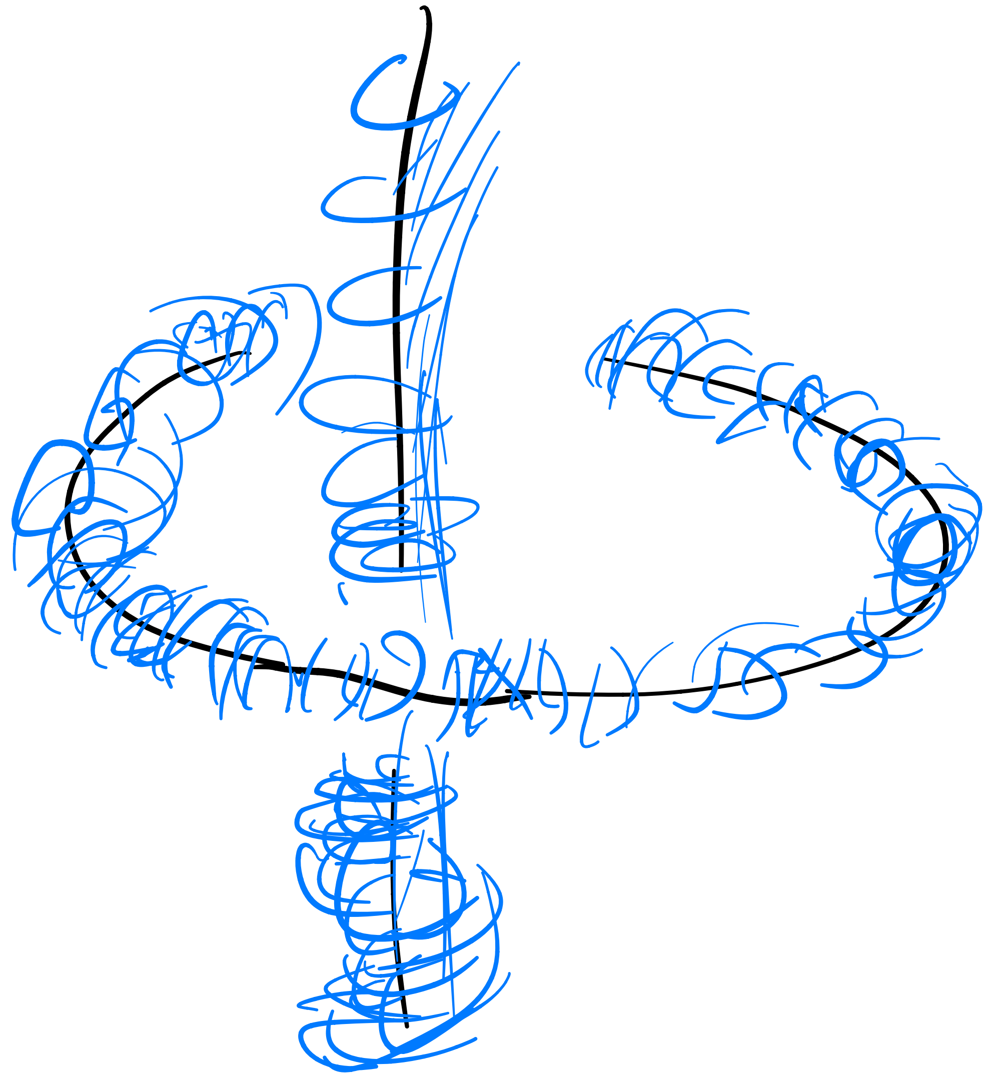
Gauge Theory for Knots

~ or ~

3 prespectives on linking #s



starting in...

1867: Kelvin's vortex theory of knots

discuss Kelvin's vortex theory, taits conjectures & how it was solved w/ the jones polynamial

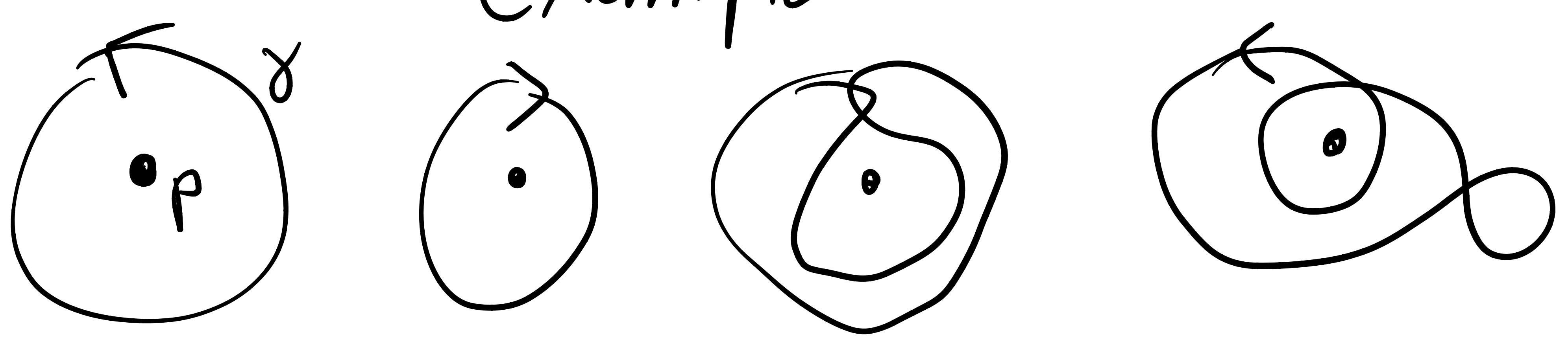
$\mathbb{R}^2$ Winding numbers

γ curve
p point

- $W_p(\gamma)$ counts # of times γ winds around p
- Want a 1-form A_p s.t. $W_p(\gamma) = \int_\gamma A_p$

examples:

p, γ:



$W_p(\gamma)$: $+1$ $\quad -1 \quad$ 0 $\quad +1$

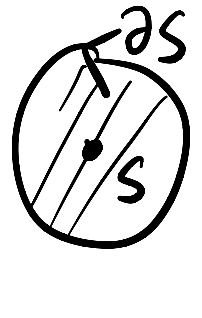
(1)

$0 = W_p(\gamma) - W_p(\gamma') = \int_\gamma A_p - \int_{\gamma'} A_p$

$0 = \int_\Sigma dA_p \quad \forall\, \Sigma$ not containing p

$\Rightarrow dA_p = 0$ away from p!

(2) $1 = W_p(\partial S) = \int_{\partial S} A_p = \int_S dA_p$

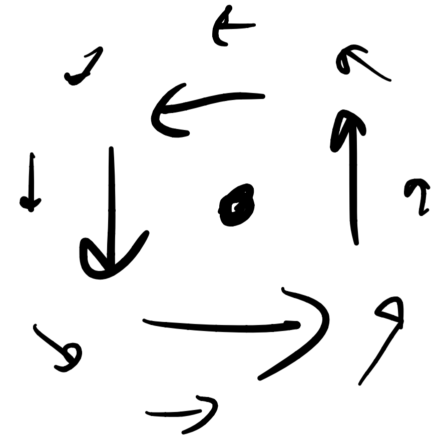


together with (1)...

$$\boxed{dA_p = \delta_p}$$

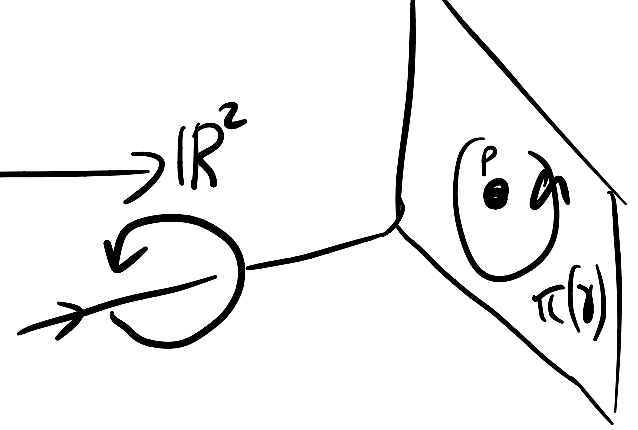
Solution:

$A =$



vortex around p

Say verbally: $\pi: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$



$L(\gamma, \pi^{-1}(p)) = w_p(\pi(\gamma))$

$\mathbb{R}^3$ Linking #s

- $L(\gamma,\gamma')$ counts # γ winds about γ'

winding # is 2D shadow of linking #

choose A_γ w/ $dA_\gamma = \delta_\gamma \star d\gamma$, $\quad d\gamma = \gamma'/|\gamma'|\,dx$

Curvature concentrated on γ

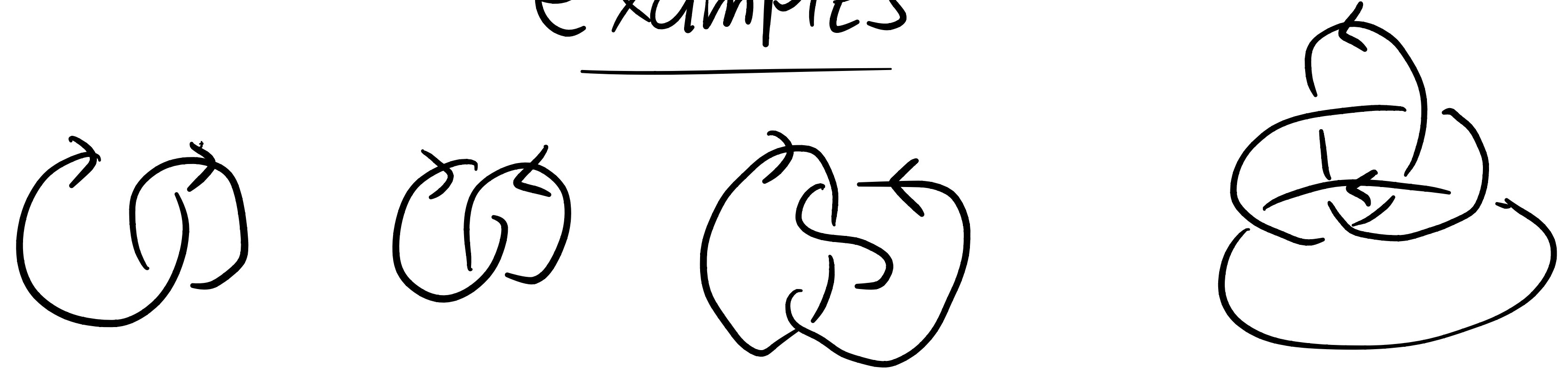
$\int_{\gamma'} A_\gamma = \int_{\Sigma} \delta_\gamma \star \dot{\gamma}\, dx =$ intersection # of Σ w/ γ

$$\boxed{L(\gamma,\gamma') := \int_{\gamma'} A_\gamma}$$

$$= \int_{\mathbb{R}^3} \delta_{\gamma'} \langle A_\gamma, d\gamma' \rangle = \int_{\mathbb{R}^3} A_\gamma \wedge (\delta_{\gamma'} \star d\gamma')$$

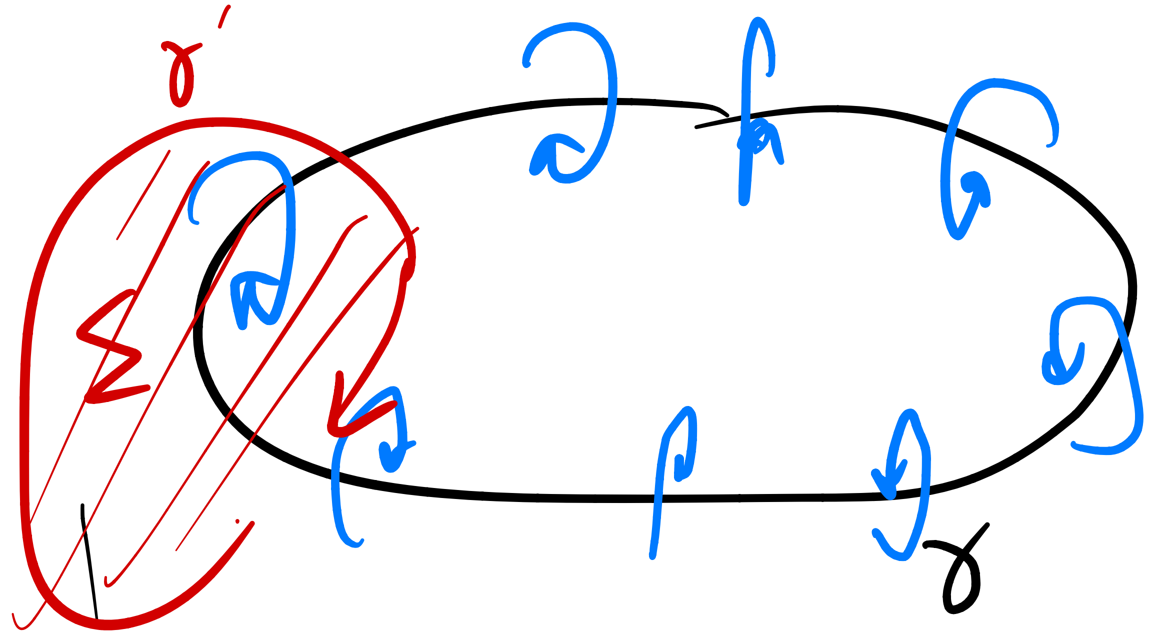
$$= \int_{\mathbb{R}^3} A_\gamma \wedge dA_\gamma$$

examples



γ, γ'				
$L(\gamma,\gamma')$	-1	$+1$	$+2$	0

solution: $A =$ knotted vortex around γ



mention gauss linking integral

$$= \int_{\gamma'} \int_{\gamma} \frac{r_1 - r_2}{|r_1 - r_2|^3}\, dr_1 \times dr_2$$

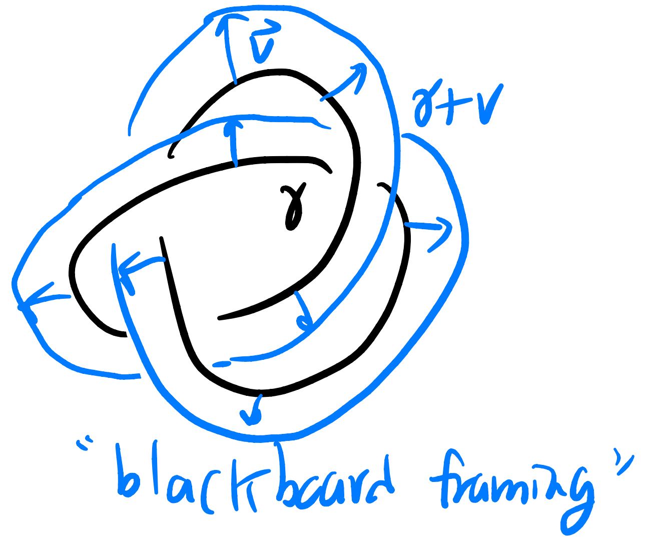
Gauss's linking integral

Pull out ribbon & show how writhe comes from straightening out ribbon

Self-Linking # (writhe)

$L(\gamma,\gamma) = \int_\gamma A_\gamma = \infty$ $\therefore$ need to regularize

(1) introduce framing: vector field $v \perp \gamma'$

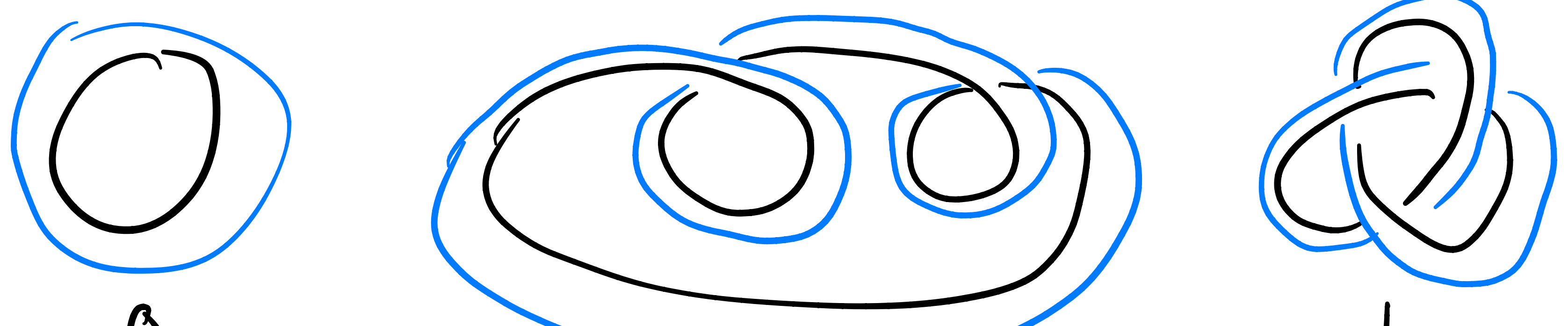


$L(\gamma,\gamma) := L(\gamma, \gamma+v)$

self-linking #, or writhe, of γ

Examples (Blackboard framing)

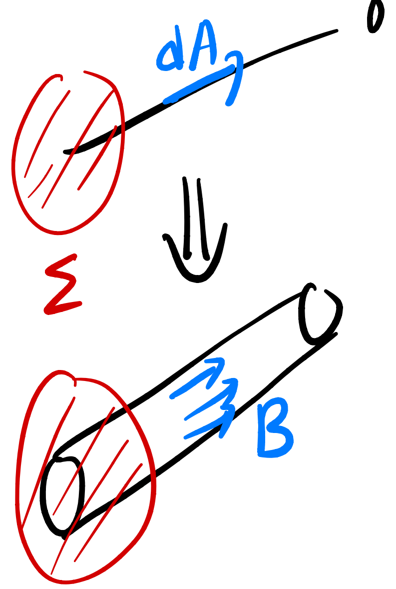
γ, v



$L(\gamma,\gamma)$ 0 2 1

(2) smooth out A_γ: replace $dA_\gamma = \delta_\gamma \star \hat{\gamma} \cdot dx$ with $dA_\gamma = B$ s.t.

- B supported on tube around γ
- $dB = 0$
- for transverse section Σ, $\int_\Sigma B = \int_\Sigma \delta_\gamma \star \hat{\gamma} \cdot dx$



Thm:

$L(\gamma,\gamma) = \int_{\mathbb{R}^3} A_\gamma \wedge B = \int_{\mathbb{R}^3} A_\gamma \wedge dA_\gamma$

Say how (2) uses a secret framing

Say how this is not topologically natural!

$x^T A x = Bx \rightarrow x^T A = B,\ x_0 = A^{-1} B^T$

$x_0^T A x_0 = B^T A^T B$

These are gaussian integrals: in finite dimensions,

$$\int_{\mathbb{R}^n} e^{i\langle x, Ax\rangle} = \frac{c}{\sqrt{\det A}}$$ ← normalization

$$\int_{\mathbb{R}^n} e^{i(\langle x, Ax\rangle - \langle x, B\rangle)} = \frac{c}{\sqrt{\det A}}\, e^{i\langle x_0, Ax_0\rangle}$$ where $Ax_0 = B$

formally applying, we get

$$\boxed{\frac{Z(S^3, \gamma)}{Z(S^3)} = e^{\frac{2\pi i}{k}\int_{S^3} A_\gamma \wedge dA_\gamma} = e^{\frac{2\pi i}{k} L(\gamma,\gamma)}}$$ computes writhe mod k

if DA exists & metric invariant this formula is:

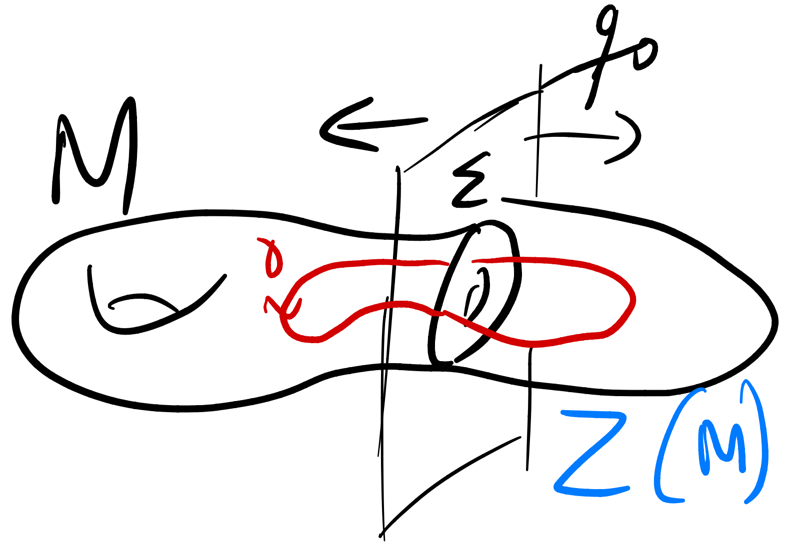
- manifest topological
- manifest 3D

$Z(M^3, \gamma) =$ self-linking # on arbitrary 3-mfld! $Z(M^3) =$ Reidmeister torsion

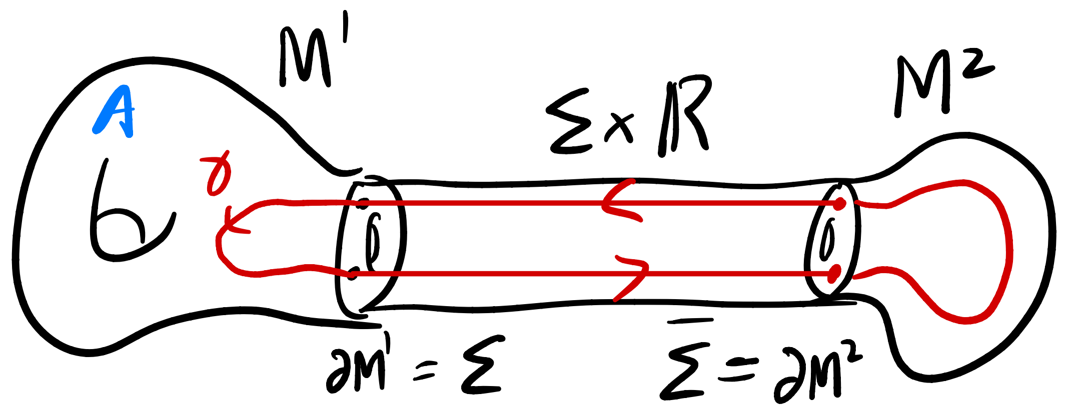
analogy: de rahm cohom. | simplical cohom. | cellular cohom

gauss linking # | path integral approach | TQFT

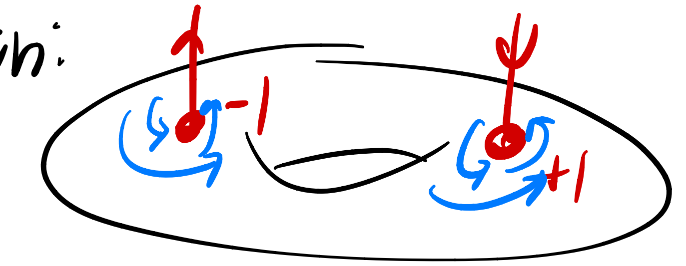
TQFT Structure



$\Rightarrow$



crossection:



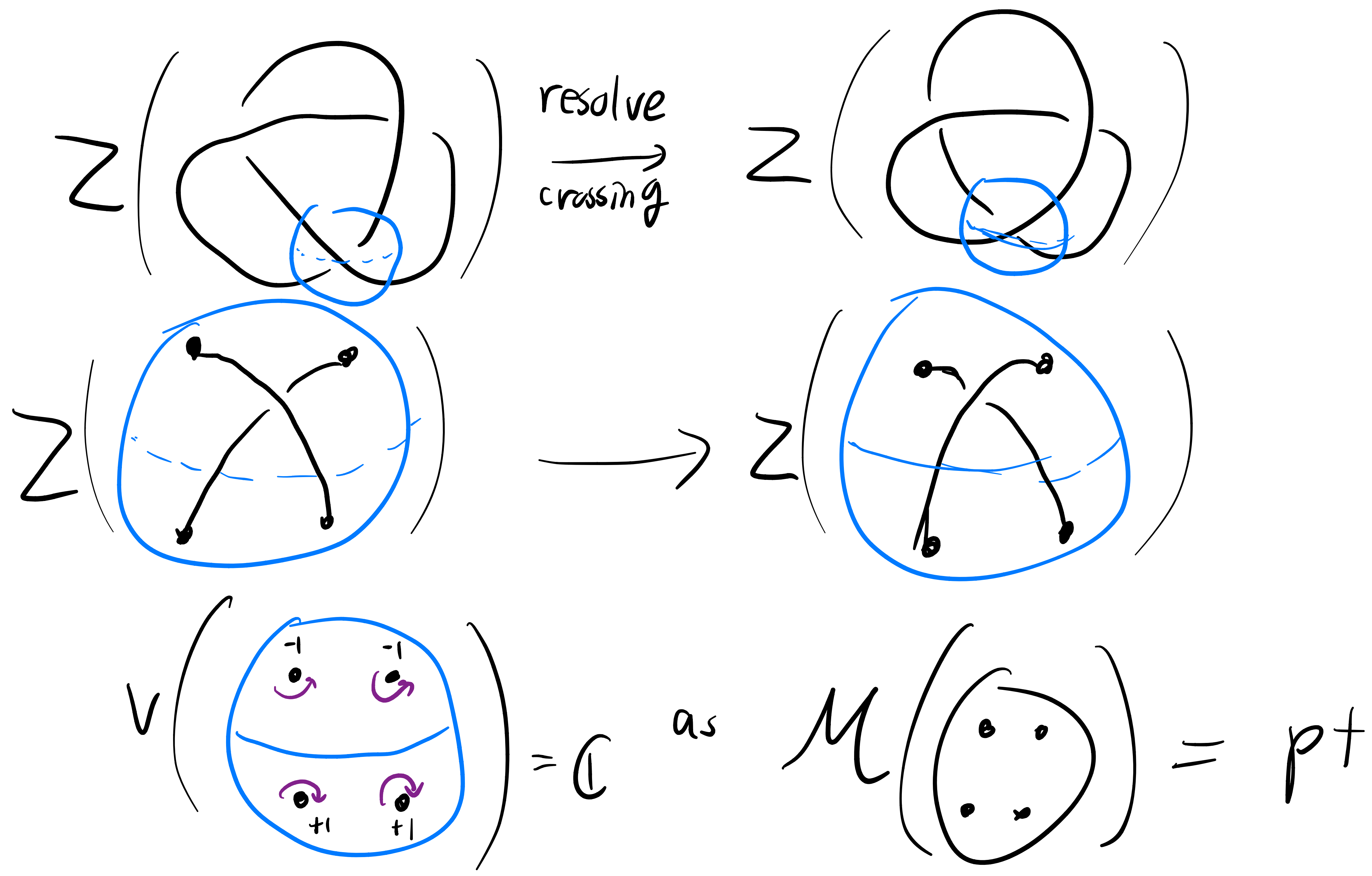
Σ space of 2D vortices $\mathcal{M}$

$Z(M^1; \Sigma)$ gives complex function on $\mathcal{M}$

$\Rightarrow$ vector in vector space $V(\Sigma)$

$$Z(M) = \langle Z(M^1;\Sigma), Z(M^2;\Sigma)\rangle_{V(\Sigma)}$$

computationally powerful!



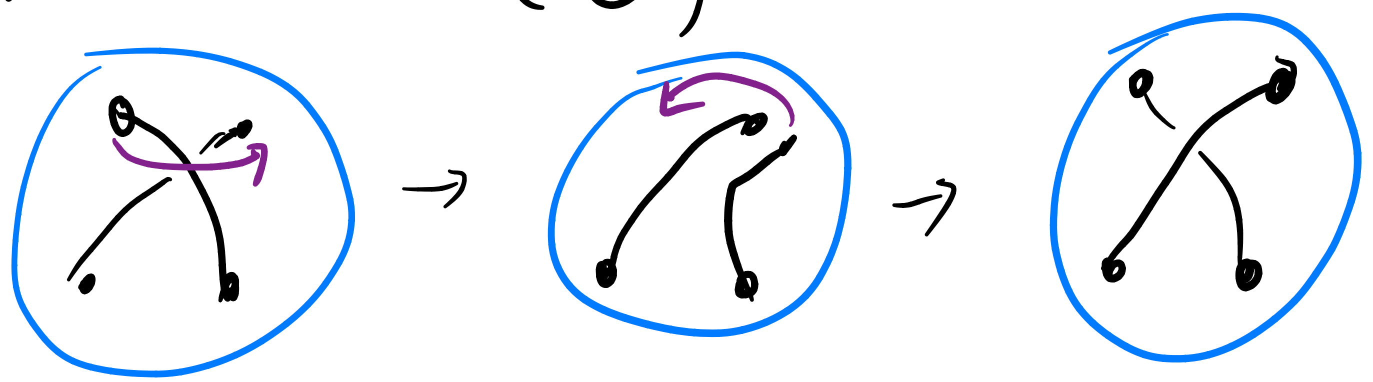
introduce holomorphic structure, making S^2 into $\mathbb{CP}^1$

$Z(\times) = \alpha Z(\times)$. what is α?

very hard to calculate $Z(\times)$.

Shortcut!

Use monodromy:

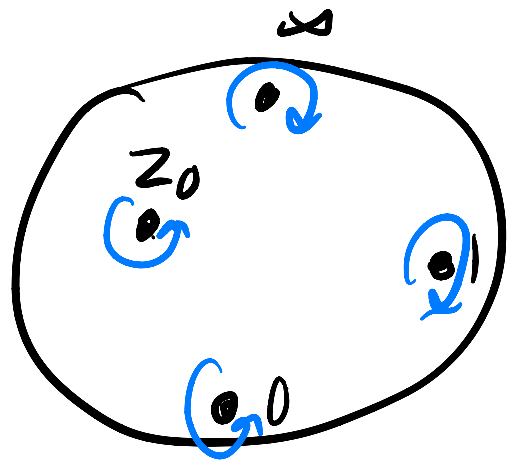


make things holomorphic: $S^2 \to \mathbb{CP}^1$

solutions are $\lambda \frac{z(z-z_0)}{(z-1)} dz$

Poles @ $1, \infty$

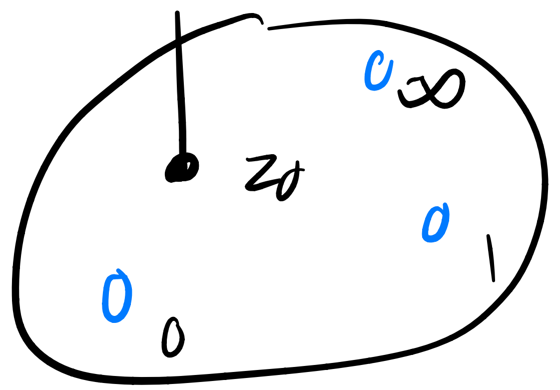
zeros @ $0, z_0$



Q: how does λ change when z_0 goes around 0?

get line bundle on $\mathbb{P}^1 \setminus \{0,1,\infty\}$

$\mathbb{C} = \langle \lambda \frac{z(z-z_0)}{z-1} dz \rangle$



Physics $\Rightarrow$ line bundle is flat

& monodromy around 0 is -1

$\Rightarrow Z(\times) = -Z(\times)$

$= Z(L) = \#$ over crossings $- \#$ undercrossings $=$ writhe(L)!

Talk Part 2

Topological Quantum field theories

& the Jones polynomial

Tell story about history of witten's involvement in the Jones polynomial

Flat connections: G Lie Group, $\mathfrak{g}$ lie algebra, Principle G-bundle

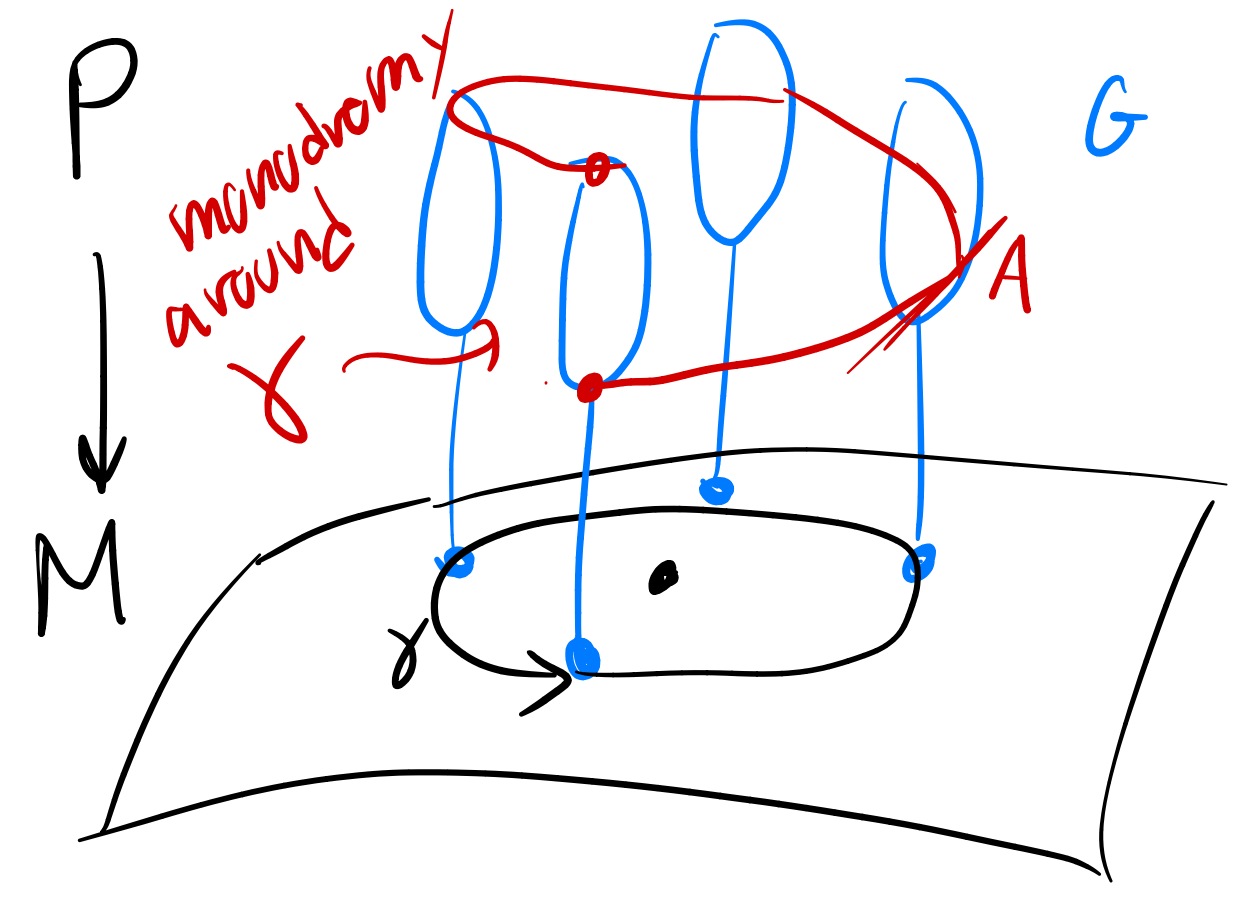
connection ∇_A defined by $A \in \mathcal{A}_G = \Omega^1(M, \mathfrak{g})$
space of connections

monodromy around γ is $\exp(2\pi i \int_\gamma A) \in G$

$\hookrightarrow$ curvature $F_A = dA + A \wedge A \in \Omega^2(M, \mathfrak{g})$

Example: $G = U(1)$, $\mathfrak{g} = \mathbb{R}$ $\quad A \in \Omega^1(M, \mathbb{R})$

$F_A = dA$



Chern-Simons theory

$$S_{CS}(A) = \int_M \mathrm{tr}\left(A \wedge dA + \tfrac{2}{3} A\wedge A \wedge A\right)$$

trace of $\mathfrak{g}$ in adjoint rep.

for loop γ, representation R of $\mathfrak{g}$,

$W_\gamma(A) = \mathrm{tr}_R \int_\gamma A$ monodromy of ∇_A

Classical: crit points of $S_{CS}(A) \Rightarrow F_A = dA + A\wedge A = 0$

crit points of $S_{CS}(A) + W_\gamma(A) \Rightarrow F_A \cong \delta_\gamma$ flat away from γ (vortex lines)

Example: $G = U(1)$, $S_{CS}(A) = \int_M A \wedge dA$

A_γ critical pt of $S_{CS}(A) + \int_\gamma A \Rightarrow F_{A_\gamma} = \delta_\gamma \star d\gamma$

Linking # $L(\gamma, \gamma') = \int_{\gamma'} A_\gamma$

Quantum:

$$Z(M, G, \gamma, R) = \int_{\mathcal{A}_G} e^{2\pi i\left(k S_{CS}(A) + W_\gamma(A)\right)} \mathcal{D}A$$

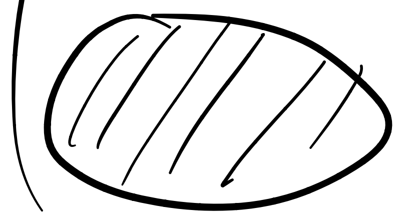
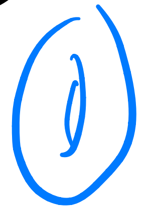
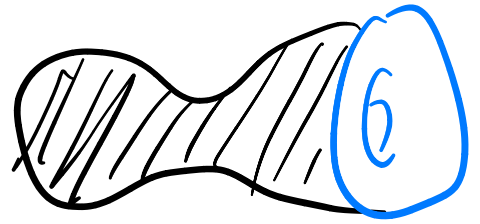
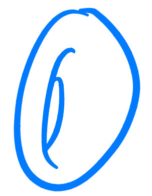
"Level" $k \in \mathbb{Z}$

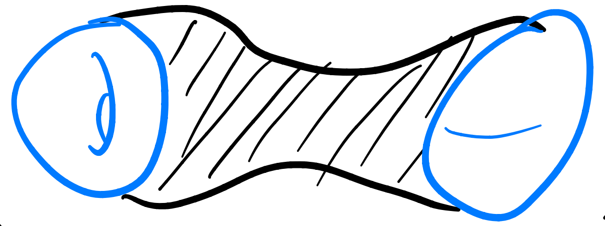
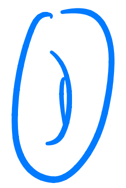
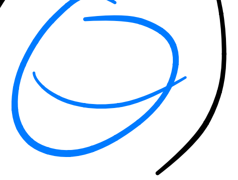
last time: $Z(S^3, U(1), \gamma, \mathbb{C}) = L(\gamma, \gamma)$

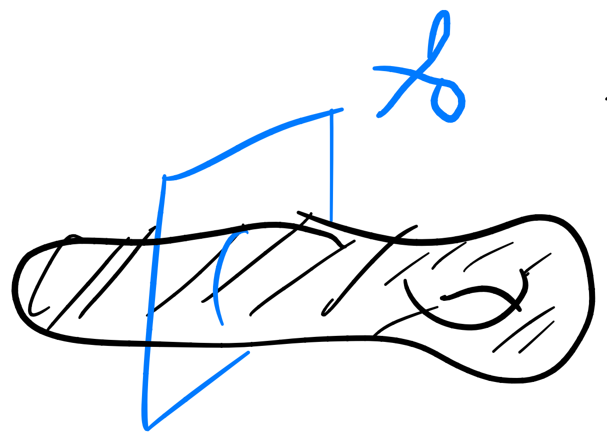
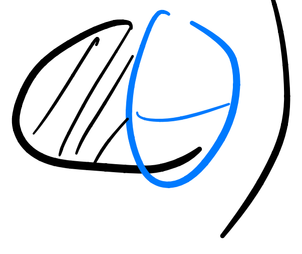
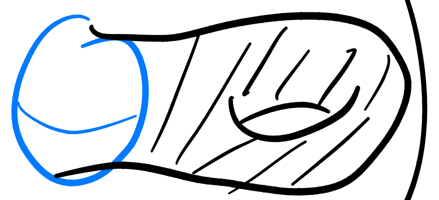
what is invariant for G nonabelian??

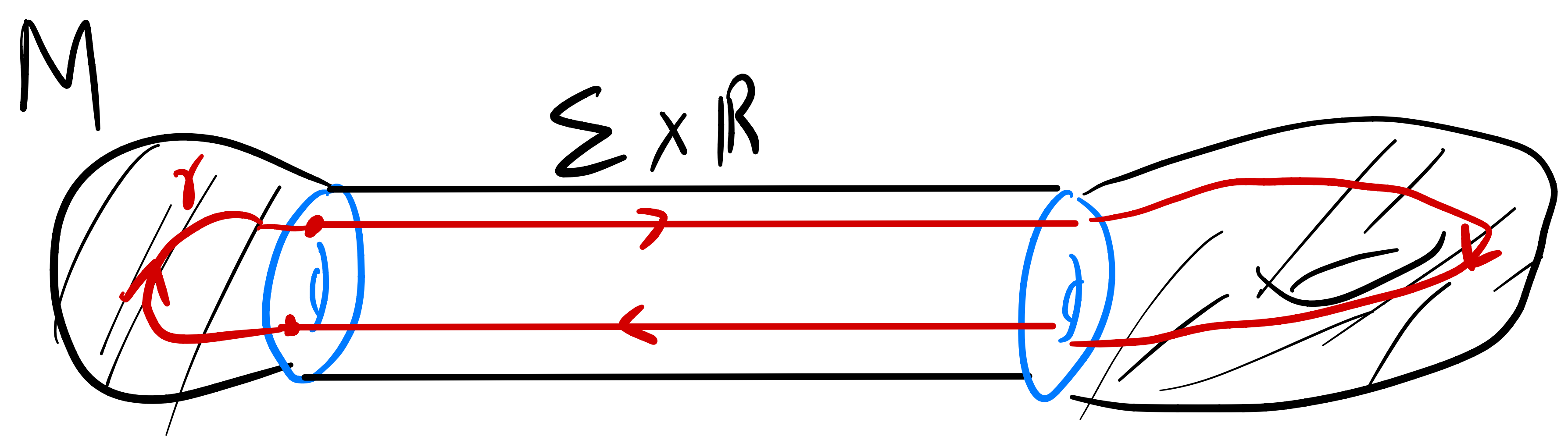
mention that we consider this integral because we expect it to localize around classical solutions

Axioms of Topological Quantum field theories (TQFT):

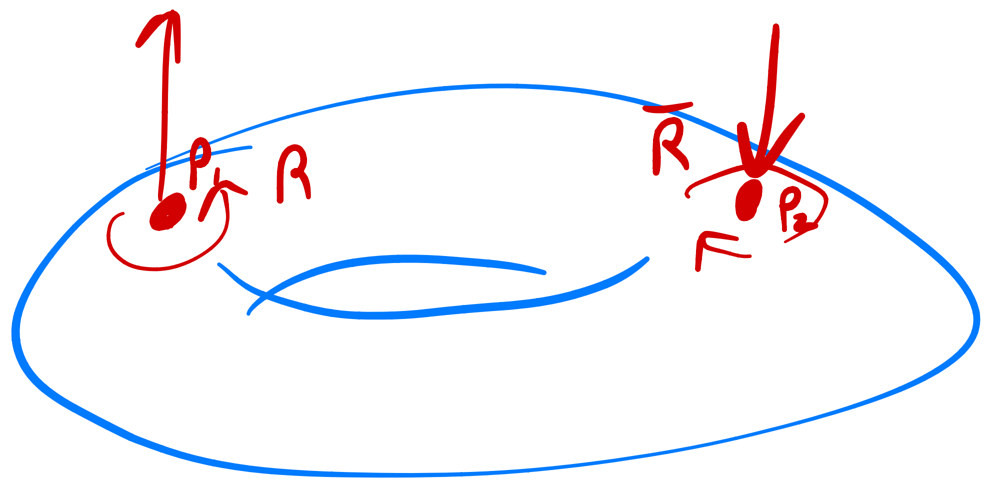
$Z\Big($ 3-mfld  $\Big) \in \mathbb{C}$ $\quad$ $Z\Big($ 2-mfld  $\Big)$ vector space $\quad$ $Z\Big($  $\Big) \in Z\Big($  $\Big)$

$Z\Big($  $\Big) : Z\Big($  $\Big) \to Z\Big($  $\Big)$ functor from 2-cobordisms to vect. spaces

for computations: $Z\Big($  $\Big) = \Big\langle Z\Big($  $\Big), Z\Big($  $\Big)\Big\rangle$



Chern-Simons Vector Space $\Longrightarrow$ Σ



Cassical solutions on $\Sigma \times \mathbb{R}$ w/ link γ, R

Flat connection on Σ w/ monodromy around marked points

moduli space $\mathcal{M}(\Sigma, G, \{P_i, R_i\})$

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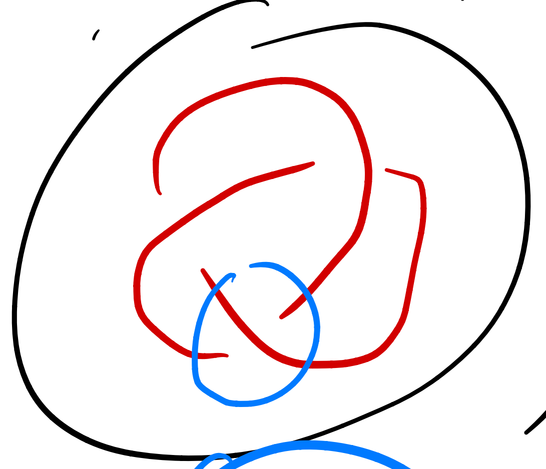
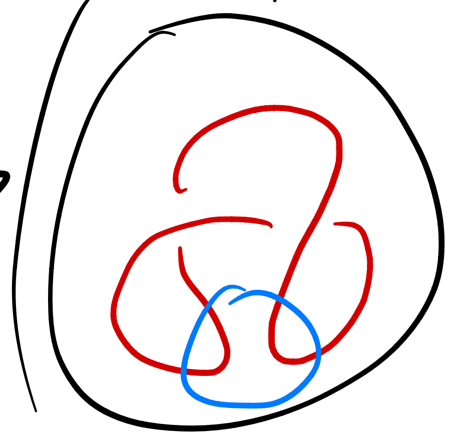
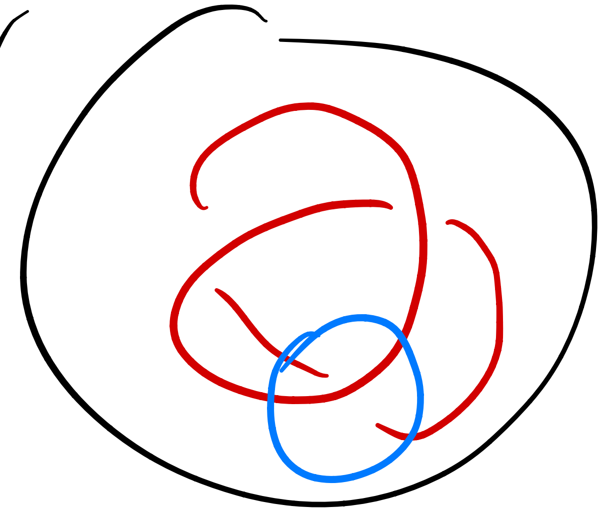
complex structure

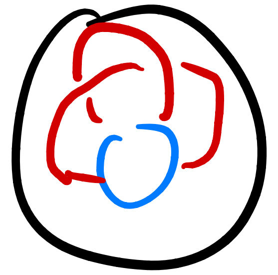
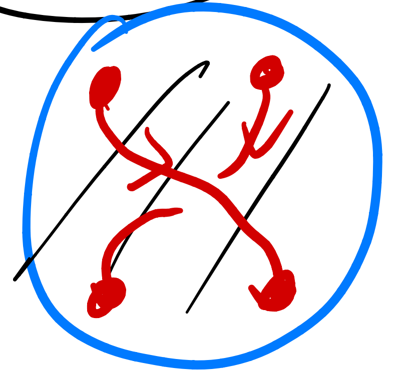
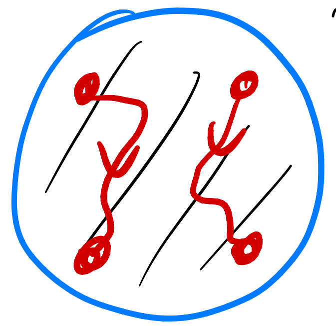
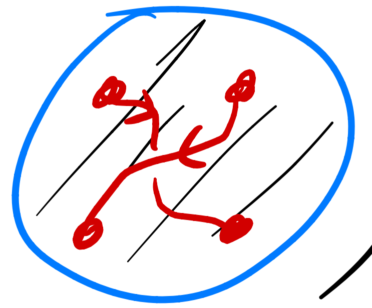
for Σ_σ Riemann surface, $\mathcal{M}(\Sigma_\sigma, \ldots)$ complex mfld w/ Line bundle L. $\boxed{Z(\Sigma_\sigma) = H^0(\mathcal{M}(\ldots), L)}$

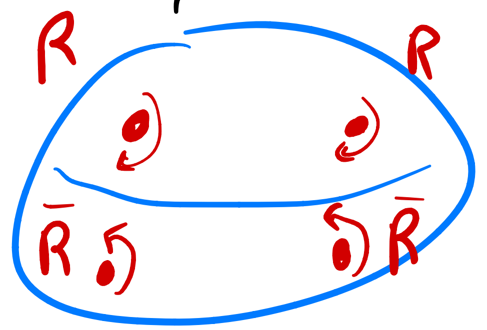
for topological invariance, $Z(\Sigma_\sigma)$ is a flat v.b over moduli space of Riemann surfaces $\mathcal{T}_\Sigma$

Skein Relations I

resolved crossings

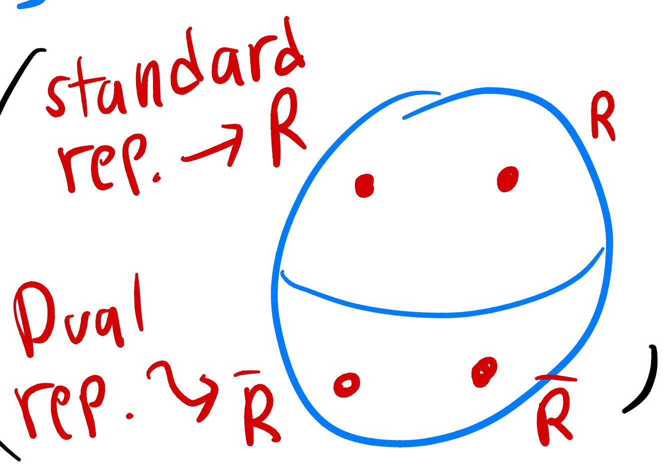
want α, β, γ s.t. $\alpha Z\Big($  $\Big) + \beta Z\Big($  $\Big) + \gamma Z\Big($  $\Big) = 0$

$= \Big\langle Z\Big($  $\Big), \alpha Z\Big($  $\Big) + \beta Z\Big($  $\Big) + \gamma Z\Big($  $\Big)\Big\rangle = 0$

$Z\Big($  $\Big)$

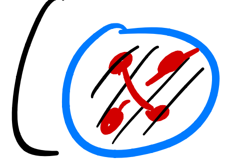
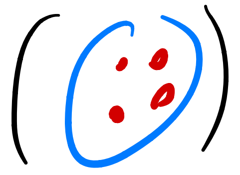
if there are α, β, γ which make this 0, get skein relations

Chern-simons on 4-holed sphere:

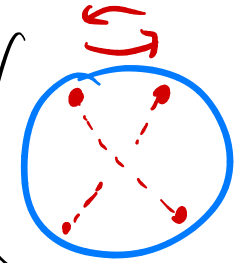
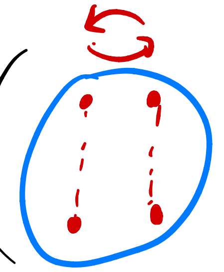
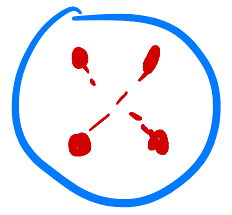
Geometric input: $Z\Big($ 

$, SU(2)\Big) \cong \mathbb{C}^2$

2 Dimensional ⇒ α, β, γ must exist!

issue: vector $Z($  $) \in Z($  $)$ very hard to find

trick: use monodromy!

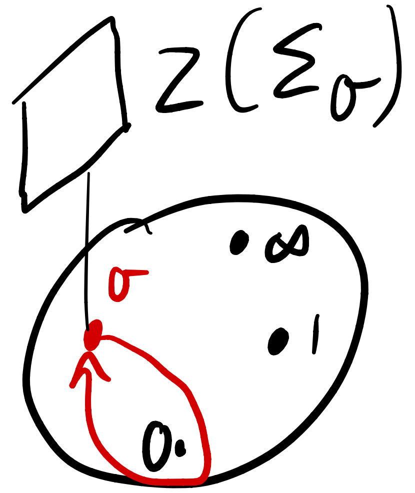
$Z($  $) \to Z($  $) \to Z($  $)$

$\Sigma = S^2 - \{4 \text{ pts}\} \longmapsto \mathbb{P}^1 - \{0, 1, \infty, z\}$

moduli space of complex structures $= \mathbb{P}^1 - \{0,1,\infty\}$

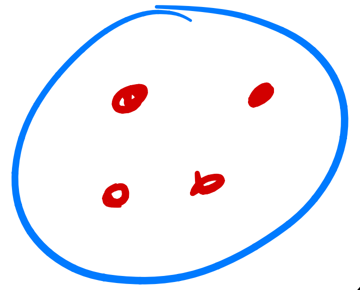
$Z(\Sigma_\bullet)$ flat, rank 2 v.b over $\mathcal{T}_\Sigma$

$\downarrow$

$\mathcal{T}_\Sigma =$ 

loop in $\mathcal{T}_\Sigma$ = action of mapping class group for $S^2 - \{P_1, \ldots, P_n\}$, Braid group B_n

monodromy of $Z(\Sigma_\bullet)$

⇒ action of B_4 on $Z\Big($  $\Big)$

Physics input gives α, β, γ:

Writing $q = e^{\frac{2\pi i}{2+k}}$

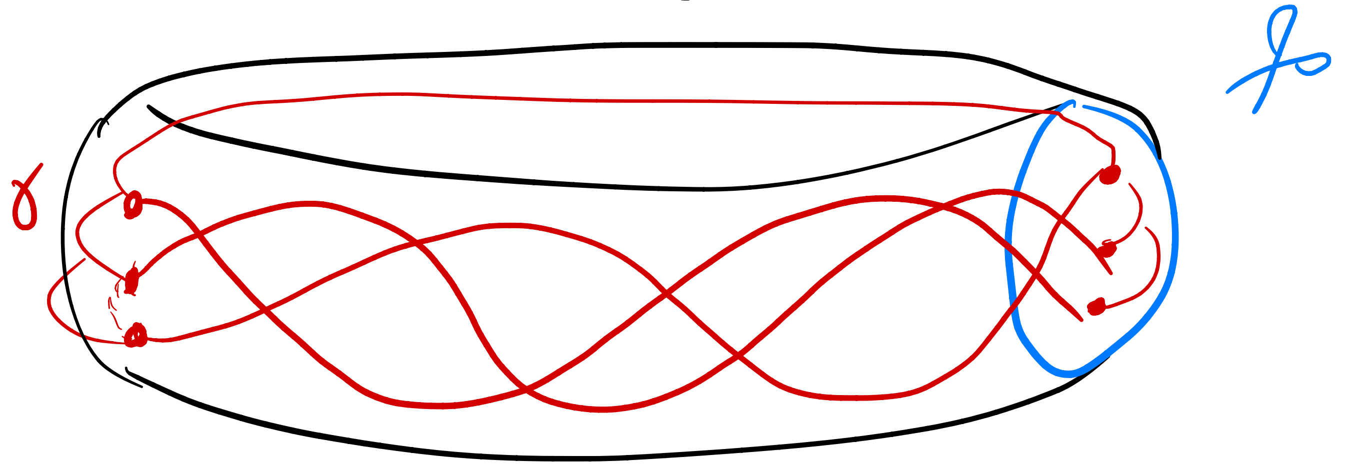
$$q\, Z\left(\;\right) + q^{-1} Z\left(\;\right) + \left(q^{-1/2} - q^{1/2}\right) Z\left(\;\right)$$

Jones polynomial

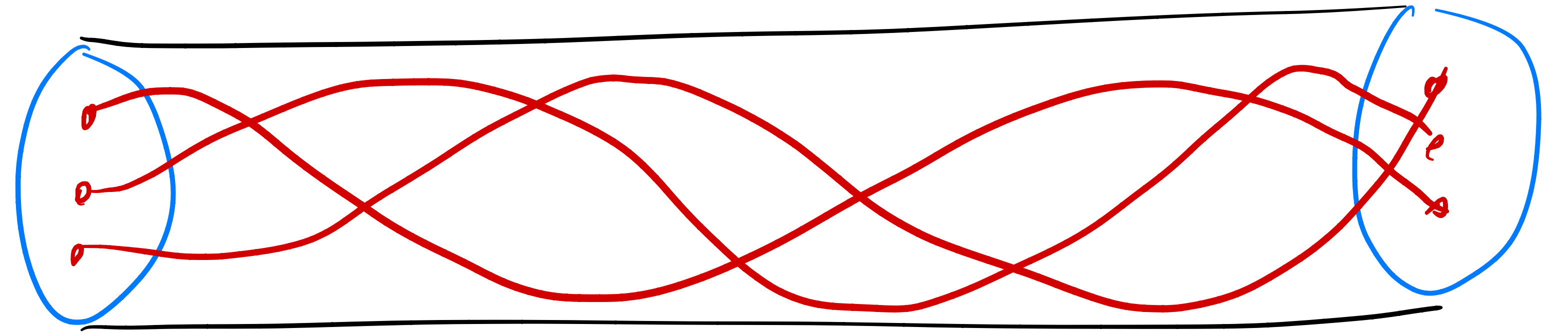
$\boxed{\text{Chern-Simons TQFT computes Jones polynomial evaluated at roots of unity}}$

for braids, Z gives trace formula!

$S^2 \times S^1$



$S^2 \times [0,1]$



$$Z(S^2 \times S^1, \gamma) = \mathrm{Tr}\left(Z(\;) \xrightarrow{\text{Braid group!}} Z(\;) \right)$$

Jones polynomial is trace of certain Braid representations