Packing Stability & The Hamiltonian Diffeomorphism Group

last week, Oliver edtmiar posted to arXiv a pair of papers:

Packing stability and the subleading asymptotics of symplectic Weyl laws
https://arxiv.org/abs/2509.15390

Smooth perfectness of Hamiltonian diffeomorphism groups
https://arxiv.org/abs/2509.16327

together these resolve the packing stability problem in dimension 4. in this talk, I will put this problem into context and give a taste of the (rather involved) proof.

Part 1: Symplectic Packing & Weyl laws

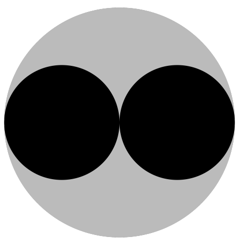
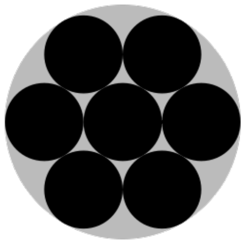
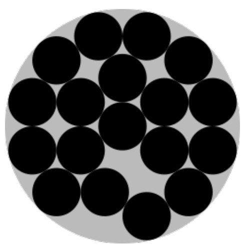
we are interested in embedding problems. let $B^{2n}(\lambda) = \{ \pi(|z_1|^2+|z_2|^2) < \lambda^2 \} \subset \mathbb{C}^2$

Packing problem: for (M^{2n}, ω) a symplectic manifold, what is the largest λ s.t there is a symplectic embedding $\bigsqcup_n B^{2n}(\lambda) \hookrightarrow M$?

Define the packing density $\nu_n(M) = \sup \left\{ \frac{\mathrm{vol}(\bigsqcup_n B^{2n}(\lambda))}{\mathrm{vol}(M)} \;\middle|\; \exists \text{ symplectic embedding } \bigsqcup_n B^{2n}(\lambda) \hookrightarrow M \right\}$

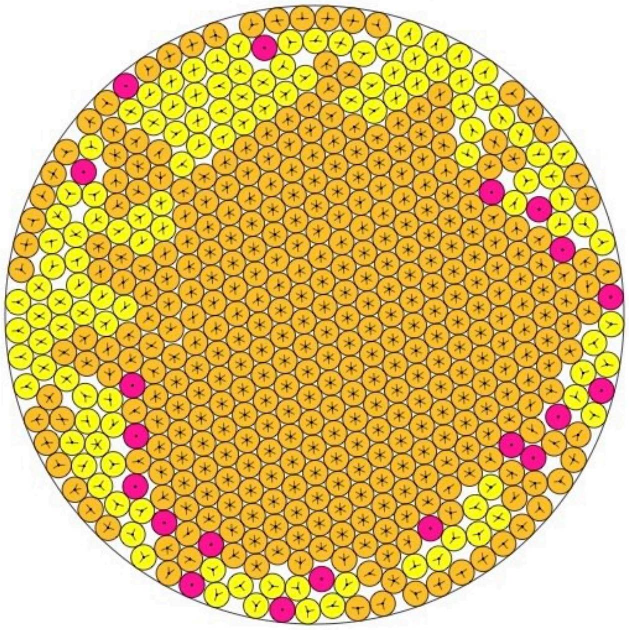
$\nu_n(M)$ measures the portion of M that can be filled by n equal-radius symplectic balls

The analagous problem for Riemannian geometry is rigid. Pack circles into a circle:

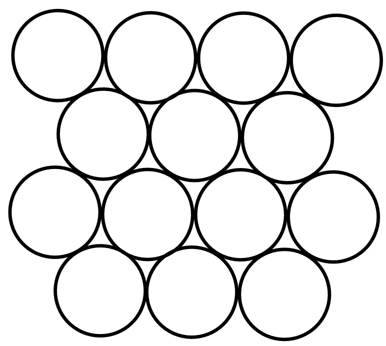
n	2	7	18
(conjectured) optimal packing			
Riemannian Packing density $\nu_n^{Riem}(B^2(1))$	0.5	0.777	0.76

as N gets very large, the optimal packing looks more like a hexagonal lattice:

$N = 510 \quad \nu^{Riem}_{510}(B^2(1)) \geq 0.85\ldots$



the hexagon lattice is the optimal Packing of circles in a plane



Density $\nu^{Riem}(\mathbb{R}^2) = \pi/\sqrt{12} \approx 0.91$

Thm: $\lim_{n\to\infty} \nu^{Riem}_n(B^2(1)) = \nu^{Riem}(\mathbb{R}^2)$

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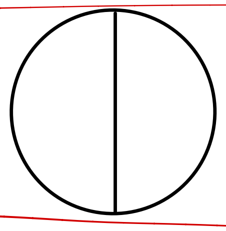
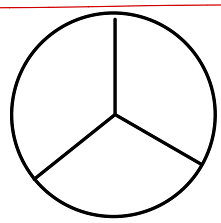
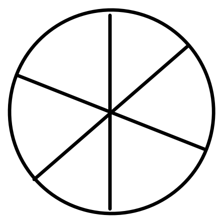
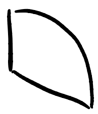
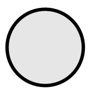
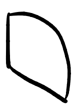
in contrast, volume preserving packing is <u>flexible</u>, & $\nu^{volume}_n = 1$ always

in 2D, this is the same as symplectic packing.

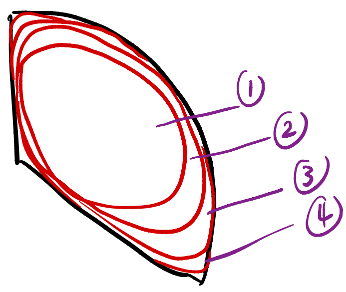
n	2	3	6	
symplectic packing				. . .
$\nu_n(B^2(1))$	1	1	1	

Note that the regions  have corners, so they are not the image of $B^2(1) =$  under a symplectomorphism...

instead, it is short hand for a sequence of embeddings

$\varphi_i : B^2(\lambda_i) \to$  $\quad \mathrm{vol}\ B(\lambda_i) \to \mathrm{vol}($  $)$

We say the region  is "fully filled" by a ball.



Packing problems of the three worlds:

Volume preserving $\supset$ Symplectic $\supset$ Riemanian

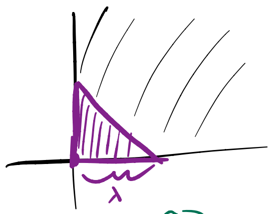
too flexible (boring) — just right :) — too Rigid (hard)

To reason pictorally about packings in 4D, we use toric domains:

$\mu : \mathbb{C}^2 \to \mathbb{R}^2_{\geq 0}$

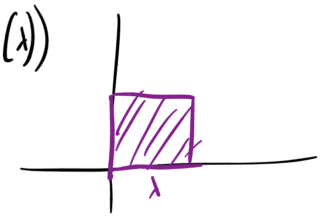
$\mu(z_1, z_2) = (\pi |z_1|^2, \pi |z_2|^2)$

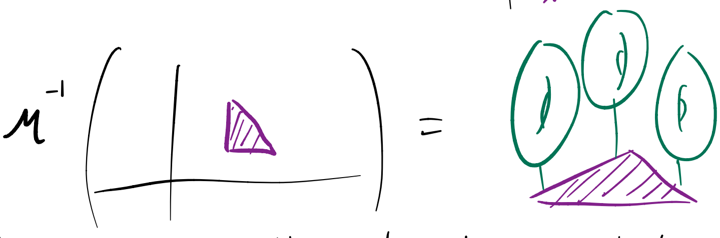
$\mu(B^4(\lambda))$



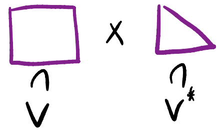
Polydisc $P^4(\lambda) = B^2(\lambda) \times B^2(\lambda)$

$\mu(P^4(\lambda))$





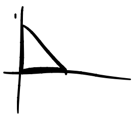
Lagrangian product of T^2 and 

in fact: $B^4(\lambda)$ is symplectomorphic to the Lagrangian product  $\subset T^*V$ is $V \times V^*$

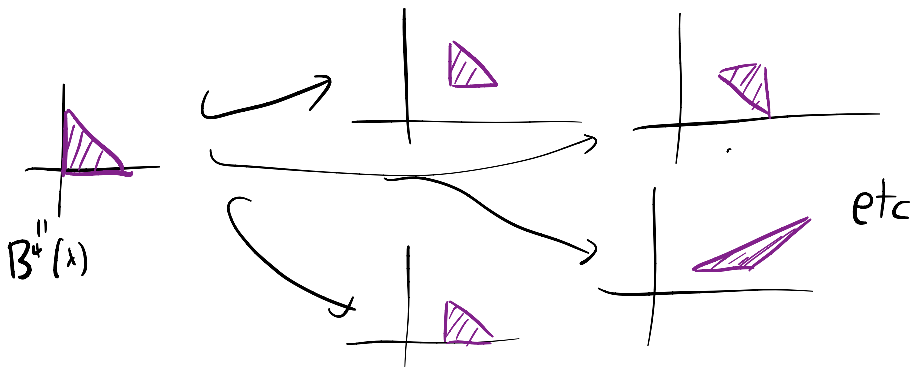
from now on, the μ^{-1} will be implicit.

Traynor's trick:

there is a full filling of $B^4(\lambda)$ into

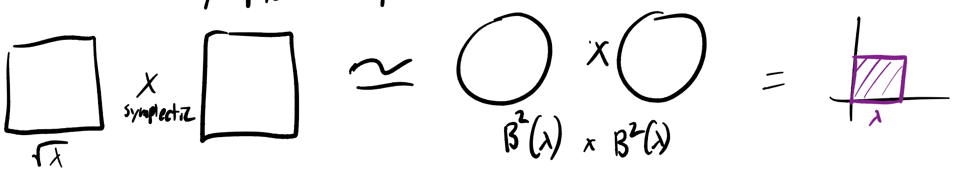
$\mu^{-1}(M \cdot$  $+ v)$

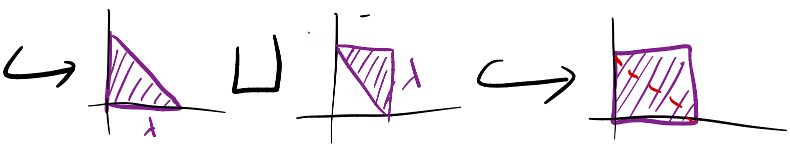
w/ $M \in SL(2,\mathbb{Z})$, $v \in \mathbb{R}^2_{\geq 0}$



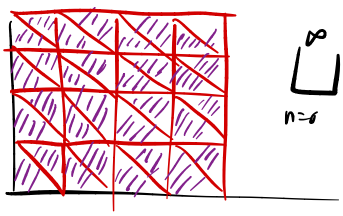
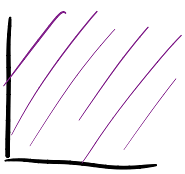
Thm: The cube $[0,\sqrt{\lambda}]^4 \subset \mathbb{C}^2$ is fully filled by $B^4(\lambda) \sqcup B^4(\lambda)$

Pf: the cube is symplectomorphic to Polydisc

$[0,\lambda]^2 =$  $B^2(\lambda) \times B^2(\lambda)$

Applying traynors trick: $B^4(\lambda) \sqcup B^4(\lambda) \hookrightarrow$ 

By tiling this cube, we can fully fill $\mathbb{C}^2$ by symplectic balls!

 $\bigsqcup_{n=0}^{\infty} B^4(\lambda) \hookrightarrow \mathbb{C}^2 =$ 

the packing density of $\mathbb{C}^2$ is $\nu(\mathbb{C}^2) = 1$

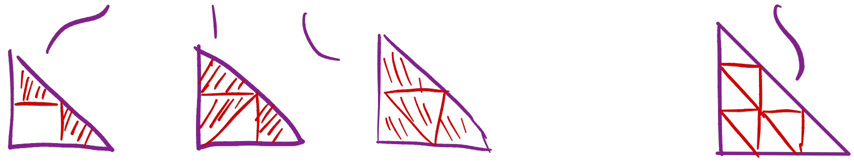
Observation (McDuff-Polterovich): for any (M,ω), $\lim_{n \to \infty} \nu_n(M) = 1$.

Let's look more specifically at symplectic packing of equal balls into a ball.

Thm: The packing densities of $B^4(1)$ are:

n	1	2 (Gromov)	3 (McDuff - Polterovich)	4	5	6	7	8	9	≥ 9 (Biran)	
$\nu_n(B^4(1))$	1	$1/2$	$3/4$	1	$20/25$	$24/25$	$63/64$	$288/289$	1	1	$\dots$

optimal packings



for all $n \geq 9$, $B^4(1)$ is fully filled by n equal balls! The limiting value $\nu_n(B^4(1)) \to 1$ is achieved after finite n.

Def: (M,ω) has Packing Stability if $\exists N$ s.t $\forall n > N$, $\nu_n(M) = 1$

At the time, it was conjectured that every (M,ω), w/ or w/o boundary, had packing stability. This was supported by a stream of results.

Thm: The following classes of symplectic manifold have packing stability:

- balls $B^4(\lambda)$ ✱
- $(\mathbb{CP}^2, \omega)$

} (Biran '96) using McDuff's connection between ball embeddings & Blow ups. Constructs J-curve on $\mathbb{CP}^2$ using SW = GR, then modifies symplectic form using J-curve to construct symplectic structure on the blowup. (inflation)

- closed, rational 4-manifolds (Biran '99) uses Donaldson submanifolds + inflation
- all closed rational 4-manifolds (Buse - Hind '13) Donaldson submanifolds w/ induction on dimension uses ECH to find embeddings between ellipsoids
- 4D ellipsoids ✱ + polydiscs ✱
- all closed 4-manifolds

} (Buse, hind, Opshtein '16) fully fills the 4-manifold by ellipsoids + "pseudo-balls", then proves packing stability for these canonical domains

the ✱ results we will use later

Inspired by Riemannian packing, we expect the rate of convergence to the bulk packing density to be controlled by the behavior of the boundary. This was confirmed in the first example of the failure of packing stability:

Thm (Dan C.G, Hind, '23): There is a Domain $X \subset \mathbb{R}^4$ without packing stability where X has fractal boundary

Oliver's paper essentially resolves the question in dimension 4:

Thm (Edtmair '25): every (M^4,ω) w/ smooth boundary has packing stability

There is a domain $X \subset \mathbb{R}^4$ w/o packing stability w/ $C^{2-\varepsilon}$ boundary.

Packing stability & Weyl laws

obstructions to packing stability are symplectic rigidity phemonema. We detect these w/ a symplectic capacity. If X a domain w/ packing stability, then there is an embedding $\coprod^n B^4(\lambda) \hookrightarrow X$ w/ $n\,\mathrm{vol}(B^4(\lambda)) = \mathrm{vol}(X)$, for every sufficently large n.

so $c(X) \geq c(\coprod^n B^4(\lambda))$ want a capacity which behaves well w/ Disjoint union.

Def: The alternate ECH capacities c_k are defined by a minimax problem on J-curves with point constraints

Energy of J-curve

$$c_k(X) = \sup_{\substack{x_1,\dots,x_k \in X \\ J \text{ compatible a.c.s}}} \ \inf_{u \in \mathcal{M}^J(\bar{X}, x_1, \dots, x_k)} \mathcal{E}(u)$$

moduli of J-curves in symplectic completion passing through $x_1, \dots, x_k$

Thm (Hutchings '22) c_k are symplectic capacities, and satisfy the same axioms as ECH capacities. in particular:

Disjoint union axiom: $$c_k(X_1 \sqcup \cdots \sqcup X_n) = \max_{\sum^n k_i = k} \sum^n c_{k_i}(X_i)$$ Distribute k pts among n disjoint regions

The Disjoint union axiom follows immediatly from the minimax.

k points in X = X_1 - - - - X_n

$\Leftrightarrow$ k_i points in X_i, w/ $\sum k_i = k$

the total energy is additive $\mathcal{E}(u) = \sum_i \mathcal{E}(u|_{\bar{X}_i})$

so maximizing over X is equivently first maximizing over X_i, then maximizing over choices of distributing $\{x_1, \dots, x_k\}$ into X_i.

Weyl law: let $X \subset \mathbb{R}^4$ be a compact domain w/ smooth boundary.

$$c_k(X) = 2\sqrt{k\,\mathrm{vol}(X)} + o(\sqrt{k})$$

define the error $e_k(X) = c_k(X) - 2\sqrt{k\,\mathrm{vol}(X)}$ sub leading term

Conjecture (Hutchings): if X star shaped & generic, $\lim_{k\to\infty} e_k(X) = \frac{-1}{2}\mathrm{Ru}(X)$

$\mathrm{Ru}(X)$ is the Ruelle invariant, a measure of avg rotation of the Reeb flow. Think of $\mathrm{Ru}(X)$ as an intrensically symplectiz measure of curvature of ∂X.

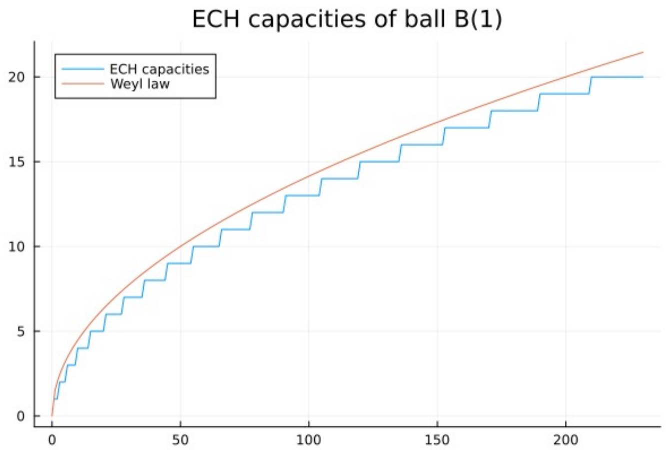
Thm (Edtmair): $c_k(X) = 2\sqrt{k\,\mathrm{vol}(X)} + o(1)$ or, $e_k(X)$ is bounded.

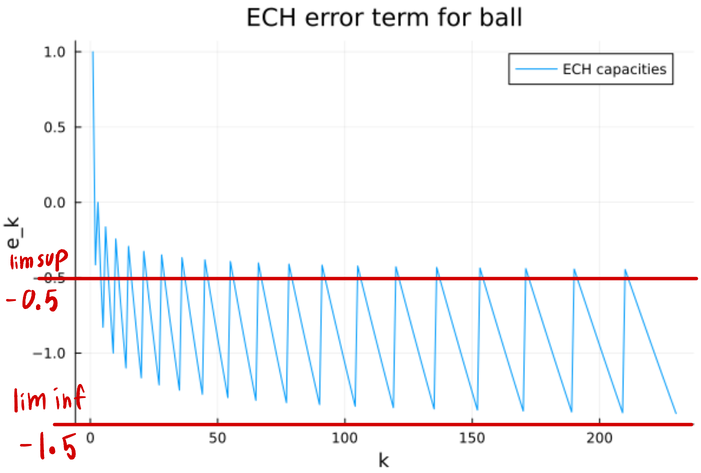
For example, take $X = B(\lambda)$: B^4 is implicit

k	$1, 2, \dots$
$\dfrac{c_k(B(\lambda))}{\lambda}$	$\underbrace{1,1}_{2\ 1s}, \underbrace{2,2,2}_{3\ 2s}, \underbrace{3,3,3,3}_{4\ 3s}, \dots$

$\mathrm{vol}(B(\lambda)) = \text{Area}$  $= \frac{1}{2}\lambda^2$

$c_k(B(\lambda)) \sim 2\sqrt{k\,\mathrm{vol}(B^4(\lambda))} = \lambda\sqrt{2k}$





write $e_k^1 = e_k(B(1))$

$e_k(B(\lambda)) = \lambda e_k^1$

$c_k(B(\lambda)) = \lambda\sqrt{2k} + \lambda e_k^B = \lambda(\sqrt{2k} + e_k^B)$

$\limsup_{k\to\infty} e_k^1 = -1/2$

$\liminf_{k\to\infty} e_k^1 = -3/2$

$\Rightarrow e_k(B(\lambda)) = o(1)$

Note $-\frac{1}{2}Ru(B(\lambda)) = -\lambda$. $B(1)$ is very nongeneric, so we don't expect $e_k(B^4(1)) \to \frac{-1}{2}Ru$

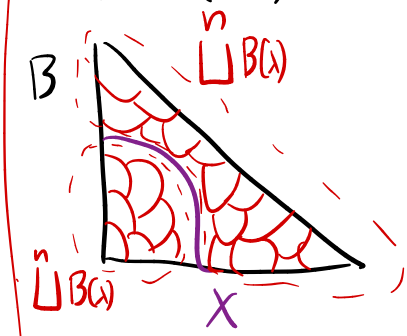
Using packing stability, we can extend the $o(1)$ bound from $B^4(1)$ to any bounded, star shaped $X \subset \mathbb{R}^4$ w/ smooth boundary.

Thm: let B be a ball & $X \subset B$ star shaped. suppose both X & $B \setminus X$ have packing stability. Then, $e_k(X) = O(1)$.

Cor: using Oliver's packing stability result, if ∂X smooth, then $e_k(X) = O(1)$

Proof: For expositional simplicity, suppose we can choose B s.t $\mathrm{vol}(B) = 2\,\mathrm{vol}(X)$

Then $\mathrm{vol}(B \setminus X) = \mathrm{vol}(X)$



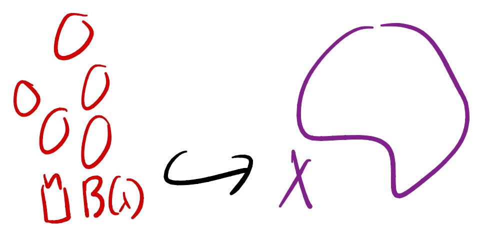
choose n the larger packing stability threshold of X or $B \setminus X$

Then, there are full fillings $\coprod^n B(\lambda) \to X$, $\coprod^n B(\lambda) \to B \setminus X$

since $\mathrm{vol}(X) = \mathrm{vol}(B \setminus X)$ by assumption, we can use the same λ for the inside & outside.

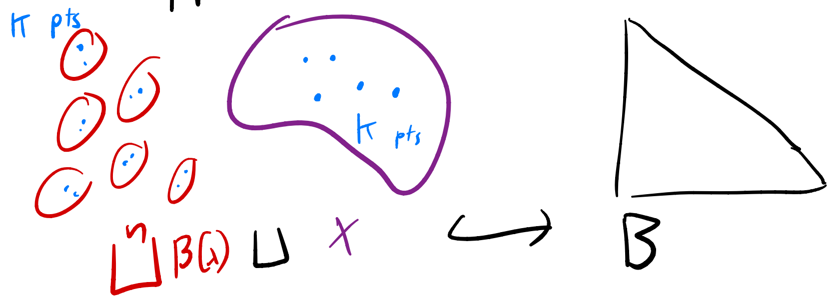
we will use this sandwiching of X between $\coprod^n B(\lambda)$ to sharply control $c_k(X)$

Lower Bound



$C_k(\bigsqcup B(\lambda)) \le C_k(X)$

Upper bound



$$C_{2k}\left(\bigsqcup^n (B(\lambda)) \sqcup X\right) \le C_{2k}(B)$$

$$\underbrace{C_k\left(\bigsqcup^n (B(\lambda))\right) + C_k(X)}_{\text{see blue pts above}} \overset{\star}{\le} \max_{i+j=2k} C_i\left(\bigsqcup^n (B(\lambda))\right) + C_j(X) \le C_{2k}(B)$$

all together, $C_k(\bigsqcup^n B(\lambda)) \le C_k(X) \le C_{2k}(B) - C_k\left(\bigsqcup^n (B(\lambda))\right)$

Remark: we expect the inequality $\star$ to be nearly sharp i.e the optimal i, j for the maximum is $i = j = k$. Asymptotically, its best to divy up the $2k$ points in C_{2k} according to the volumes of the connected components. So, this is close to the best bound we can get on $C_k(X)$.

Next we compute $C_k(\bigsqcup^n B(\lambda)) = \max_{\sum^n k_i = k} \sum^n C_{k_i}(B(\lambda))$

lower bound: (schematic) $\max_{\sum k_i = k} \sum^n C_{k_i}(B(\lambda)) \ge \sum^n C_{k/n}(B(\lambda))$

Distribute points evenly amongst the Balls. we expect this to give the best bound.

$$C_k(\bigsqcup^n B(\lambda)) \ge n \cdot C_{k/n}(B(\lambda)) = n \cdot \lambda\left(\sqrt{2k/n} + e'_k\right) = \lambda\left(\sqrt{2kn} + n e'_k\right)$$

$$\ge \underbrace{2\sqrt{k \,\mathrm{vol}(\bigsqcup^n B(\lambda))}}_{\text{weyl law}} + \lambda n e'_k \qquad O(1) \text{ terms}$$

Upper Bound: use packing stability! If $n \ge 9$, $\exists$ full filling

$\bigsqcup^n B(\lambda) \hookrightarrow B(\alpha)$ where $\mathrm{vol}\, B(\alpha) = \mathrm{vol}(\bigsqcup^n B(\lambda))$

$\alpha^2/2 = n\lambda^2/2 \Rightarrow \alpha = \lambda\sqrt{n}$

$$C_k\left(\bigsqcup^n (B(\lambda))\right) \le C_k(B(\lambda\sqrt{n})) = \lambda\sqrt{n}\left(\sqrt{2k} + e'_k\right)$$

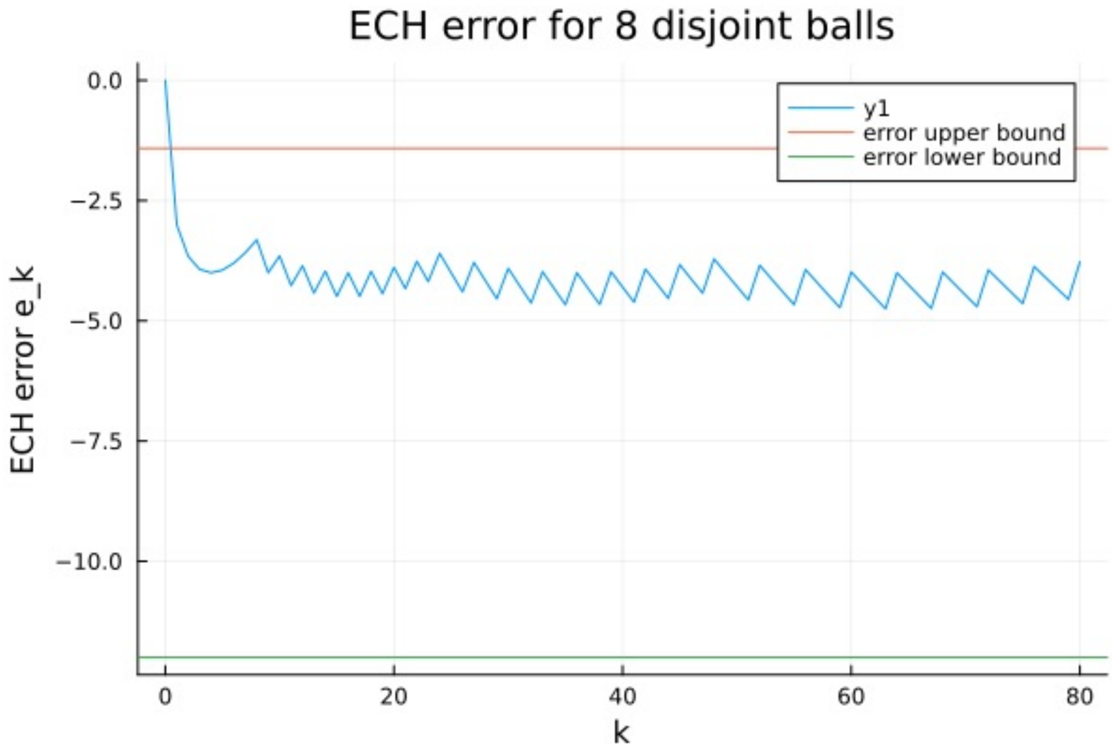
All together: $\lambda\sqrt{2kn} + \lambda n e'_k \le C_k(\bigsqcup^n B(\lambda)) \le \lambda\sqrt{2kn} + \lambda\sqrt{n}\, e'_k$

$$\Rightarrow C_k\left(\bigsqcup^n B(\lambda)\right) = \sqrt{2k \,\mathrm{vol}(\bigsqcup^n B(\lambda))} + O(1)$$

note that we have control over the error terms in terms of n.

using $\liminf e^B_k = -3/2 \qquad \limsup e^B_k = -1/2$

we get $\liminf e_k(\coprod^n B(\lambda)) \geq -\frac{3}{2}\lambda n \qquad \limsup e_k(\coprod^n B(\lambda)) \leq -\frac{1}{2}\lambda\sqrt{n}$



Putting everything together:

$$\underbrace{c_k(\coprod^n B(\lambda))}_{} \leq c_k(X) \leq \underbrace{c_{2k}(B)}_{} - \underbrace{c_k(\coprod^n (B(\lambda))}_{}$$

$$2\sqrt{k \operatorname{vol}(\coprod^n B(\lambda))} + \lambda n e'_k \leq c_k(X) \leq \left(2\sqrt{2k\operatorname{vol}(B)} + e_{2k}(B)\right) - \left(2\sqrt{k\operatorname{vol}(\coprod^n B(\lambda))} + \lambda\sqrt{n}\, e'_k\right)$$

using $\operatorname{vol}(\coprod^n B(\lambda)) = \operatorname{vol}(X) = \operatorname{vol}(B)/2$,

$$2\sqrt{k\operatorname{vol}(X)} + \lambda n e'_k \leq c_k(X) \leq \left(\cancel{2}\sqrt{4k\operatorname{vol}(X)} + e_{2k}(B)\right) - \left(2\cancel{\sqrt{k\operatorname{vol}(X)}} + \lambda\sqrt{n}\, e'_k\right)$$

$e_k(X) = c_k(X) - 2\sqrt{k\operatorname{vol}(X)}$ satisfies $\lambda n e'_k \leq e_k(X) \leq e_{2k}(B) - \lambda\sqrt{n}\, e'_k$

$\Rightarrow e_k(X) = o(1)$ $\blacksquare$

While we're here, let's figure out bounds on $e_k(X)$. we need to express above in terms of n & $\operatorname{vol}(X)$

$n\lambda^2/2 = \operatorname{vol}(\coprod^n B(\lambda)) = \operatorname{vol}(X) \quad \Rightarrow \quad \lambda = \sqrt{\frac{2\operatorname{vol}(X)}{n}}$

if $B = B(\alpha)$, then $\alpha^2/2 = \operatorname{vol}(B(\alpha)) = \operatorname{vol}(X) \Rightarrow \alpha = \sqrt{2\operatorname{vol}(X)}$. so, $e_k(B) = \sqrt{2\operatorname{vol}(X)}\, e'_{2k}$

$$\sqrt{2n\operatorname{vol}(X)}\, e'_k \leq e_k(X) \leq \sqrt{2\operatorname{vol}(X)}\, e'_{2k} - \sqrt{2\operatorname{vol}(X)}\, e'_k$$

using $\liminf e_k' = -3/2$ & $\limsup e_k' = -1/2$, get

$$\boxed{\begin{aligned} \liminf_{k\to\infty} e_k(x) &\geq \sqrt{n}\cdot\frac{-3}{2}\sqrt{2\,\mathrm{vol}(x)} \\ \limsup_{k\to\infty} e_k(x) &\leq +\sqrt{2\,\mathrm{vol}(x)} \end{aligned}}$$

Remarks:

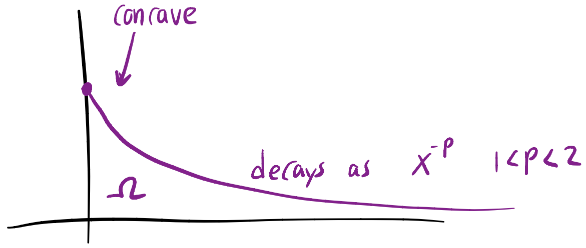
- The $O(1)$ bound obtained from the proof depends on the packing stability threshold N. smaller threshold $\Rightarrow$ better bound on error
- if $e_k(x) \to Ru(x)$, then the $N \gtrsim Ru(X)^2$
  larger Ruelle invariant $\Rightarrow$ larger stability threshold

~ Counterexamples to packing Stability ~

We saw X has Packing stability $\Rightarrow e_k(x) = o(1)$

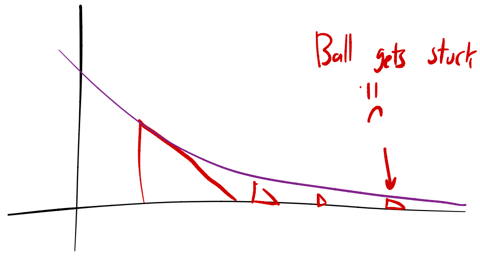
contrapositivly, $\lim_{k\to\infty} |e_k(x)| = \infty \Rightarrow X$ does not have packing stability.

Thm (Dan C.G, Hind '23): The toric domain $X_\Omega = \mu^{-1}(\Omega)$ does not have packing stability



Intuitivly, this domain is like a symplectic cylinder whose width limits to zero as you go to infinity. By Gromov non-squeezing, a fixed width ball can only get so deep into the tube before getting stuck. No matter how small the balls, you can't fill the end of the tube, so you don't expect packing stability

X_Ω is unbounded, so is hard to work with. Though, X_Ω is symplectomorphic to a bounded domain.



"spaghetti morphism"

$X_\Omega \Rightarrow Z$

fit Ω into unit disc by "swirling" around

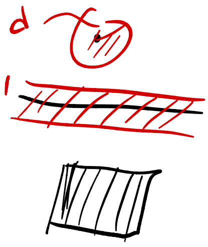
Can do something similar to X_Ω w/ a symplecto morphism

∂X_Ω smooth unbounded $\xLeftrightarrow[\text{symplectomorphism}]{}$ ∂Z not smooth bounded no free lunch :(

X_Ω has $e_k(z) \to \infty$ by solving combinatorial problem from the toric domain

in fact, if Ω decays like x^{-p}, $e_k(z) \sim k^{1/2p}$, because ∂Z has minkowski dimension ≥ 3!

measure dimension of a set by the volume of points within distance d of the set.



vol $\propto d^2 \Rightarrow$ codim. 2

vol $\propto d \Rightarrow$ codim. 1

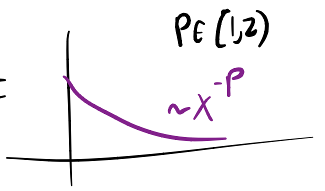
vol $\propto d^0 \Rightarrow$ codim. 0

let $V_d(\partial z)$ be the volume of Z within distance d of ∂Z

Def: The inner minkowski dimension of ∂Z

$$\dim_{min}(\partial z) = 4 - \liminf_{d \to 0} \frac{\ln(V_d(\partial z))}{\ln(d)}$$

Remarks: – if $V_d(z) = d^\alpha + o(d^\alpha)$, $\dim_{min}(\partial z) = 4 - \alpha$. the minkowski dimension measures decay rate of volume near the boundary.

- if $M \subset Z$ is smooth of codimension α, then $V_d(M) \sim d^\alpha$.
- this is independent of metric on $\mathbb{R}^4$
- for X_Ω = , $\dim_{min}(\partial X_\Omega) = 2 + \frac{2}{p} \in \underbrace{(3,4)}$

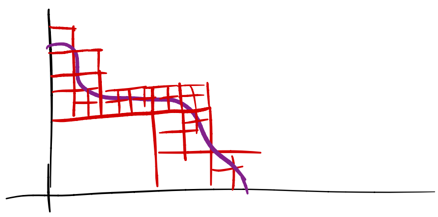
Dimension > 3 bdry!

The minkowski dimension of ∂Z limits the growth rate of e_k:

Thm (Fractal Weyl law): if $\dim_{min}(\partial z) = d$, then $|e_k(z)| \lesssim k^{\frac{d-2}{4}}$ $d \in [3,4]$

or... if you ever see a domain Z w/ $e_k(z) \sim k^\alpha$ for $\alpha > 1/4$, then ∂Z must have minkowski dimension > 3. Detecting fractals w/ ECH.

"Proof": follow usual proof of weyl law: efficiently pack inside w/ poly discs, by increasing resolution near boundary. for smooth boundary, this gets us to $|e_k| < k^{1/4}$.



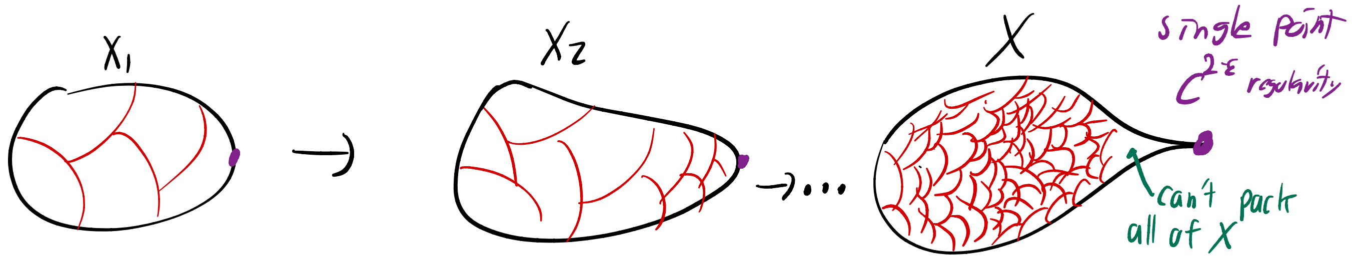
If the volume decays slowly (extra large minkowski dimension), then we need to keep many more polydiscs. This makes our ball packing less efficient, & gives us a worse bound on e_k.

Edtmair constructed a domain $X \subset \mathbb{R}^4$ w/ $\partial X \in C^{1,\alpha}$ for all $\alpha \leq 1$, w/ $e_k(X) \to \infty$. Hence X does not have packing stability, showing that you need at least a C^2 boundary for guarenteed packing stability.

Properties of this example:

- ∂X is smooth outside of 1 point
- X is constructed from a sequence of C^∞ domains X_n. The curvature of $\partial X_n \to \infty$ only at the point of non-smoothness
- The packing stability threshold of $X_n \to \infty$ as $n \to \infty$
- $Ru(X_n) \to \infty$, in accordance with micheals conjecture

X_n:



Curvature $\to \infty$ at non regular point

Threshold for packing stability increases

Ruelle invariant increases

Moral of the Story:

- Packing stability $\Rightarrow$ uniform error bound on ECH weyl law
- Packing stability / ECH error are Boundary Phenomena
- As boundary curvature is larger, stability / ECH error gets worse
  - ↳ packing stability requires smaller balls
  - ↳ ECH error term gets larger

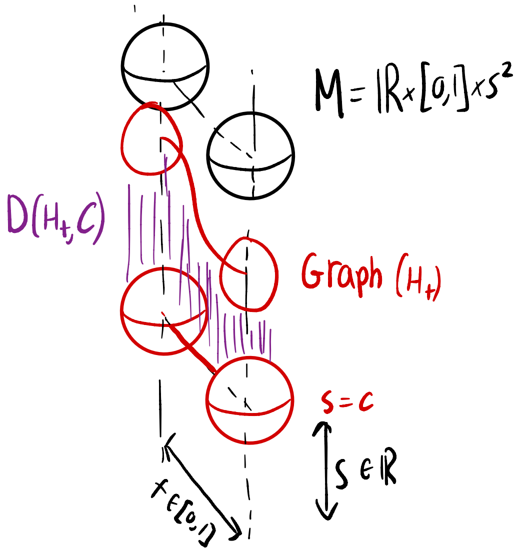
Part 2: Proving Packing Stability

Recall Oliver's packing stability theorem

<u>Thm</u>: every Symplectic 4-manifold (M,ω) with smooth boundary has packing stability

to prove this, we need to decompose (M,ω) into simple pieces with packing stability.

The key idea: Decompose (M,ω) using the algebraic structure of the Hamiltonian diffeomorphism group $\mathrm{Ham}(M)$



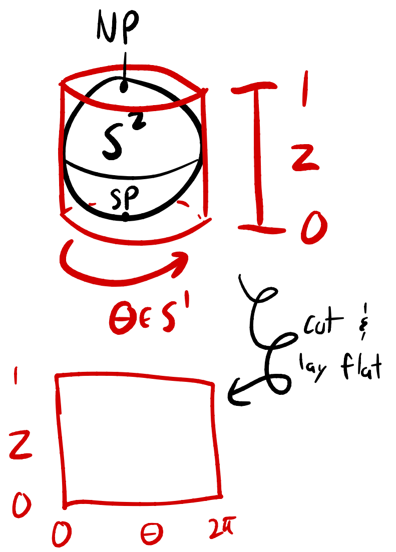
<u>Toy model</u>: let H_t be a family of hamiltonians on S^2, $H_t: \underset{t}{[0,1]} \times \underset{p}{S^2} \to \mathbb{R}$

fix ω an area form on S^2. let $M = \underset{s}{\mathbb{R}} \times \underset{t}{[0,1]} \times \underset{p}{S^2}$ with symplectic form $\Omega = ds\wedge dt + \omega$

$$D(H_t, C) = \{(s,t,p)\in M \mid C \le s \le H_t(p)\} \subset M$$

Region below Graph of H_t, above C

<u>Thm</u>: for C sufficiently negative, $D(H_t, C)$ can be fully filled by Balls and polydiscs



<u>Example</u>: choose polar coordinates (z,θ) on S^2, $z\in[0,1]$, $\theta\in S^1$

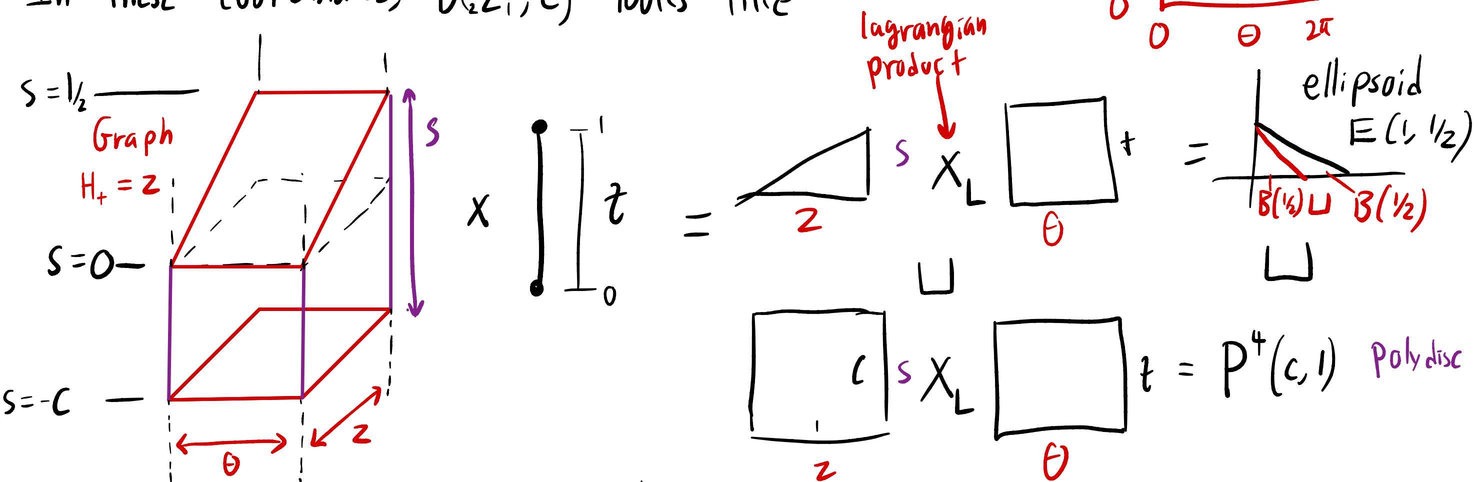
define $H_t(z,\theta) = \frac{1}{2}z$ the standard height function

we can describe the packing of $D(\frac{1}{2}z, C)$ explicitly

first, use polar coordinates to project $S^2 - \{NP, SP\} \to \underset{\theta}{S^1} \times \underset{z}{(0,1)}$

Next, cut along $\theta=0$ to obtain a rectangle $(0,1)\times(0,2\pi)$

In these coordinates, $D(\frac{1}{2}z, C)$ looks like



That is, $D(z,C)$ is fully filled by $B^4(1/2) \sqcup B(1/2) \sqcup P^4(c,1)$.

Remark: Both balls and polydiscs have packing stability. It's not so simple to show that something fully filled by balls / polydiscs has packing stability

e.g $B^4(1)$ is only fully filled by $\bigsqcup^n B^4(\lambda)$ when $\begin{array}{l} - n \geq 9 \text{ (Packing stability)} \\ - \operatorname{vol}(B^4(\lambda)) = \operatorname{vol}(B^4(1))/n \end{array}$

if disjoint copies of $B^4(\lambda)$ fully fill $B^4(1) \sqcup P^4(c,1)$ then $\operatorname{vol}(B^4(\lambda)) = \frac{\operatorname{vol}(B^4(1))}{n} = \frac{\operatorname{vol}(P^4(c,1))}{n'}$

This is only possible if $\frac{\operatorname{vol}(B^4(1))}{\operatorname{vol}(P^4(1,c))}$ is rational !! a little too much to hope for.

Oliver circumvents this by using the decomposition into balls & polydiscs to fully fill (M,ω) w/ a single ellipsoid, then packing the ellipsoid.

When are two domains $D(H_+^0, c)$ & $D(H_+^1, c)$ symplectomorphic?

If there is a family of symplectomorphisms $\psi^\lambda : M \circlearrowleft$ s.t $\begin{cases} \psi^0 = id \\ \psi^1 : D(H_+^0, c) \xrightarrow{\sim} D(H_+^1, c) \end{cases}$, then:

1. $\operatorname{vol} D(H_+^0, c) = \operatorname{vol} D(H_+^1, c) = \int_{S^2 \times [0,1]} (H_+ - c)\, \omega \wedge dt$

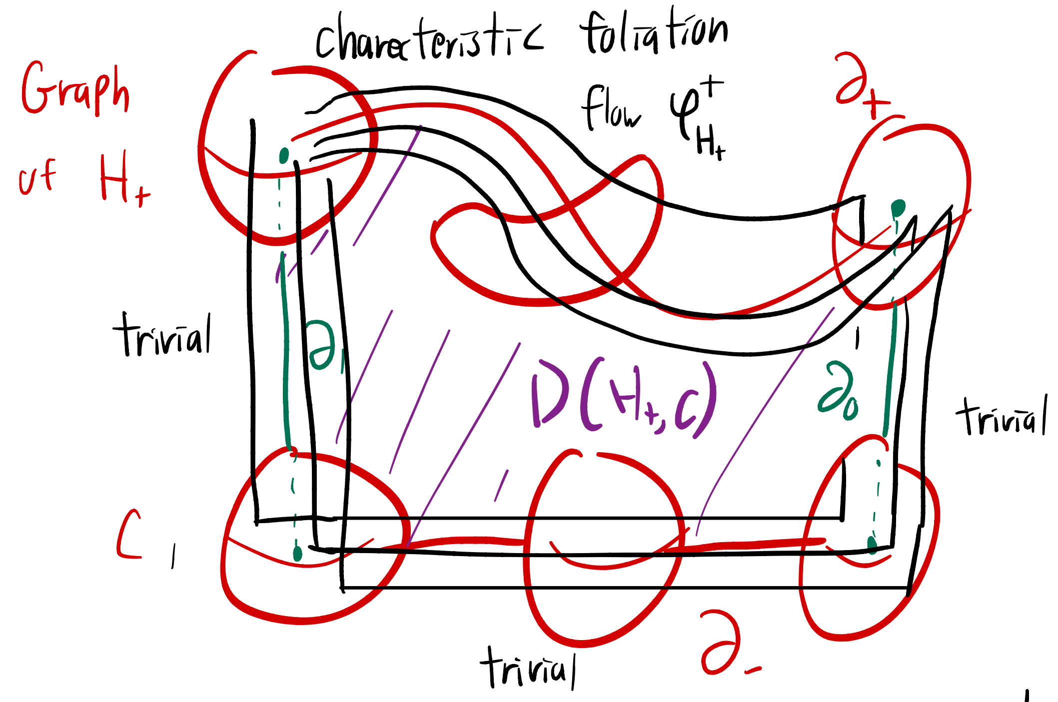
2. The charateristic foliations of $D(H_+^0, c)$ & $D(H_+^1, c)$ agree.

Recall for $X \subset (M,\omega)$, the charecteristic foliation $\mathcal{I}$ on ∂X is the foliation integrating the distrabution $\ker \omega|_{T\partial X}$

e.g, if ∂X is contact type, $\mathcal{I}$ is foliation by Reeb orbits.

we can explicitly describe the charectistic foliation of $D(H_+, c)$:

on Graph (H_t), $\mathcal{I} = \langle \dot{H}_t \partial_s + \partial_t + X_{H_t} \rangle$



$\partial D(H_+, c)$ has 4 parts:

- $\partial_+ D =$ Graph H
- $\partial_- D = C$
- $\partial_0 D = S^2 \times \{0\} \times \mathbb{R} \cap D$
- $\partial_1 D = S^2 \times \{1\} \times \mathbb{R} \cap D$

on $\partial_{0,1} D$, $\mathcal{I} = \langle \partial_s \rangle$

on $\partial_- D$, $\mathcal{I} = \langle \partial_t \rangle$

First return map of $\mathcal{I}$ is flow $\varphi^1_{H_+}$ of X_{H_+}

So we should think of $\partial D(H_t, c)$ as the mapping torus of $\varphi^1_{H_t}$

$D(H^0_t, c)$ symplectomorphic to $D(H^1_t, c) \Rightarrow \varphi^1_{H^0_t}$ & $\varphi^1_{H^1_t}$ have same mapping torus

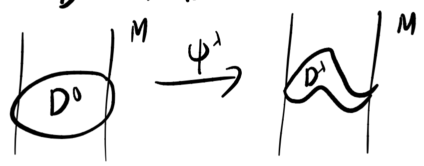
$\Rightarrow \varphi^1_{H^0_t} = \varphi^1_{H^1_t}$ (up to conjugation)

Thm: let $D^\lambda = D(H^\lambda_t, c)$. if H^λ_t satisfies:

A. vol (D^λ) independant of λ

B. $\varphi^1_{H^\lambda_t}$ independant of λ

$\Rightarrow$ $\exists$ path of symplectomorphisms $\psi^\lambda : D^0 \to D^\lambda$



Proof Sketch:

Step 1: use condition B to build a map $\psi^\lambda : \partial D^0 \to \partial D^\lambda$ which preserves $\omega|_{\partial D}$. This extends to a symplectomorphism of nbhds of ∂D, & similarly to the complement $M \setminus D^0 \to M \setminus D^\lambda$

Step 2: use moser argument to extend ψ^λ into the interior of D^λ. this uses condition A to ensure $[\psi^{\lambda *}\Omega] \in H_2(D^\lambda, \partial D^\lambda)$ is constant.

for our example $H_t(z,\theta) = \frac{1}{2}z$, time 1 flow is rotation by π around the poles

for any H_t w/ time 1 flow conjugate to rotation by π, $\int H_t = \int \frac{1}{2}z = \frac{1}{4}$,

$D(H_t, C)$ symplectomorphic to $D(\frac{1}{2}z, C)$, so $D(H_t, C)$ decomposes as balls & polydiscs (& has packing stability)

Key insight: an arbitrary $\psi \in Ham(S^2, \omega)$ can be decomposed into terms $\chi \circ \varphi^1_{\frac{1}{2}z} \circ \chi^{-1}$

Thm (Banyaga) $Ham(S^2, \omega)$ is a simple group. I.e, it has no normal proper subgroups

Corrolarry: for a fixed $\psi \in Ham(S^2, \omega)$, consider the subgroup $N(\psi_0)$ generated by $\chi \circ \psi_0 \circ \chi^{-1}$ for $\chi \in Ham(S^2, \omega)$. $N(\psi_0)$ is conjugation invariant, hence normal. Since $Ham(S^2, \omega)$ is simple, $Ham(S^2, \omega) = N(\psi_0)$

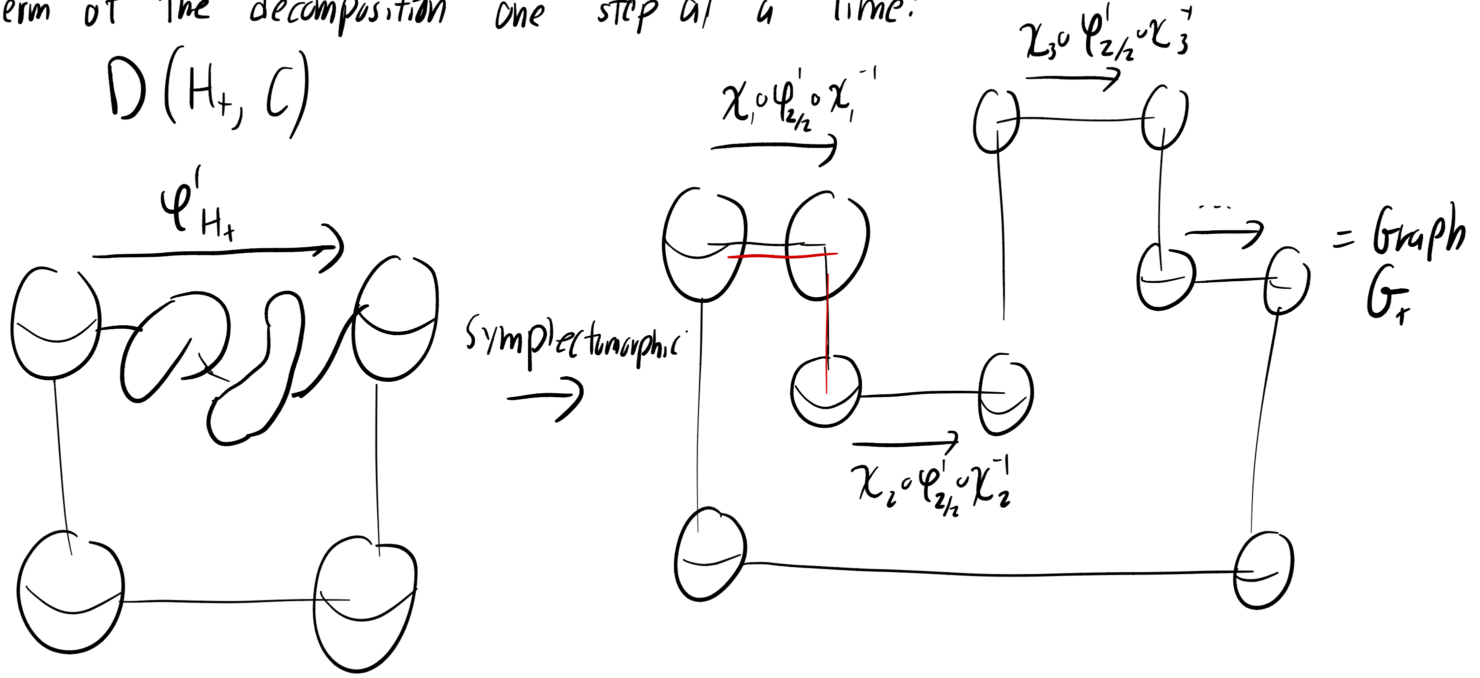
That is, every $\psi \in Ham(S^2, \omega)$ can be written as

$$\psi = (\chi_n \circ \psi_0 \circ \chi_n) \circ \dots \circ (\chi_1 \circ \psi_0 \circ \chi_1)$$

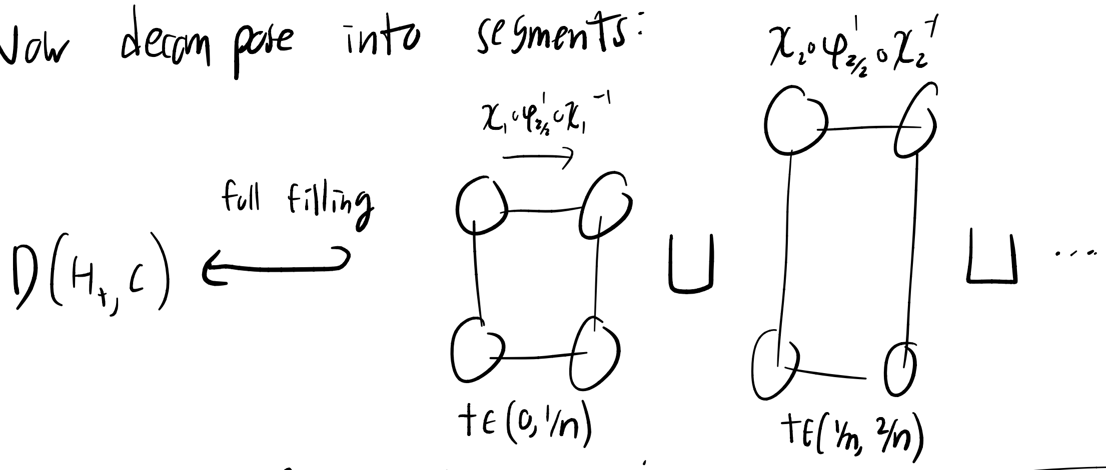
In particular, take $\psi_0 = \varphi^1_{\frac{1}{2}z}$

Let's apply this to $D(H_+, C)$. decompose $\varphi^1_{H_+} = (\chi_n \circ \varphi^1_{z/2} \circ \chi_n) \circ \dots \circ (\chi_1 \circ \varphi^1_{z/2} \circ \chi_1)$

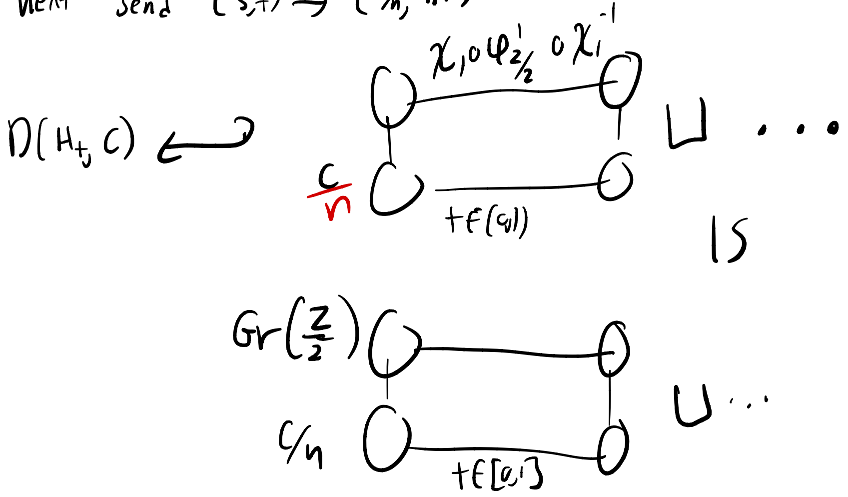
Then, choose a careful Hamiltonian $\tilde{H}_+$ w/ same time 1 map, which runs through each term of the decomposition one step at a time:



Now decompose into segments:



next send $(s,t) \rightsquigarrow (s/n, nt)$



$$D(H_n, C) \hookrightarrow \bigsqcup^n D\left(\frac{z}{2}, \frac{c}{n}\right)$$

each $D\left(\frac{z}{2}, \frac{c}{n}\right)$ decomposes into balls & polydiscs so $D(H_+, C)$ decomposes too, for sufficiently negative C.

The constructed hamiltonian G_+ osscilates of order n, # of terms in the Hamiltonian decomposition.

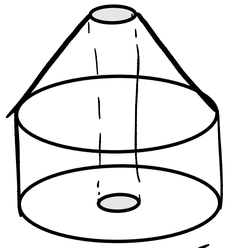
The larger n, the smaller the needed C.

This proves subgraphs of Hamiltonians on S^2 can be decomposed ▨

To Decompose an arbitrary (M^4, ω) into balls & polydiscs, need to first split (M, ω) into subgraphs of hamiltonians. we can't always use spheres.

Instead, we use subgraphs over annuli.

The Hamiltonian which generates rigid rotation has subgraph



"Symplectic Frustum"

Ingredients for Proving packing stability:

- Analogue of Banyaga's simplicity result for Ham( ) —— Olivers other paper
- Control over # of terms in Hamiltonian decomposition, using C^∞ norm.
- Generalize above argument from spheres to annuli, to split a symplectic frustum into balls & poly discs
- Method of decomposing (M, ω) into subgraphs of annuli C^∞-close to the standard rotation (split into symplectic frustum)
- Method for gluing together filling by balls & polydiscs into full filling by single, very long & skinny ellipsoid.