Student Symplectic Seminar

Quasihamiltonian G-spaces & loop groups

§1: my favorite manifold

my favorite manifold is the moduli space of flat G-bundles on a Riemann surface

fix a Riemann Surface Σ, & a trivial principle G-bundle $\begin{array}{c} P=\Sigma\times G \\ \downarrow \\ \Sigma \end{array}$

a connection on P is defined by a $\underline{\mathfrak{g}}=Lie(G)$-valued 1-form $A\in\Omega^1(\Sigma,\underline{\mathfrak{g}})$

this has curvature $F_A=dA+A\wedge A\in\Omega^2(\Sigma,\underline{\mathfrak{g}})$

define $\mathcal{M}_G^{flat}(\Sigma)=\{A\in\Omega^1(\Sigma,\underline{\mathfrak{g}})\mid F_A=0\}/\text{Gauge transforms}$

Thm (Atiyah-Bott): $\mathcal{M}_G^{flat}(\Sigma)$ is a symplectic manifold

we define $\mathcal{M}_G^{flat}(\Sigma)$ as an infinite dimensional symplectic reduction

let $\mathcal{A}=\Omega^1(\Sigma,\underline{\mathfrak{g}})$ be the space of connections, $\mathcal{G}=Aut(P)=Maps(\Sigma,G)$ the group of gauge transforms

$\mathcal{A}$ is symplectic with symplectic form $\omega(\alpha,\beta)=\int_\Sigma\langle\alpha\wedge\beta\rangle$ killing form

$\mathcal{G}\circlearrowright\mathcal{A}$ is hamiltonian action, with moment map $\mu:\mathcal{A}\to Lie(\mathcal{G})^*$

$Lie(\mathcal{G})=\Omega^0(\Sigma,\underline{\mathfrak{g}})$, $Lie(\mathcal{G})^*=\Omega^2(\Sigma,\underline{\mathfrak{g}})$, via poincare duality

The moment map is curvature: $\mu: A\mapsto F_A\in\Omega^2(\Sigma,\underline{\mathfrak{g}})=Lie(\mathcal{G})^*$

explicitly, $\xi\in\Omega^0(\Sigma,\underline{\mathfrak{g}})$ generates vector field $V_\xi|_\alpha\in T_\alpha\mathcal{A}=\Omega^1(\Sigma,\underline{\mathfrak{g}})$ by $V_\xi=[\xi,\alpha]+d\xi$

$$i_{V_\xi}\omega=\int_\Sigma\langle F_A,\xi\rangle$$

symplectic reduction $\mathcal{A}/\!/\mathcal{G}=\mu^{-1}(0)/\mathcal{G}=\{\text{flat connections}\}/\text{Gauge transforms}=\mathcal{M}_G^{flat}(\Sigma)$

endows $\mathcal{M}_G^{flat}(\Sigma,G)=\mathcal{A}/\!/\mathcal{G}$ with a symplectic form

As a manifold, we can construct a finite dimensional model via the Riemann-Hilbert correspondence: The only gauge invariants of a connection come from monodromy

For a flat connection, monodromy is topological

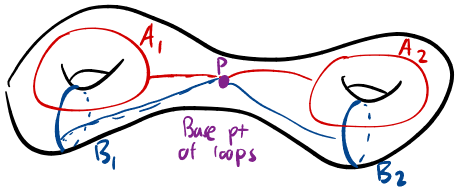
The monodromy perspective gives a finite dimnl model:

$$\mathcal{M}_G^{flat}(\Sigma) = \frac{\mathrm{Hom}(\pi_1(\Sigma), G)}{G}$$

Description of monodromy around every loop

"change of coordinates", gauge transform acting on base point

G acts on $\mathrm{Hom}(\pi_1(\Sigma), G)$ by overall conjugation



Explicitly: $\pi_1(\Sigma) = \dfrac{\langle A_1, \dots, A_k, B_1, \dots, B_k \rangle}{\langle [A_1, B_1] \cdots [A_k, B_k] \rangle}$ (genus k)

for flat connection α, denote monodromies as $\mathrm{Hol}_{A_i}\alpha = a_i$, $\mathrm{Hol}_{B_i}\alpha = b_i$

$a_i, b_i \in G$. $\mathrm{Hom}(\pi_1(\Sigma), G) = \{(a_1, b_1, \dots, a_k, b_k) \in G^{2k} \mid \underbrace{a_1 b_1 a_1^{-1} b_1^{-1} \cdot \dots \cdot a_k b_k a_k^{-1} b_k^{-1}}_{\mu(a_1, \dots, a_k, b_1, \dots, b_k)} = e\}$

$= \mu^{-1}(e)$ for $\mu : G^{2k} \to G$

for gauge transform $g(x) \in \mathcal{G}$, $g : \Sigma \to G$, action on monodromy is $g . a_i = g(p)^{-1} a_i g(p)$, $g . b_i = g(p)^{-1} b_i g(p)$ acts by conjugation

$$\mathcal{M}_G^{flat}(\Sigma) = \frac{\mathrm{Hom}(\pi_1(G))}{G} = \mu^{-1}(e)/G$$ for G acting on G^{2k} by conjugation

Q: can we recover the symplectic form on $\mathcal{M}$ from this finite dimnl model?

idea: $\mu^{-1}(e)/G$ looks like a "symplectic reduction" of G^{2k}

issue is, μ isn't $\underline{g}^*$-valued. it's G-valued.

§2: Quasihamiltonian G-spaces

Let's review the classical definiton carefully

Def a Hamiltonian G-space is a manifold M with structures

1. 2-form ω
2. G action, encoded infintesimally: each $\xi \in \underline{g}$ acts on M by a vector field v_ξ s.t $[v_\xi, v_\zeta] = v_{[\xi,\zeta]}$
3. G-equivariant map $\mu : M \to \underline{g}^*$, where G acts on $\underline{g}^*$ by the coadjoint action

satisfying

a) $i_{v_\xi}\omega|_x = d\langle \mu(x), \xi \rangle$ moment map condition

b) $d\omega = 0$
c) $\ker(\omega) = 0$ $\}$ symplectic condition

We wish to construct an analagous definiton, where $\mu : M \to G$. a "lie group valued moment map"

Def a Quasi-Hamiltonian G-space is a manifold M with structures

1. 2-form ω
2. G action.
3. G-equivariant map $\mu : M \to G$, where G acts on G by conjugation

Satisfying

a) $i_\xi \omega = \langle \mu^*(\theta^L + \theta^R), \xi \rangle$

b) $d\omega = \mu^* \chi$

c) $\ker \omega = \{ V_\xi \mid Ad_\xi + \xi = 0 \}$

} fill these in during talk

first we need the moment map type condition. Instead of specifying V_ξ as hamiltonian vector fields, we suffice to describe their pairing with ω. This should depend only on ξ & the geometry of the moment map μ.

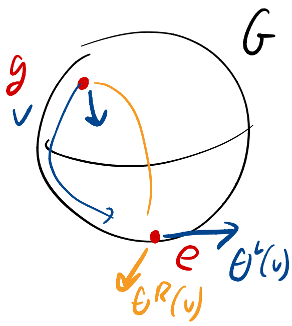
Define the $\mathfrak{g}^*$-valued 1-form $i_{\mathfrak{g}}\omega(\xi) = i_{V_\xi}\omega$ $\quad i_{\mathfrak{g}}\omega \in \Omega^1(M, \mathfrak{g}^*)$

moment map condition should be of the form $i_{\mathfrak{g}}\omega = \mu^*(\alpha)$ for some $\alpha \in \Omega^1(G, \mathfrak{g}^*)$

in particular, $i_{\mathfrak{g}}\omega(V_\xi) = \alpha(\mu_* V_\xi)$

by G-equivariance, $\mu_* V_\xi$ is the vector field on G generating the conjugation action. denote this by ξ^{Ad}. Note $\xi^{Ad} = \xi^L - \xi^R$, w/ ξ^L canonical left invariant vect field, ξ^R right invariant vect field

The natural choice of α is the Maurer-Cartan form



$\theta^L = g^{-1}dg \in \Omega^1(G, \mathfrak{g})$ is "Left-invariant" Maurer cartan form

$\theta^L|_g$ moves vectors in T_gG to $T_eG \cong \mathfrak{g}$ via left multiplication

$\Rightarrow \theta^L(\xi^L)|_g = \xi \qquad \theta^L(\xi^R)|_g = -Ad_g \xi$

Similarly $\theta^R = dg\, g^{-1}$ is "Right invariant" Maurer cartan form

$\theta^R(\xi^R) = \xi \qquad \theta^R(\xi^L)|_g = Ad_g \xi$

Let's guess & check until we find a good definition for α

try $\alpha = \theta^L$, $\quad i_{\mathfrak{g}}\omega \overset{?}{=} \mu^*\langle \theta^L, \cdot \rangle$: then

$i_{V_{\xi_1}}\omega(V_{\xi_2})|_x \overset{?}{=} \mu^*\langle \theta^L, \xi_1 \rangle (V_{\xi_2})$

$\overset{?}{=} \langle \theta^L(\xi_2^L - \xi_2^R)|_{\mu(x)}, \xi_1 \rangle$

$\omega(V_{\xi_1}, V_{\xi_2}) \overset{?}{=} \underbrace{\langle \xi_2 - Ad_{\mu(x)}\xi_2, \xi_1 \rangle}_{\text{not antisymmetric in } \xi_1, \xi_2}$ ✗

try $\alpha = \frac{1}{2}(\theta^L + \theta^R)$, $\quad i_{\mathfrak{g}}\omega = \mu^*\langle \frac{1}{2}(\theta^L + \theta^R), \cdot \rangle$

$i_{V_{\xi_1}}\omega(V_{\xi_2}) = \frac{1}{2}\langle (\theta^L + \theta^R)(\xi_2^L - \xi_2^R), \xi_1 \rangle$

$= \frac{1}{2}\langle (\cancel{\xi_2} + Ad_{\mu(x)}\xi_2) + (Ad_{\mu(x)}\xi_2 - \cancel{\xi_2}), \xi_1 \rangle$

$\omega(V_{\xi_1}, V_{\xi_2}) = \frac{1}{2}(\langle Ad_{\mu(x)}\xi_2, \xi_1 \rangle - \langle \xi_2, Ad_{\mu(x)}\xi_1 \rangle)$ ✓

antisymmetric

Moment map condition (condition a): $\boxed{i_{\mathfrak{g}}\omega = \mu^*(\langle \frac{1}{2}(\theta^L + \theta^R), \cdot \rangle)}$

but moment map condition + G-invariance of ω $\Rightarrow$ $d\omega \neq 0$

$$0 = \mathcal{L}_{v_\xi}\omega = d\iota_{v_\xi}\omega + \iota_{v_\xi}d\omega$$

$$\Rightarrow \iota_{v_\xi} d\omega = d\mu^*(\langle \tfrac{1}{2}(\theta^L+\theta^R), \xi\rangle = \mu^*(\langle \tfrac{1}{2}(d\theta^L + d\theta^R), \xi\rangle)$$

Solution: impose $d\omega = \mu^*\chi$ for $\chi \in H^3(G,\mathbb{R})$: χ must satisfy $\iota_{\xi_{ad}}\chi = \langle \frac{1}{2}d(\theta^L+\theta^R), \xi\rangle$

Maurer Cartan equation: $d\theta^L + [\theta^L, \theta^L] = 0$ (θ defines flat connection on G-bundle $\begin{matrix} G\times G \\ \downarrow \\ G \end{matrix}$)
$d\theta^R + [\theta^R,\theta^R] = 0$

$$\Rightarrow -\chi = \frac{1}{12}([\theta^L,\theta^L],\theta^L) = \frac{1}{12}([\theta^R,\theta^R],\theta^R)$$

"Cartan 3-form"
generator of $H^3(G,\mathbb{Z}) \simeq \mathbb{Z}$ for G simple

derivative condition: $\boxed{d\omega = -\mu^*\chi}$

The moment map condition also breaks nondegeneracy. Indeed, if $(\theta^L+\theta^R)\xi^{Ad}\big|_{\mu(x)} = 0$ then $\iota_{v_\xi}\omega|_x = 0$. Lie theoretically, $(\theta^L+\theta^R)(\xi^{Ad})\big|_{\mu(x)} = 0$ whenever $Ad_{\mu(x)}\xi + \xi = 0$

impose maximal nondegeneracy: such ξ yield the only kernel of ω

condition C: $\boxed{\ker \omega|_x = \{\xi \in \mathfrak{g} \mid Ad_{\mu(x)}\xi + \xi\}}$

Examples of Q-Hamiltonian G-spaces

i) Conjugacy classes of G: denote by $\mathcal{C} \hookrightarrow G$

these are the Q Ham version of the coadjoint orbit

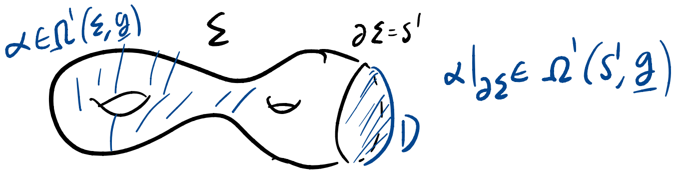
- G-action: Conjugation
- moment map: the inclusion $\mu: \mathcal{C} \to G$
- $\omega(\xi_1^{Ad}, \xi_2^{Ad})|_g = \frac{1}{2}(\langle \xi_1, Ad_g \xi_2\rangle - \langle \xi_2, Ad_g \xi_1\rangle)$ ($T\mathcal{C}$ generated by ξ^{Ad})

ii) The Double $D(G)$: $D(G) = G\times G$ as a manifold

- G-action: $g.(a,b) = (gag^{-1}, gbg^{-1})$
- moment map $\mu(ab) = aba^{-1}b^{-1}$
- $\omega = \frac{1}{2}(a^*\theta^L, b^*\theta^R) + \frac{1}{2}(a^*\theta^R, b^*\theta^L) + \frac{1}{2}((ab)^*\theta^L, (ab)^{-1*}\theta^R)$

Encouraging for realizing $\mathrm{Hom}(\pi_1(\Sigma), G)$ as moment map condition

§3: Loop groups



idea for constructing $\mathcal{M}_G^{flat}(\Sigma)$: drill out a point

let $\Sigma' = \Sigma \setminus D$ for D a disc. This has boundary!

$\mathcal{G}_\partial(\Sigma') \subset \mathcal{G}(\Sigma')$ is kernel of restriction to $\partial\Sigma'$ (i.e gauge transform preserves bdry)

Define $\mathcal{M}(\Sigma') = \mathcal{A}_{flat}(\Sigma') / \mathcal{G}_\partial(\Sigma')$ ~ infinite dimensional symplectic manifold

residual gauge transform: $\mathcal{G}(\Sigma)/\mathcal{G}_\partial(\Sigma') = \mathrm{Maps}(S^1, G) := LG$ "Loop group"

$\mathcal{M}(\Sigma')$ is a hamiltonian LG space!

What is LG like as a lie group?

Assume G simple, $\langle , \rangle$ killing form

$g \in LG$ $\quad g: S^1 \to G,$ $\quad$ multiplication pointwise $\quad g_1 \cdot g_2(\theta) = g_1(\theta) \cdot g_2(\theta)$
$\qquad g: \theta \mapsto g(\theta)$

Lie algebra $\quad L\mathfrak{g} = \Omega^0(S^1, \mathfrak{g}) \quad L\mathfrak{g}^* = \Omega^1(S^1, \mathfrak{g})$

for $\xi \in L\mathfrak{g}$, $\alpha \in L\mathfrak{g}^*$ $\quad \langle \xi, \alpha \rangle = \int_{S^1} \langle \xi, \alpha \rangle = \int_{S^1} \langle \xi(\theta), \alpha(\theta) \rangle d\theta$

LG acts on $\Omega^1(S^1, \mathfrak{g})$ by gauge transforms: $\quad g \cdot \alpha = g^{-1} \alpha g + \overbrace{g^{-1} dg}^{g^*\theta^L}$

The (co) adjoint action acts as $\quad \mathrm{ad}_g^* \alpha = g^{-1} \alpha g \quad$ affine coadjoint action

we want to realize the LG-action on $\Omega^1(S^1, \mathfrak{g})$ as a sort of coadjoint action

LG has a canonical central extention $\widetilde{LG}$

i.e there is a SES $\quad S^1 \overset{i}{\hookrightarrow} \widetilde{LG} \to LG \quad$ s.t $\quad i(S^1) \subset Z(\widetilde{LG})$

Think of $\widetilde{LG}$ as the total space of a circle bundle over LG

This has LG-invariant curvature form $\quad F \in \Omega^2(LG, \mathbb{R})^{LG}$

F is determined by it's value at the identity: $\quad F|_e := \omega \in \Lambda^2 L\mathfrak{g}^*$

canonical central extention defined by $\quad \boxed{\omega(\xi_1, \xi_2) = \int_{S^1} \langle \xi_1, d\xi_2 \rangle}$

let $\widetilde{L\mathfrak{g}} := \mathrm{Lie}(\widetilde{LG}) = \underset{(\xi,}{L\mathfrak{g}} \oplus \underset{\lambda)}{\mathbb{R}}$

lie bracket $\quad [(\xi, \lambda), (\xi', \lambda')] = \left([\xi, \xi'], \omega(\xi, \xi') \right)$

choose $(\alpha,\lambda)\in \widetilde{L\mathfrak{g}}^*$. affine coadjoint action: for $\tilde{g}\in\widetilde{LG}$, w/ g its projection to LG

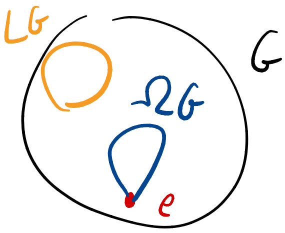
preserves G central part

$$Ad^*_{\tilde g}(\alpha,\lambda) = \Big(g^{-1}\alpha g + \lambda\,\underbrace{g^{-1}dg}_{\text{comes from central extention } \omega}\,,\ \lambda\Big)$$

coadjoint action preserves λ. Restricted to hyperplane $\lambda=1$, this gives the action of gauge transforms on $L\mathfrak{g}^*\simeq\Omega^1(S^1,\mathfrak{g})$

The affine action is important because it makes the LG action nearly free

Define the based loop group $\Omega G=\{g(\theta)\in LG \mid g(0)=e\}$



Fact: the affine coadjoint action $\Omega G \circlearrowright L\mathfrak{g}^*$ is free

Thm: $L\mathfrak{g}^*/\Omega G \simeq G$

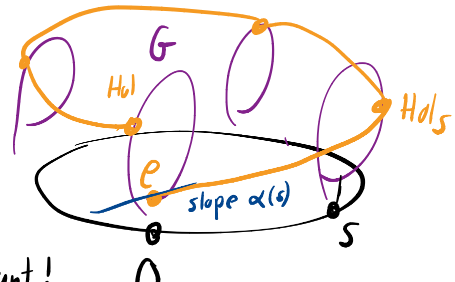
best to think of this gauge theoretically: $L\mathfrak{g}^*=\Omega^1(S^1,\mathfrak{g})=\mathcal{A}(S^1,\mathfrak{g})$

$\Omega G=$ gauge trans froms preserving $0\in S^1$

As always, the only gauge invariants are monodromy

for $\alpha\in\Omega^1(S^1,\mathfrak{g})$ Define $Hol_s(\alpha)\in G$ as the parralel transport from 0 to s

explicitly, $Hol_s^{-1}\partial_s Hol_s=\alpha(s)$



Denote overall monodromy by $Hol_1 := Hol : L\mathfrak{g}^*\to G$

$g\in LG$ acts on Hol as $Hol(g.\alpha)=g(0)^{-1}\,Hol(\alpha)\,g(0)$

if $g\in\Omega G$, $Hol(g.\alpha)=Hol(\alpha)$: Hol is gauge invariant!

also, $Hol^{-1}(e)=\Omega G$

Gives principle fibration

$$\begin{array}{ccc}\Omega G & \hookrightarrow & L\mathfrak{g}^* \\ & & \downarrow{\scriptstyle Hol} \\ & & G\end{array}$$

note $L\mathfrak{g}^*$ contractible, so proves that $G=B\Omega G$

Hence $L\mathfrak{g}^*/\Omega G=G$

§4: Hamiltonian LG Spaces are QuasiHamiltonian G-spaces

Now the punchline: The Holonomy map lets us trade infinite dimensional $L\underline{g}^*$ with finite dimensional G. Turns out, with careful definitions, This induces an equivalence of Hamiltonian LG spaces and quasihamiltonian G-spaces

Def: a Hamiltonian LG space is a banach manifold $\mathcal{M}$ along with:

- an LG action, generated by $V_\xi \in \Gamma(T\mathcal{M})$ for $\xi \in L\underline{g}$
  (Really, we are defining a $\widehat{LG}$ space $\widehat{\mathcal{M}}$, which is a line bundle over $\mathcal{M}$ where the central S^1 of $\widehat{LG}$ acts w/ weight 1 on the line bundle direction)
- a symplectic form σ: i.e, $\sigma \in \Omega^2(\mathcal{M})$ s.t
  - $d\sigma = 0$
  - σ induces an injection $T\mathcal{M} \xrightarrow{\sigma^b} T^*\mathcal{M}$
  needed b/c generally $T\mathcal{M}^*$ is much larger than $T\mathcal{M}$ in infinite dimensions. This ensures all vector fields have generating Hamiltonians
- a moment map $\Phi: \mathcal{M} \to L\underline{g}^*$, equivariant wrt affine coadjoint action on $L\underline{g}^*$
  s.t $i_{V_\xi}\sigma|_x = d\langle \Phi, \xi \rangle = d\int_{S^1} \langle \Phi(x), \xi \rangle$

example: $\mathcal{M}(\Sigma')$ for $\Sigma' = \Sigma \setminus D$
LG acts on $\alpha \in \mathcal{A}(\Sigma)$ via gauge transforms restricted to $\partial\Sigma'$
moment map is $\Phi: \alpha \mapsto \alpha|_{\partial\Sigma'} \in \Omega^1(S, \underline{g}) \simeq L\underline{g}^*$

if $\mathcal{M}$ is a hamiltonian LG space, by equivariance of Φ, we know ΩG acts freely

Define

$$\begin{tikzcd} \mathcal{M} \arrow[r, "Hol"] \arrow[d, "\Phi"'] & \mathcal{M}/\Omega G \simeq M \arrow[d, "\mu"] \\ L\underline{g}^* \arrow[r, "Hol"] & L\underline{g}^*/\Omega G \simeq G \end{tikzcd}$$

induced equivariant map

M & G carry induced $LG/\Omega G = G$ actions, where G acts on G by conjugation

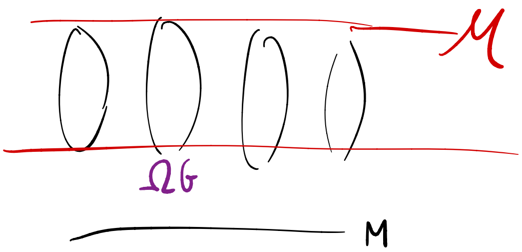
Thm: M is a Quasihamiltonian G-space, w/ moment map μ

We have the G-action & the momement map. we need to construct a 2-form ω on M s.t

a) $i_{V_\xi}\omega = \mu^*\left(\tfrac{1}{2}(\theta^L + \theta^R), \xi\right)$

b) $d\omega = \mu^*\chi$

we want ω to descend from σ on M. But, there is no canonical lift from TM to $T\mathcal{M}$:



modify σ by pulling back a form $\varpi \in \Omega^2(L\underline{g}^*)$, so that it is basic

i.e, want $\sigma + \Phi^*\varpi = Hol^*\omega$ for ω satisfying (a) & (b) need $d(\sigma + \Phi^* \varpi) = Hol^* d\omega = Hol^* \chi$, so $\Phi^* d\varpi = Hol^* \chi$. This holds iff $d\varpi = Hol^* \chi$ on $L\mathfrak{g}^*$. There is a canonical choice for such a primitive:

$$\varpi = \int_0^1 ds \, \langle Hol_s^* \theta^L, \partial_s Hol_s^* \theta^L \rangle \in \Omega^2(L\mathfrak{g}^*)$$

Facts: $d\varpi = \Phi^* \chi$

$\iota_\xi(\sigma - \Phi^* \varpi) = \langle Hol^*(\theta^L + \theta^R), \xi \rangle$ in particular, for $\xi \in \Omega\mathfrak{g}$, $\iota_\xi(\sigma - \Phi^*\varpi) = 0$. so, $\sigma - \Phi^*\varpi$ is basic, & $\sigma - \Phi^* \varpi = Hol^* \omega$ s.t

This constructs ω s.t $\sigma - \Phi^* \varpi = Hol^* \omega$,

$d\omega = \mu^* \chi$

$\iota_{v_\xi} \omega = \langle \mu^*(\theta^L + \theta^R), \xi \rangle$ which gives Quasihamiltonian structure

I tried so hard to figure out what ϖ means, why it's canonical, etc. :(

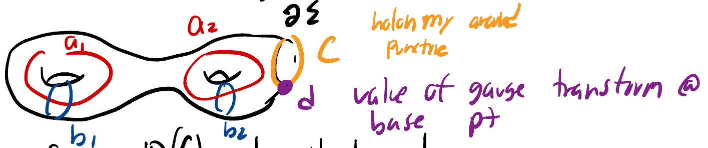
- It should ultimately come out of the form defining the canonical central extension
- It should be a sort of curvature of the symplectic structure of ΩG. (a coadjoint orbit) fibered above M
- It should measure the projective hamiltonian action of $\widetilde{LG}$ on $\mathcal{M}$, relating to the prequantum line bundle
- It should be close to the KKS symplectic form on $L\mathfrak{g}^*$

Conversely: every Quasihamiltonian G-space $M \xrightarrow{\mu} G$ comes from $\mathcal{M} \xrightarrow{\Phi} L\mathfrak{g}^*$ via this construction

- G is $B\Omega G$: pulling back $\begin{smallmatrix} L\mathfrak{g}^* \\ \downarrow \\ G \end{smallmatrix}$ to M defines ΩG bundle $\begin{smallmatrix} \mathcal{M} \\ \downarrow Hol \\ M \end{smallmatrix}$
- endow $\mathcal{M}$ w/ symplectic form $Hol^* \omega - \Phi^* \varpi$

example: The Quasihamiltonian space associated to $\mathcal{M}^{flat}(\Sigma')$ is

$M(\overset{\text{genus } k}{\Sigma'}) = G^{2k} \times G \times G$, w/ $\mu : (a_1, b_1, \dots, a_k, b_k, c, d) \mapsto ([a_1, b_1] \cdots [a_k, b_k], c)$



the form ω on G^{2k} is built inductively from $D(G)$ described above

$G^{2k} = D(G) \circledast \cdots \circledast D(G)$

where $(M_1, \omega_1) \xrightarrow{\mu_1} G \circledast (M_2, \omega_2) \xrightarrow{\mu_2} G$ is the fusion product $(M_1 \times M_2, \mu_1 \mu_2, \omega_1 + \omega_2 + \langle \mu_1^* \theta^L, \mu_2^* \theta^R \rangle)$

$M(\Sigma)$ is $\mathcal{M}^{flat}(\Sigma') /\!/ LG = \Phi^{-1}(0)/LG$ force zero monodromy around bdry, & connections to be flat

= "Quasi hamiltonian reduction"

Thm: if $M \xrightarrow{(\mu_G, \mu_H)} G \times H$ is a Q-Ham $G \times H$ space then for any conjugacy class $\mathcal{C} \subset H$, $M /\!/_{\mathcal{C}} H = \mu_H^{-1}(\mathcal{C})/H$ is a Q-ham G-space

G only acts on c & d by conjugation

$$M(\Sigma')/\!/_e G\times G = \left(\{c=e\}/G\right)/\!/G$$ "Reduction in stages"

$$= \mathbb{D}(G)^{2K}/\!/_e G = \mu^{-1}(e)/G = \{(a_1,b_1,\dots,a_n,b_n) \mid [a_1,b_1]\dots[a_n,b_n]=e\}/G = \frac{\mathrm{Hom}(\pi_1(\Sigma),G)}{G}$$

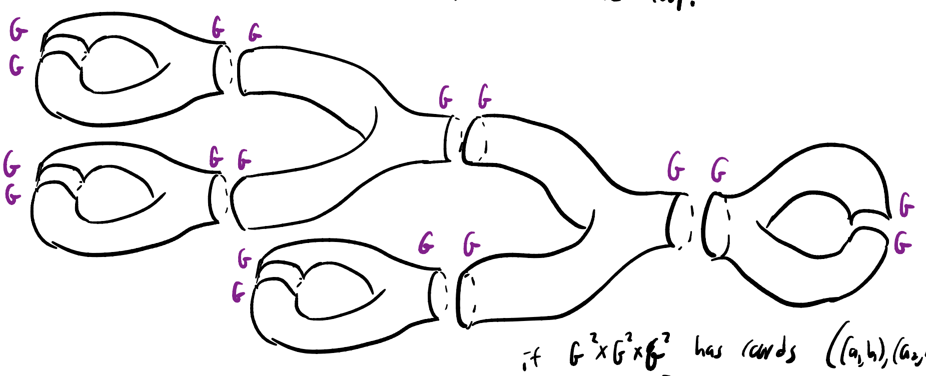
$M(\Sigma)$ is a Quasihamiltonian $\{e\}$-space - we reduced away all the symmetries

So, $M(\Sigma)$ carries 2 form ω satisfying:

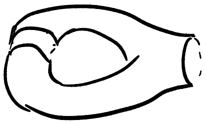
1. $d\omega = \mu^* \chi = 0$
2. $\ker \omega = \{\xi \mid \mathrm{Ad}_g \xi = -\xi\} = \emptyset$

$\Rightarrow$ $\boxed{\omega \text{ is symplectic}}$

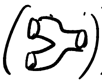
$g = 4$ each bdry carries a copy of G, meaning holonomy around the loop.



Repeatedly fuse together copies of $\mathbb{D}(G)$ = build pants decomp. of surface

 $= G^2\times G^2\times G^2 /\!/_e G\times G = \mathbb{D}(G)$

if $G^2\times G^2\times G^2$ has coords $((a_1,b_1),(a_2,b_2),(a_3,b_3))$
$(g_1,g_2)\in G^2$ acts by $(g_1.(a_1,b_1), g_2.(a_2,b_2), (g_1g_2)^{-1}.(a_3,b_3))$

$M_1 \circledast M_2 = (M_1 \times M_2 \times M($$))/\!/_e G^2$ divide out by incoming 2 circles, leaving only the outgoing circle

a Quasihamiltonian G space is the boundary datum for a 2D CFT? Boundary conditions have an algebra structure