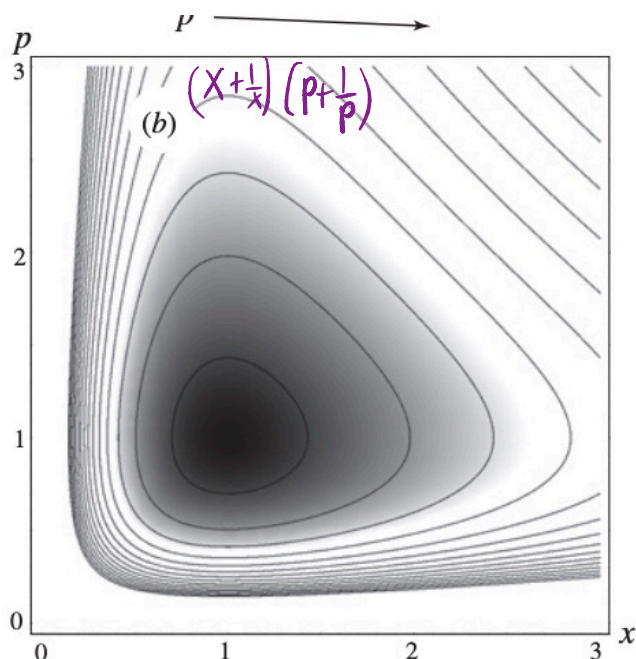
$$N_Z(E) = \underbrace{N_Z^{\text{sm}}(E)}_{\text{controlled by nice fn.}} + \underbrace{N_Z^{\text{osc}}(E)}_{\text{controlled by primes}}$$

$$N^{\text{sm}}(E) = \frac{-E}{2\pi} \log \pi + \frac{\text{Im } \log \Gamma \left(\frac{1}{4} - \frac{1}{2} iE \right)}{\pi}$$

$$N_Z^{\text{sm}}(E) - N_H^{\text{sm}}(E) = N_Z^{\text{sm}}(E) - \frac{E}{2\pi} \left(\log \left(\frac{E}{2\pi} \right) - 1 \right) + \frac{7}{8} \rightarrow 0 \text{ as } E \rightarrow \infty!$$

Therefore, $\hat{H}$ (if it can be defined) has the same asymptotic distribution of its eigenvalues as the zeros of ζ !

More recently, Berry-Kesting found $H(x,p) = (x + \frac{1}{x})(p + \frac{1}{p})$



this also has the correct spectral asymptotics!

- it doesn't require an ad hoc regularization procedure
- it has a well-defined quantization

Unfortunately, the smoothed eval counting fn only agrees w/ the zero counting fn asymptotically and the oscillations are totally different. so, does not solve RH.

My proof (pending) of the Riemann hypothesis.

The xp hamiltonian is good because it has the right area sublevel sets. we can transfer this property to higher dimensions.

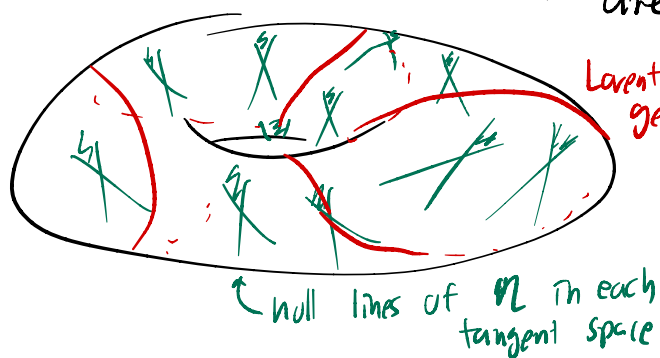
If (M, ω) is symplectic and $\downarrow_{M, \omega}$ is a symplectic complex line bundle, then

we can consider a Hamiltonian which is fiberwise xp . This should share the semiclassical spectral asymptotics with xp , if $\text{vol}(M)=1$.

A good example is $T^*\Sigma$ for a surface Σ . since Σ is symplectic, the fibers $T^*\Sigma$ inherit a symplectic structure, so this is a symplectic bundle.

note that, after a change of coordinates, $xp = x^2 - p^2$.

A function on $T^*\Sigma$ which is fiberwise $x^2 - p^2$ is the symbol of a Lorentzian inner product. (Σ, η) . The orbits of this hamiltonian are Lorentzian geodesics.



The Quantization, pre-regularization, is the De'Lambertian: the Laplacian in signature (1,1). This is hyperbolic, giving the wave equation on a compact 2D spacetime. Nasty stuff.

IF we can find a 2D Lorentzian surface with geodesics of length $\ln(p)$ for p prime that has a shot at being the long-sought Riemann dynamics.

Questions for you:

- can we attack Lorentzian geodesic flows with hamiltonian / contact dynamics? I couldn't find any floor-theoretic approaches.
 - can we find / rule out Riemannian dynamics using Floer theory?
- ↳ something fun: if primitive orbits have action $\ln p$, consider the orbit sets in ECH. They have action $\ln(k)$ for every integer k , each action appearing from exactly one orbit set.
- I wonder what the ECH indices are...