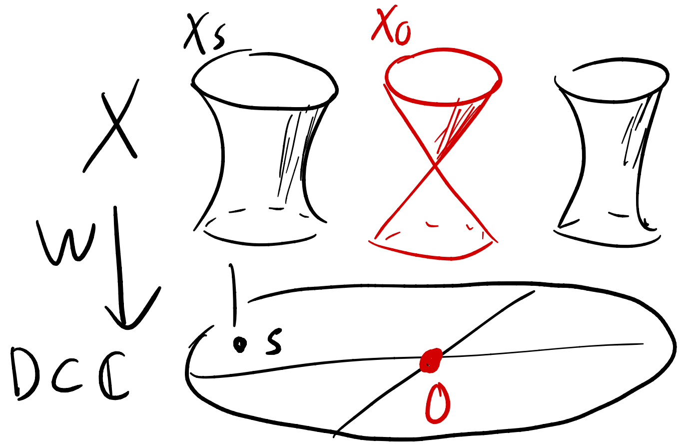
Matrix factorizations and Topology



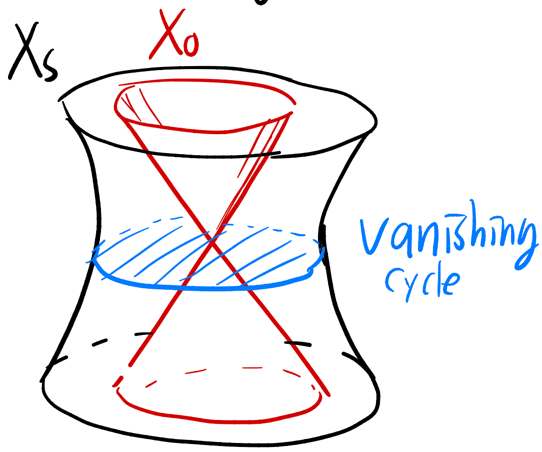
let $W: X \to D$ be a holomorphic function, with a unique critical value 0.

$X_0 = W^{-1}(0)$ is the singular fiber

$X_s = W^{-1}(s)$ is the nearby fiber

Classically, we have a long exact sequence in homology

$$\cdots \to \underset{\substack{\text{is} \\ H^n(X_0)}}{H^n(X)} \longrightarrow \underset{\text{"nearby cycles"}}{H^n(X_s)} \to \underset{\text{"vanishing cycles"}}{H^n(X, X_s)} \to \cdots$$



this is categorified by the exact sequence of categories

$$\mathrm{Perf}(X_0) \hookrightarrow D^b(X_0) \longrightarrow D^b(X_0)/\mathrm{Perf}(X_0) = \mathrm{Sing}(X_0)$$

We learned last time that $\mathrm{Sing}(X_0) \simeq MF(X, W)$

Def: A Matrix factorization of W is a pair of vector bundles P_0, P_1 & maps $P_0 \underset{p_1}{\overset{p_0}{\rightleftarrows}} P_1$ s.t

$p_0 p_1 = W \cdot id_{P_0} \qquad p_1 p_0 = W \cdot id_{P_1}$

or, $P = P_0 \oplus P_1 \quad p: P \circlearrowleft \quad p = \begin{bmatrix} 0 & p_0 \\ p_1 & 0 \end{bmatrix} \qquad p^2 = W \cdot id$

Example: $W = x^2 + y^2$ ← can't factor this over the reals

$$\underbrace{\begin{bmatrix} x & y \\ -y & x \end{bmatrix}}_{f} \underbrace{\begin{bmatrix} x & -y \\ y & x \end{bmatrix}}_{g} = \begin{bmatrix} x^2+y^2 & 0 \\ 0 & x^2+y^2 \end{bmatrix} = W \cdot id$$ can factor it with matrices!

Definition $MF(X,W)$ is a category with objects matrix factorizations & morphisms $\mathbb{Z}_2$-graded maps $\mathrm{Hom}^\bullet(P,Q)$

Hom^0: degree 0
$$\begin{tikzcd} P_0 \arrow[r, shift left, "p_0"] \arrow[d, "\varphi_0"'] & P_1 \arrow[l, shift left, "p_1"] \arrow[d, "\varphi_1"] \\ Q_0 \arrow[r, shift left, "q_0"] & Q_1 \arrow[l, shift left, "q_1"] \end{tikzcd}$$

Hom^1: degree 1
$$\begin{tikzcd} P_0 \arrow[r, shift left] \arrow[dr, "\phi_1"] & P_1 \arrow[l, shift left] \arrow[dl, "\phi_0"'] \\ Q_0 \arrow[r, shift left] & Q_1 \arrow[l, shift left] \end{tikzcd}$$

the complex carries a differential $\partial\phi = \phi p + (-1)^{|\phi|} q\phi$

define the homotopy category $[MF(X,W)]$ with objects matrix factorizations and morphisms $\mathrm{Mor}^\bullet(P,Q) = H^\bullet(\mathrm{Hom}^\bullet(P,Q),\partial)$

explicitly:

$\mathrm{Mor}^0(P,Q)$

ϕ commutes w/ f, g

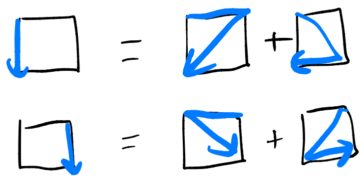
"cycles in $\mathrm{Hom}^0(P,Q)$"

$$\begin{tikzcd} P_0 \arrow[r, shift left] \arrow[d, "\phi_0"'] \arrow[dr, "\varphi_1"] & P_1 \arrow[l, shift left] \arrow[d, "\phi_1"] \arrow[dl, "\varphi_0"'] \\ Q_0 \arrow[r, shift left] & Q_0 \arrow[l, shift left] \end{tikzcd}$$

$\phi \sim \phi$ if

$$\phi_0 - \phi_0' = \varphi_1 p + q\varphi_0$$
$$\phi_1 - \phi_1' = q\varphi_1 + \varphi_0 p$$

"modulo boundary"



i.e chain homotopies of degree-p

Questions: - why does $MF(X,W)$ see only the critical locus?
- How do we extract topology $H^*(X,X_s)$ from $MF(X,W)$?

Part 1: Boring Matrix factorization categories

at the outset, $MF(X,W)$ seems to have many objects. at least $\Gamma(\mathcal{O}_X)$ worth. but, by defining morphisms up to homotopy, many objects become isomorphic. This "cuts down" $MF(X,W)$ to a managable size. If W has no critical points, every object in $MF(X,W)$ is isomorphic to the 0 object

Lemma: an object $Q_0 \rightleftarrows Q_1$ is isomorphic to the 0 object $0 \underset{g}{\overset{f}{\rightleftarrows}} 0$ if the morphisms id & 0 are identified in $[MF(X,W)]$

Pf: $\phi \in \mathrm{Mor}(Q,0)$ is an isomorphism $\Leftrightarrow$ $\exists$ $\phi^{-1} \in \mathrm{Mor}(0,Q)$ s.t $\phi^{-1}\phi = id \in \mathrm{Mor}(Q,Q)$
but any $\phi^{-1} \in \mathrm{Mor}(0,Q)$ is the 0 morphism, so $\phi^{-1}\phi = 0$.
Hence, $Q \simeq 0 \Leftrightarrow id = 0$ as morphisms from Q to Q.

$$\begin{tikzcd} Q_0 \arrow[r] \arrow[d, shift left] \arrow[rr, bend left, "id"] & 0 \arrow[r] \arrow[d, shift left] & Q_0 \arrow[d, shift left] \\ Q_1 \arrow[r] \arrow[u, shift left] & 0 \arrow[r] \arrow[u, shift left] & Q_1 \arrow[u, shift left] \end{tikzcd}$$
ϕ $\quad$ ϕ^{-1}

explicitly: $\exists$ φ_0, φ_1 s.t

$$\begin{tikzcd} P_0 \arrow[r, shift left, "f"] \arrow[d, "id"'] \arrow[dr, "\varphi_0"] & P_1 \arrow[l, shift left, "g"] \arrow[d, "id"] \arrow[dl, "\varphi_1"'] \\ P_0 \arrow[r, shift left, "f"] & P_1 \arrow[l, shift left, "g"] \end{tikzcd}$$

$$\varphi_1 f + g\varphi_0 = id$$
$$\varphi_0 g + f\varphi_1 = id$$

Thm: Every object of the following categories are isomorphic to 0

Category	objects	Chain homotopy $\mathrm{id} - 0 = \partial\varphi$
$MF(X,1)$	$P_0 \underset{g}{\overset{f}{\rightleftarrows}} P_1$	$\varphi_0 = f,\ \varphi_1 = 0$
$MF(X,w)$ $w \neq 0$	$P_0 \underset{g}{\overset{f}{\rightleftarrows}} P_1$	$\varphi_0 = \frac{f}{w}$ $\varphi_1 = 0$
$MF(\mathbb{C}, z)$	$\mathbb{C} \underset{g}{\overset{f}{\rightleftarrows}} \mathbb{C}$	$fg = z \Rightarrow$ WLOG f vanishes to degree 1 $\varphi_0 = f/z,\quad \varphi_1 = g$
$MF(\mathbb{C}^n, z_i)$ ← linear	$P_0 \underset{g}{\overset{f}{\rightleftarrows}} P_1$	$z_i \cdot \mathrm{id} = fg \Rightarrow \mathrm{id} = \partial_{z_i} f \cdot g + f\,\partial_{z_i} g$ $\varphi_0 = \partial_{z_i} f \quad \varphi_1 = \partial_{z_i} g$
$MF(\mathbb{C}^n, w)$ $dw \neq 0$ no critical points	$P_0 \underset{g}{\overset{f}{\rightleftarrows}} P_1$	Suppose I find v a hol. vect. field s.t $v(w) = 1 = dw(v)$ then $w \cdot \mathrm{id} = fg \Rightarrow \mathrm{id} = v(f)g + f\,v(g)$ $\varphi_0 = v(f) \quad \varphi_1 = v(g)$ I couldn't find a way to construct v, but it's probably possible on $\mathbb{C}^n$...

Moral of the story:

- defining morphisms up to homotopy lets us "cross our eyes" and see many more objects as isomorphic
- The MF category is basically trivial w/o critical points

Part 2: First Interesting MF: $W = z_1^2 + \dots + z_n^2$, $X = \mathbb{C}^n$

~ a history lesson ~

in 1920s, Dirac needed a first-order operator $\not{D}$ such that $\not{D}^2 = \Delta = \partial_{x_1}^2 + \dots + \partial_{x_n}^2$. This is impossible on scalar functions. Dirac's idea was to use matrices. Define a "spinor bundle" $\begin{array}{c} S \times \mathbb{R}^n \\ \downarrow \\ \mathbb{R}^n \end{array}$ & $\not{D} \in TM \otimes End(S)$ s.t $\not{D}^2 = \Delta \otimes id_S$

replacing $\partial_{x_i} \mapsto x_i$. $\not{D}$ becomes a matrix of linear functions $\not{z}$, s.t $\not{z}^2 = (x_1^2 + \dots + x_n^2)\, id_S$

This was the first matrix factorization. Following Dirac's lead, we use

~ Clifford algebras ~

Definition The complex clifford algebra on a vector space V w/ Quadratic form Q, denoted $Cl(V,Q)$ is the tensor algebra on V w/ relations $v \otimes v = Q(v,v) \cdot 1$

Up to isomorphism, $Cl(V,Q) \simeq Cl_{\underset{\dim V}{n}} = \dfrac{\langle e_1, \dots, e_n \rangle}{\langle e_i e_j + e_j e_i = \delta_{ij} \rangle}$

as a vector space, $Cl(V,Q) \simeq \Lambda^\bullet V$

as an algebra, the product is deformed by Q, breaking $\mathbb{Z}$-grading to $\mathbb{Z}_2$-grading

$Cl^0(V) \simeq \bigoplus_k \Lambda^{2k} V \qquad Cl^1(V) \simeq \bigoplus_k \Lambda^{2k+1} V$ $\quad$ $\mathbb{Z}_2$-graded algebra

$Cl(V)$ is uniquely designed to have many square roots of 1

for any representation $c: Cl(V) \to End(M)$, $c(1) = id$. if $v^2 = 1$ in $Cl(V,Q)$, then $c(v)^2 = id$. These square roots of the identity gives a matrix factorization

Thm: every $\mathbb{Z}_2$-graded module $M^\bullet$ of $Cl(V,Q)$ defines an object in $MF(V,Q)$

Pf: Identify M^0, M^1 with trivial vector bundles $\begin{array}{c} M^0 \times V \\ \downarrow \\ V \end{array}$, $\begin{array}{c} M^1 \times V \\ \downarrow \\ V \end{array}$

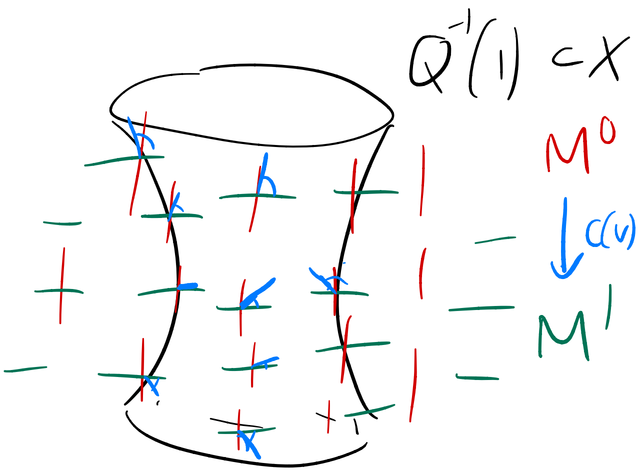
Define matrix factorization $M^0 \underset{c(z)}{\overset{c(z) \leftarrow \text{degree } 1}{\rightleftarrows}} M^1$ as pointwise clifford multiplication by the base point $z \in V$

for $m(z) \in H^0(M^\bullet)$, $c(z)\big(c(z) \cdot m(z)\big) = c(z^2)\, m(z) = Q(z,z)\, m(z) \Rightarrow c(z)^2 = Q \cdot id_M$ ∎

Remark: in coordinates, $c(z)m = \sum z_i\, c(e_i)\, m$ for a basis e_i. This is the fourier transform of the dirac operator $\not{D} m = \sum c(e_i) \frac{\partial}{\partial z_i} m$

~topology from that MF~

the matrix factorization $M^0 \underset{c(v)}{\overset{c(v)}{\rightleftarrows}} M^1$ gives two (trivial) vector bundles on X.
But, along $Q^{-1}(1)$, the maps $c(v)$ must be an isomorphism: after all, they square to the identity. So, along $Q^{-1}(1)$, we get a (nontrivial) isomorphism between M^0 & M^1. We can extract topology from this.
Eventually, a class in $H^\bullet(X, w^{-1}(1))$



Topological K-Theory

see Lawson-Michelson spin geometry, §1.9

Define $K^0(X)$ of a manifold X by the group with elements

$[V] - [W]$ topological complex vector bundles

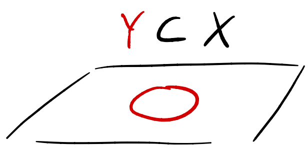
and relations $[V \oplus W] = [V] + [W]$

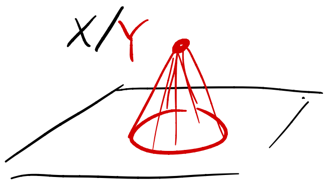
K theory can be extended to a generalized cohomology theory

define $K^{-n}(X) = K(\Sigma^n X)$ n-fold suspension

relative K-groups $K(X,Y) = \tilde{K}(X/Y)$

where X/Y is a pointed space w/ point $[Y]$, &
$\tilde{K}((X,\cdot)) = \ker r: K(X) \to K(\cdot)$
"K-theory up to trivial bundles"
or "K-theory but I don't care about the rank of the vector bundle"





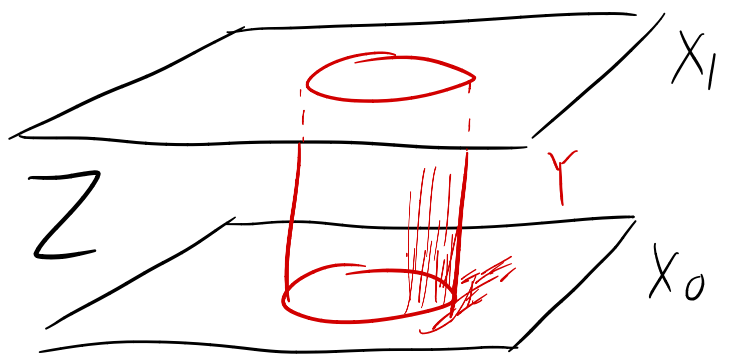
I can extract ordinary cohomology from K-theory using the chern charecter:

ch: $K(X) \to H^\bullet(X, \mathbb{Z})$
$K(X,Y) \to H^\bullet(X, Y, \mathbb{Z})$ is a ring homomorphism

Given $V_0 \xrightarrow{\sigma} V_1$ V_0, V_1 v.bs on X, $\sigma: V_0|_Y \to V_1|_Y$ an iso along Y, I can produce a class in $K(X,Y)$

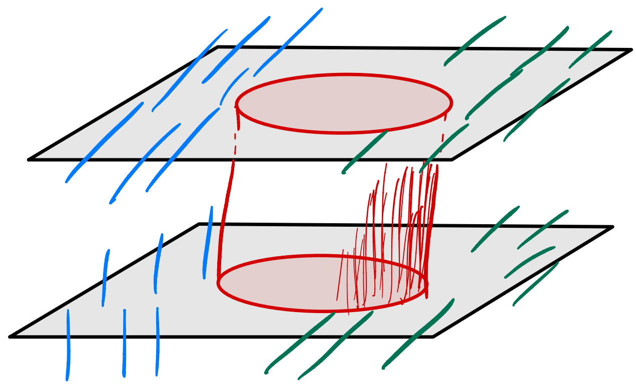
differences bundles as $K(X,Y)$ classes

1. create "Quilt" Z from 2 layers of X, sewed along Y

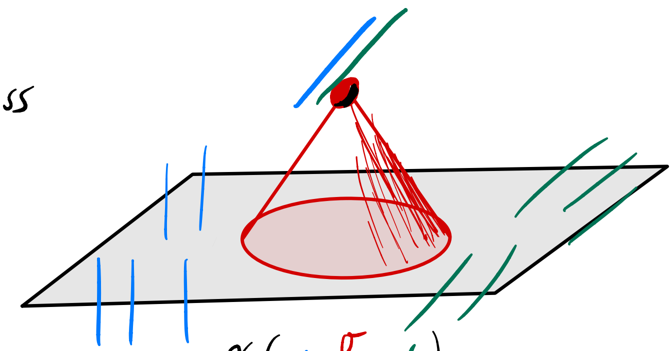


2. use data $V_0 \xrightarrow{\sigma} V_1$ to create vector bundle W, which equals V_0 on lower layer, V_1 on upper layer, & is glued by σ.

3. subtract off the v.b W_1, which equals V_1 on both layers

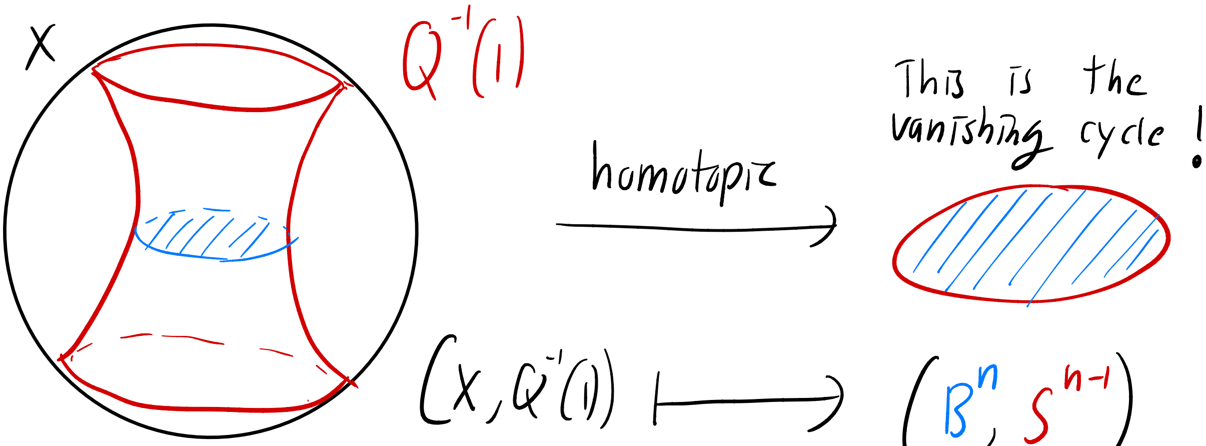


4. restricted to top layer X_1,
$[W]-[W_1] = [V_1]-[V_1] = 0$
$\Rightarrow$ class $[W]-[W_1]$ pulls back from class in $K(Z,X_1) \simeq K(X,Y)$
call this $\chi(V_0 \xrightarrow{\sigma} V_1)$



roughly, $\chi(V_0 \xrightarrow{\sigma} V_1) = [V_0]-[V_1]$ on $K(X)$, pulled back to $K(X,Y)$ using σ to force $[V_0]-[V_1]=0$ along Y

Lets apply this construction to $M^0 \xrightarrow{\sigma(v)} M^1$



$$(X, Q^{-1}(1)) \longmapsto (B^n, S^{n-1})$$

Clifford module $M^\bullet$ $\rightsquigarrow$ matrix factorization $M^0 \underset{\longleftarrow}{\overset{c(v)}{\longrightarrow}} M^1$ $\rightsquigarrow$ K-theory class $K(B^n, S^{n-1}) \simeq \tilde{K}(S^n)$
$\simeq \tilde{K}^{-n}(*)$

Thm: (Atiyah-Bott-shapiro '64)

let $\mathcal{M}(\mathbb{C}l_n)$ denote the ring of modules of $\mathbb{C}l_n$
we have an inclusion $i: \mathbb{C}l_n \hookrightarrow \mathbb{C}l_{n+1} \Rightarrow$ restriction $i^* \mathcal{M}(\mathbb{C}l_{n+1}) \to \mathcal{M}(\mathbb{C}l_n)$
The above construction is a ring homomorphism $\mathcal{M}(\mathbb{C}l_n) \to \tilde{K}(S^n)$
with kernel $i^* \mathcal{M}(\mathbb{C}l_{n+1})$

i.e $\tilde{K}(S^n) = \mathcal{M}(\mathbb{C}l_n) / i^* \mathcal{M}(\mathbb{C}l_{n+1})$

We've seen that each object in $MF(V, Q)$ gives a K-theory class on the sphere. But we can do better.

Thm: let $CLMOD(V,Q)$ denote the category of modules of $Cl(V,Q)$
The categories $MF(V,Q)$ & $CLMOD(V,Q)$ are equivelent

Original ref: R.O. Buchweitz, D. Eisenbud, J. Herzog, Cohen-Macaulay modules on quadrics.

exposition in this language: Clifford Algebras and Matrix Factorizations, by Jos´e Bertin

Physical Derivation: D-BRANES IN LANDAU-GINZBURG MODELS AND ALGEBRAIC GEOMETRY by kapustin-li section 7

https://arxiv.org/abs/hep-th/0210296

Complex clifford algebras & modules have a simple structure theory

consider the category of clifford modules $\mathbb{C}LMOD(n)$

Fact: $\mathbb{C}\ell_n$ is <u>semisimple</u>. That is, every module M splits into a direct sum of irriducible representations of $\mathbb{C}\ell_n$.

- for odd n, there are two irreps
- for even n there is a unique irrep

n	$\mathbb{C}\ell_n$	ring of modules $M(\mathbb{C}\ell_n)$
0	$\mathbb{C}$	$\mathbb{Z}$
1	$\mathbb{C}\oplus\mathbb{C}$	$\mathbb{Z}\oplus\mathbb{Z}$
2	$Mat_2(\mathbb{C})$	$\mathbb{Z}$
3	$Mat_2(\mathbb{C})\oplus Mat_2(\mathbb{C})$	$\mathbb{Z}\oplus\mathbb{Z}$
4	$Mat_4(\mathbb{C})$	$\mathbb{Z}$
$\vdots$		

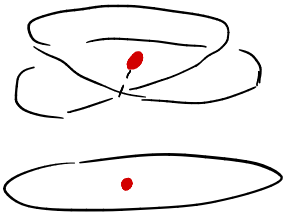
Thm (Clifford algebra periodicity) $\mathbb{C}\ell'_{n+2} \simeq Mat_2(\mathbb{C}\ell_n)$.

Cor: $\mathbb{C}LMOD(n+2) \simeq \mathbb{C}LMOD(n)$ algebraic manifestation of bott periodicity

from any object in $MF(V,Q)$, we can build a clifford module. Let's start by computing the endomorphisms of one object:

Example: $X = \mathbb{C}$, $W = z^2$ $\quad \mathbb{C} \underset{z}{\overset{z}{\rightleftarrows}} \mathbb{C}$, $\quad fg = z^2$

graph of W



sheaf $\delta_0 \in Sing(W^{-1}(0))$

$\delta_0 = \operatorname{Ker}\ \mathbb{C} \xrightarrow{z} \mathbb{C}$

"periodic resolution"

in physics language, this object is a "D_0 brane centered at the origin"

$$Mor^0\left(\mathbb{C} \underset{z}{\overset{z}{\rightleftarrows}} \mathbb{C}\right)$$
$$\|$$
$$\mathbb{C}[z]/z = \mathbb{C}$$

$$\begin{tikzcd}
\mathbb{C} \arrow[r, shift left, "z"] \arrow[d, "\phi_0"'] \arrow[dr, "\varphi_0"'] & \mathbb{C} \arrow[l, shift left, "z"] \arrow[d, "\phi_1"] \arrow[dl, "\varphi_1"] \\
\mathbb{C} \arrow[r, shift left, "z"] & \mathbb{C} \arrow[l, shift left, "z"]
\end{tikzcd}$$

$\phi_0, \phi_1 \in \Gamma(\mathcal{O}_X)$

$z\phi_0 = z\phi_1 \Rightarrow \phi_0 = \phi_1$

trivial morphisms: $\partial\varphi = z\varphi_0 + z\varphi_1 = z(\varphi_0+\varphi_1)$

ϕ is trivial if $\phi = z\varphi$, $\varphi \in \Gamma(\mathcal{O}_X)$

$\mathrm{Mor}^1(\mathbb{C} \underset{z}{\overset{z}{\rightleftarrows}} \mathbb{C})$: $\quad \varphi_0, \varphi_1 \in \Gamma(\mathcal{O}_X) \quad \varphi_0 = \varphi_1$

$\quad \| \qquad$ trivial morphisms: $\partial\phi = z(\phi_0 - \phi_1)$

$\mathbb{C}[z]/z = \mathbb{C}$

All together: $\mathrm{Mor}^\bullet(\mathbb{C} \underset{z}{\overset{z}{\rightleftarrows}} \mathbb{C}) = \langle 1, \theta \rangle$ w/ 1 in degree 0, θ in degree 1, $\quad \theta^2 = 1$

$= \mathbb{C}\ell_1$

Now we extend this computation to objects in $MF(\mathbb{C}^n, z_1^2 + \dots + z_n^2)$

$$S^n = (\mathbb{C} \underset{z}{\overset{z}{\rightleftarrows}} \mathbb{C})^{\boxtimes n} \in MF(\mathbb{C}^n, z_1^2 + \dots + z_n^2)$$

Under $MF(V,Q) \simeq \mathrm{Sing}(Q^{-1}(0))$, S corresponds to the skyscraper sheaf at 0.

$$\mathrm{End}^\bullet(S^n) = (\mathrm{End}^\bullet(S))^{\otimes n} = (\mathbb{C}\ell_1)^{\otimes n} \overset{\text{Fact}}{\simeq} \mathbb{C}\ell_n$$

Prop: Every object $P \in [MF(\mathbb{C}^n, z_1^2 + \dots + z_n^2)]$ defines a module of $\mathbb{C}\ell_n^{\mathbb{C}}$

define $\mathbb{Z}_2$ graded vector space $M^\bullet = \mathrm{Mor}^\bullet(S^n, P)$

then $\mathbb{C}\ell_n \simeq \mathrm{End}^\bullet(S^n)$ acts on $\mathrm{Mor}^\bullet(S^n, P)$ by composition

$$\begin{tikzcd} S^n \arrow[r, "\mathrm{End}(S^n)"] \arrow[rr, bend right, "\mathrm{Mor}(S^n, P)"'] & S^n \arrow[r, "\mathrm{Mor}(S^n, P)"] & P \end{tikzcd}$$

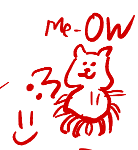
to summarize:

$$\begin{tikzcd} CLMOD(V,Q) \arrow[r, bend left, "\text{Dirac operator}"] & MF(V,Q) \arrow[l, bend left, "\mathrm{Mor\ w/}\ S^n"] \end{tikzcd}$$

Thm: these two operations are inverse functors, & define an equivalence of categories (see Kapustin-Li)

Cor $\quad CLMOD(n) \simeq CLMOD(n+2) \iff MF(V,Q) \simeq MF(V \oplus \mathbb{C}^2, Q + x^2 + y^2)$

Applying atiyah-Bott-Shapiro, this implies $\tilde{K}(S^n) \simeq \tilde{K}(S^{n+2})$ Bott Periodicity



Warning: ABS used Bott periodicity to prove their isomorphism. instead of proving bott periodicity, we are seeing it manifest in MF periodicity in MFs holds in more generality than Quadradics...

Thm (Kronner Periodicity): $MF(X,w) \simeq MF(X\times\mathbb{C}^2, W+x^2+y^2)$

$$P_0 \rightleftarrows P_1 \longmapsto P_0 \rightleftarrows P_1 \otimes (\mathbb{C} \overset{z}{\underset{z}{\rightleftarrows}} \mathbb{C})^{\boxtimes 2}$$

how does this manifest topologically? for $P_0 \overset{f}{\underset{g}{\rightleftarrows}} P_1 \in MF(X,w)$,

on $w^{-1}(1)$ $f: P_0 \to P_1$ is an isomorphism. applying the construction locally (in a ball B around an isolated singularity of w),

$$\chi(P_0 \xrightarrow{f} P_1) \in K(B, w^{-1}(1)) \simeq K(B, X_s) \quad \text{nearby fiber} \quad X_s \simeq F_w \ \text{milnor fiber}$$

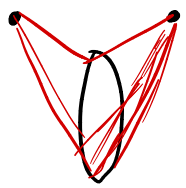
compare this to the object in $MF(X\times\mathbb{C}^2, W+x^2+y^2)$ related by Kronner periodicity

$$\chi((P_0 \xrightarrow{f} P_1)\boxtimes(\mathbb{C}\rightleftarrows\mathbb{C})^{\boxtimes 2}) \in K(B, (W+x^2+y^2)^{-1}(1)) \simeq K(B, F_{W\boxplus x^2\boxplus y^2})$$

Thom-Sebastiani: $F_{W\boxplus x^2} = F_w \overset{\text{join}}{*} \underbrace{F_{x^2}}_{2 \text{ points}}$

so $F_{W\boxplus x^2 \boxplus y^2} = S^2 F_w$

& $S^2 B^n = B^{n+2}$



$$F_{x^2} * F_w = S F_w$$

join w/ 2 pts = Suspension!

so $K(B^{n+2}, F_{W\boxplus x^2+y^2}) = K(S^2 B^n, S^2 F_w) \overset{\sim}{=} K(B^n, F_w)$

Bott Periodicity

Thm (Brown) Kronner & Bott periodicity are compatible

$$\begin{array}{ccc} MF(X,w) & \xrightarrow{ABS} & K(B^n, F_w) \\ \simeq \downarrow \text{Kronner periodicity} & \circlearrowright & \downarrow \text{Bott periodicity} \\ MF(X\times\mathbb{C}^2, W+x^2+y^2) & \xrightarrow{ABS} & K(B^{n+2}, F_{W\boxplus x^2+y^2}) \end{array}$$

Moral: periodicity of $MF(X,w)$ is categorified Bott periodicity