Orbit Method review:

G compact semisimple lie grp, T a maximal torus

$\mathfrak{g}$ = Lie G $\qquad \mathfrak{t}$ = Lie T cartan

Weyl group $W \curvearrowright \mathfrak{t}^* \subseteq \mathfrak{g}^*$

using killing form, freely identify $\mathfrak{t} \simeq \mathfrak{t}^*$

$G \curvearrowright \mathfrak{g}^*$ coadjoint action, orbit thru $\lambda \in \mathfrak{g}^*$ is $\mathcal{O}_\lambda$

Thm: $\mathcal{O}_\lambda$ carries canonical symplectic structure ω_λ

Fact: $\mathcal{O}_\lambda \cap \mathfrak{t}^*$ is a W-orbit in $\mathfrak{t}^*$

$\Rightarrow$ coadjoint orbit defined by λ in Dominant weyl chamber $\mathcal{C}^+ \subset \mathfrak{t}^*$

Thm of Heighest weight: irrep π_λ defined by $\lambda \in \mathcal{C}^+ \cap \mathfrak{t}_{\mathbb{Z}}$ weight lattice

Orbit method: integral $\mathcal{O}_\lambda \Longleftrightarrow$ irriducible π_λ

Drawing coadjoint orbits

As a manifold, $\mathcal{O}_\lambda = G/G_\lambda$ ← stabalizer of λ

for $\lambda \in \operatorname{int}(\Delta^+)$, $G_\lambda = T$, $\mathcal{O}_\lambda = G/T$

$T \circlearrowright \mathfrak{g}^*$ induces $T \circlearrowright \mathcal{O}_\lambda$

moment map $\mu : \mathcal{O}_\lambda \to \mathfrak{t}^*$ orthogonal projection

$G/T = G_{\mathbb{C}}/B$ ← Borel, so $\mathcal{O}_\lambda$ is Kahler

Thm (Moment map convexity) $\{p\}$ fixed pts of $T \circlearrowright M$

$\operatorname{im}(\mu) \subset \mathfrak{t}^*$ is convex hull of $\{\mu(p)\}$

fixed set of $T \circlearrowright \mathfrak{g}^*$ is $\mathfrak{t}^*$

$\Rightarrow$ fixed pts of $T \circlearrowright \mathcal{O}_\lambda$ is $\mathcal{O}_\lambda \cap \mathfrak{t}^* = W \cdot \lambda$

$\Rightarrow \mu(\mathcal{O}_\lambda) =$ convex hull of weyl orbit

$\mathcal{O}_\lambda$ decomposes into $\mu^{-1}(x)$, w/ $\mu^{-1}(x)/T$ symplectic, $= \mathcal{O}_\lambda /\!/_x T$

Examples

$G = SU(2)$

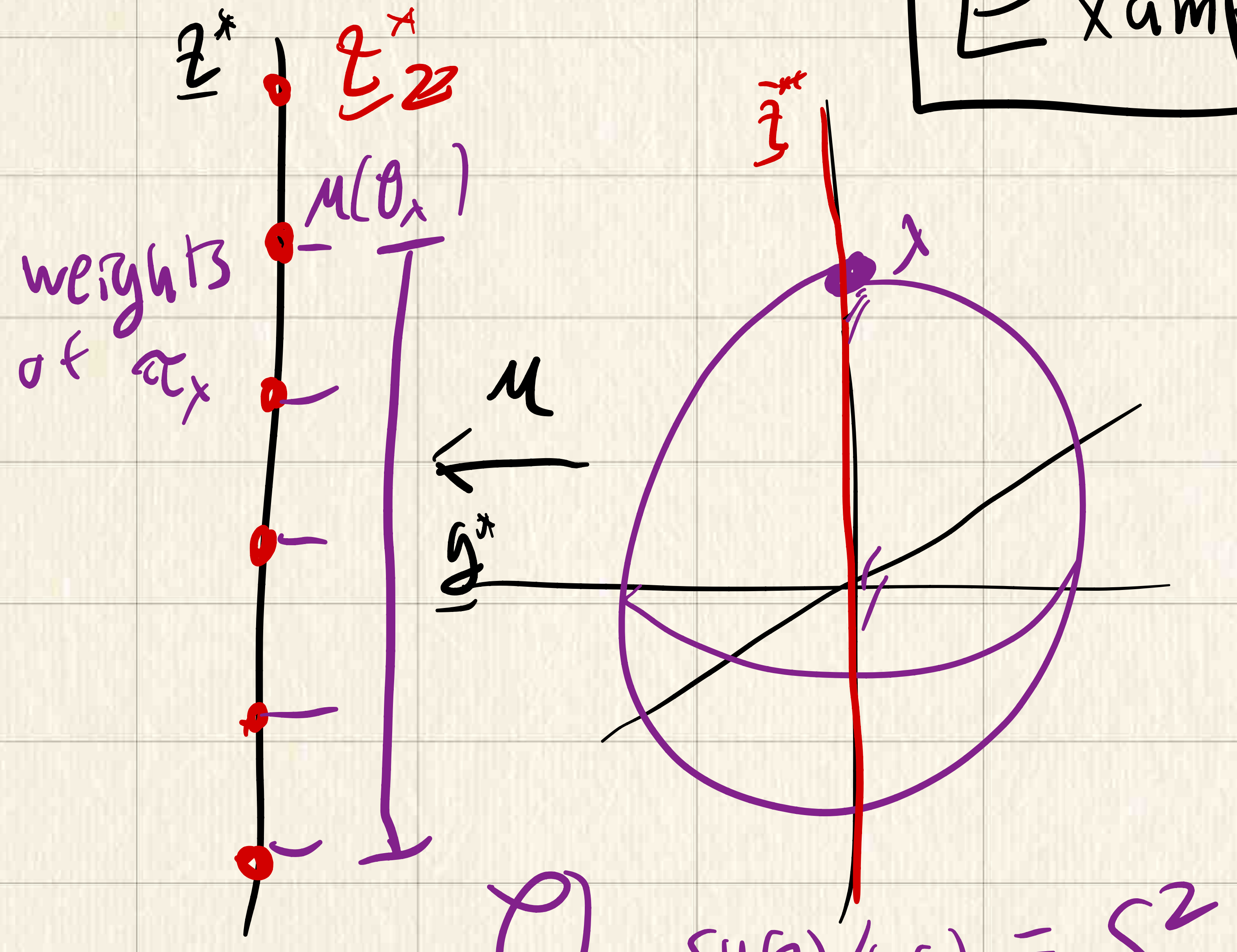
$\mathfrak{g}^* = \mathfrak{su}(2) \simeq \mathbb{R}^3$

$G \circlearrowright \mathfrak{g}^*$ rotation

$T = U(1)$

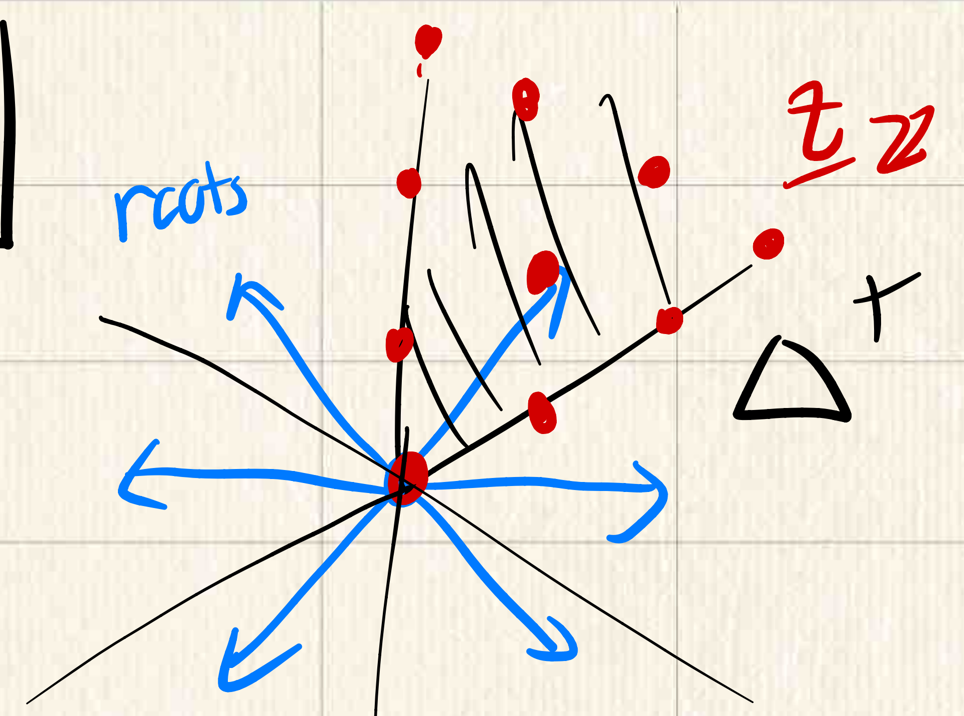
$\mathfrak{t}^* = \mathbb{R}$

$W = \mathbb{Z}_2$

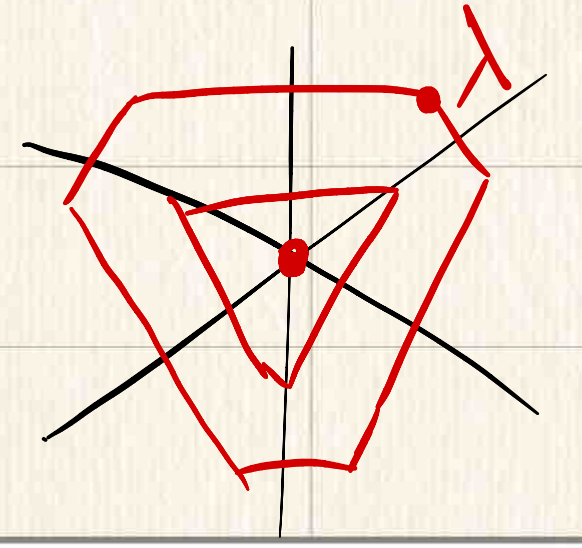


$\mathcal{O}_\lambda = SU(2)/U(1) = S^2$

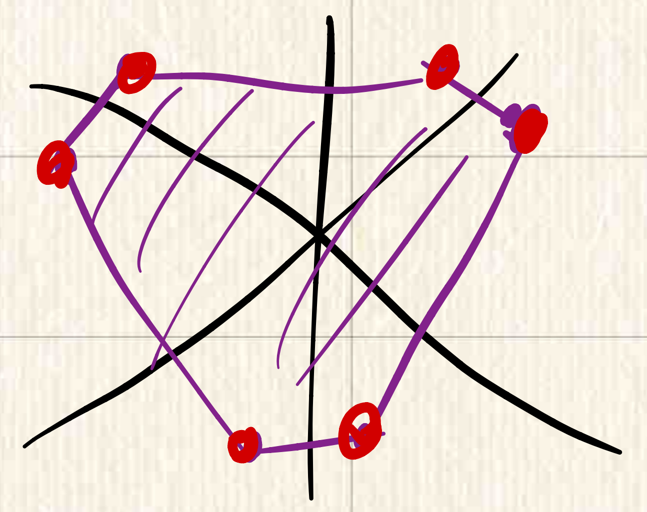
$G = SU(3)$



weights of irrep π_λ:



$\mu(\mathcal{O}_\lambda)$ $\quad$ $\mathcal{O}_\lambda \cap \mathfrak{t}^*$



To be filled out over the course of fall 17

Geometric Quantization shopping list

symplectic manifold (M, ω)

prequantum line bundle $\mathcal{L} \to M$ w/ curvature ω

polarization $\mathcal{P} \subset T_{\mathbb{C}}M$

choice of $\sqrt{K}$ (metaplectic correction) symplectic Geometry $\rightsquigarrow$ $\boxed{\text{Geometric Quantization}}$ $\rightsquigarrow$ Rep. Theory

- (M, ω) symplectic
- $H \in C^\infty(M)$ generates hamiltonian vect. field X_H
  $\omega(X_H, \cdot) = dH$
- $\{f, g\} = h$
- Hamiltonian G-action $G \curvearrowright M$
  Generated by moment map $\mu : M \to \mathfrak{g}^*$

- Hilbert space $\mathcal{H}(M)$ (function space)
- operator $D_H \curvearrowright \mathcal{H}$ (Differential operator)
- $[D_f, D_g] = D_h$
- representation $\pi : G \curvearrowright \mathcal{H}$

Geometric Quantization - A Quantization cookbook

Geometric quantization gives one answer for this trouble

Step 1: Prequantization

Pick hermitian line bundle $L \to M$ w/ connection ∇, curvature ω

!! Requires $[\omega] \in H^2(M, \mathbb{Z})$! symplectic form is integral

- pre quantum Hilbert space $\mathcal{H} = L^2(M, L)$
- operator $\widetilde{D}_H = \nabla_{X_H} + H$

derivative in direction generated by H (good!)

Constructing $\mathcal{L}$ on $\mathcal{O}_\lambda$: need $\lambda \in \mathfrak{t}^*_{\mathbb{Z}}$

$\iff \exp(\lambda) : T \to S^1$ is a character χ_λ

Line bundle $\mathcal{L} = G/T \times_{\chi_\lambda} \mathbb{C}$ $\quad (x, s) \sim (gx, \chi_\lambda(g)s)$

Fact: $\mathcal{L}_\lambda$ has curvature ω_λ

$$\mathcal{H}(\mathcal{O}_\lambda) = \operatorname{ind}_T^G(\chi_\lambda)$$

Step 2: Polarization

Cut down to half the variables

e.g. want $\mathcal{H}(T^*X) = L^2(X)$, but $\tilde{\mathcal{H}}(T^*X) = L^2(T^*X)$

Demand fn constant on cotangent fibers (Lagrangian foliation)

<u>Def</u> a polarization P is an integral, Lagrangian subbundle of $TM \otimes \mathbb{C}$ $\quad [P,P]=0 \quad \omega|_P = 0$

P is real if $P = \bar{P}$ $\quad P$ is complex if $P \cap \bar{P} = 0$

Quantum Hilbert space: flat in P direction

$$\mathcal{H} = \{ s \in L^2(\mathcal{L}) \mid \nabla_X s = 0 \ \forall X \in P \}$$

Example: Kahler polarization $P_{\bar{\partial}}$

M, ω Kahler $\Rightarrow T_{\mathbb{C}}M = T^{1,0}M \oplus \underbrace{T^{0,1}M}_{\text{polarization}}$

for holomorphic line bundle $\mathcal{L}$,

$$\mathcal{H} = \{ s \in L^2(\mathcal{L}) \mid \bar{\partial} s = 0 \} = H^0(M, L)$$

finite dimensional for M compact

Kahler polarization on $\mathcal{O}_\lambda$:

$G/T \cong G_{\mathbb{C}}/B$ (Borel), so $\mathcal{O}_\lambda$ Kahler

for $\lambda \in \mathfrak{t}_{\mathbb{Z}}$, get $\mathcal{L}_\lambda$ holomorphic

$\Rightarrow$ Get representation $G \curvearrowright H^0(\mathcal{O}_\lambda, L_\lambda)$

Thm (Borel-Weil): This rep. is irreducible, w/ heighest weight λ.

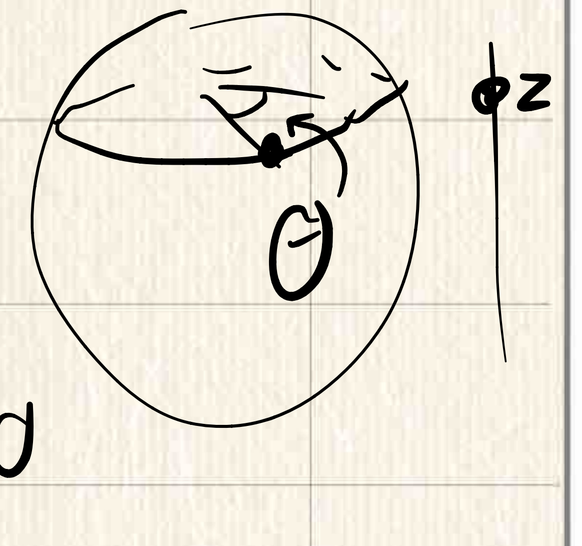
e.g. $G = SU(2)$, $\mathcal{O}_\lambda = \mathbb{P}^1$, $\mathcal{L} = \mathcal{O}(\lambda)$, $\mathcal{H} = H^0(\mathbb{P}^1, \mathcal{O}(\lambda))$ $\cong$

Real polarization on $S^2 = \mathcal{O}_\lambda \subset \mathfrak{su}(2)$ $\lambda \in \mathbb{Z}$

Torus moment map fibers $\mu^{-1}(p)$ are lagrangians

in coordinates (θ, z), $P = \langle \partial_\theta \rangle$

$z \in [-\lambda, \lambda]$



leaf $(\cdot, z)$ has sections f s.t. $\nabla_{\partial_\theta} f = 0$

only when $z \in 2\mathbb{Z}$

$\mathcal{H}$ = "distributional sections" supported on integral leafs

$\mathcal{H} = L^2(\{\lambda, \lambda-2, \dots, -\lambda\})$

This makes connections to rep theory clearer

Mixed polarization on $\mathcal{O}_\lambda$

Quantization should reflect $\mu(\mathcal{O}_\lambda)$ polytope

Generally, $\mu^{-1}(pt)$ not lagrangian.

construct mixed polarization: $\mathcal{P}_{mix}$

- on $\mathfrak{t}^* \oplus X_{\mathfrak{t}^*} \subset T\mathcal{O}_\lambda$ use real polarization

  symplectic dual directions

- on $(\mathfrak{t}^* \oplus X_{\mathfrak{t}^*})^\perp \simeq T\, \mathcal{O}_\lambda /\!/ T$, use holomorphic polarization

Distributional sections supported on $\mu^{-1}(\mathfrak{t}^*_{\mathbb{Z}})$

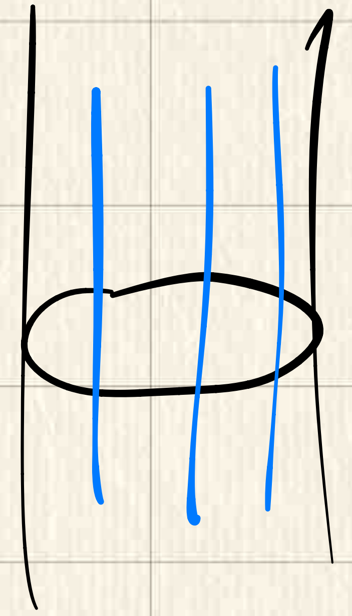
Thm (Leung-Wang 2023)

$\mathcal{H}(\mathcal{O}_\lambda, \mathcal{P}_{\bar\partial}) \simeq \mathcal{H}(\mathcal{O}_\lambda, \mathcal{P}_{mix})$ "invariance of polarization"

ω wt. space $\mathcal{H}_\omega(\mathcal{O}_\lambda, \mathcal{P}_{mix}) \simeq \mathcal{H}(\mathcal{O}_\lambda /\!/_\omega T, \mathcal{P}_{\bar\partial})$ "Quantization commutes w/ reduction"

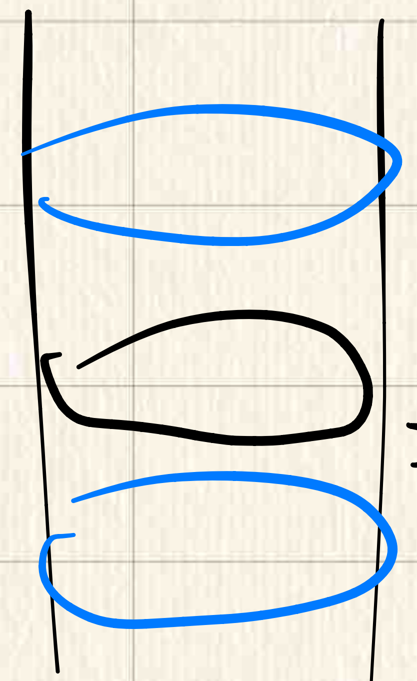
Examples: $M = T^*S^1 = \{(\theta, p)\}$ $\mathcal{L}$ trivial ∇ has 1-form $p\,d\theta$

$P = \langle \partial_p \rangle$



$\nabla_{\partial_p} f = 0$
$\Rightarrow f$ constant on $T^*_p S^1$

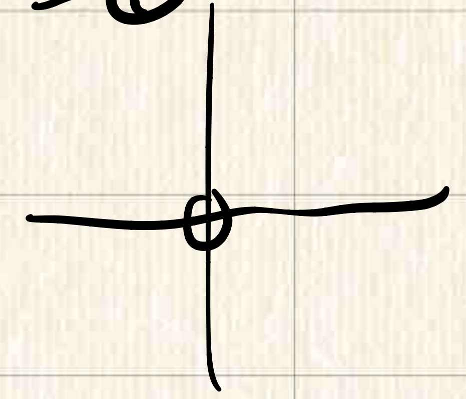
$\mathcal{P} = \langle \partial_\theta \rangle$ $\nabla_{\partial_\theta} f = 0$ has solutions on leaf $(\cdot, p)$ only when $\int_{(\cdot, p)} p\,d\theta \in \mathbb{Z}$



$\Rightarrow f$ supported on integer leaves
"Distributional section"

$\mathcal{H} = L^2(S^1) \xleftrightarrow{\text{fourier transform}} \mathcal{H} = \ell^2(\mathbb{Z})$

$T^*S^1 \simeq \mathbb{C}^*$ $P = \langle \partial_p - i\partial_\theta \rangle$, $\partial_{\bar z} = \partial_p - i\partial_\theta$



$\nabla_{\partial_{\bar z}} f = 0 \Rightarrow$
f holomorphic

$\mathcal{H} = \mathbb{C}[z, z^{-1}]$ with bounded L^2 norm (for weird measure)

"Generalized Segal-Bergmann space"

Step 3: Half-form correction

need to integrate polarized sections somehow

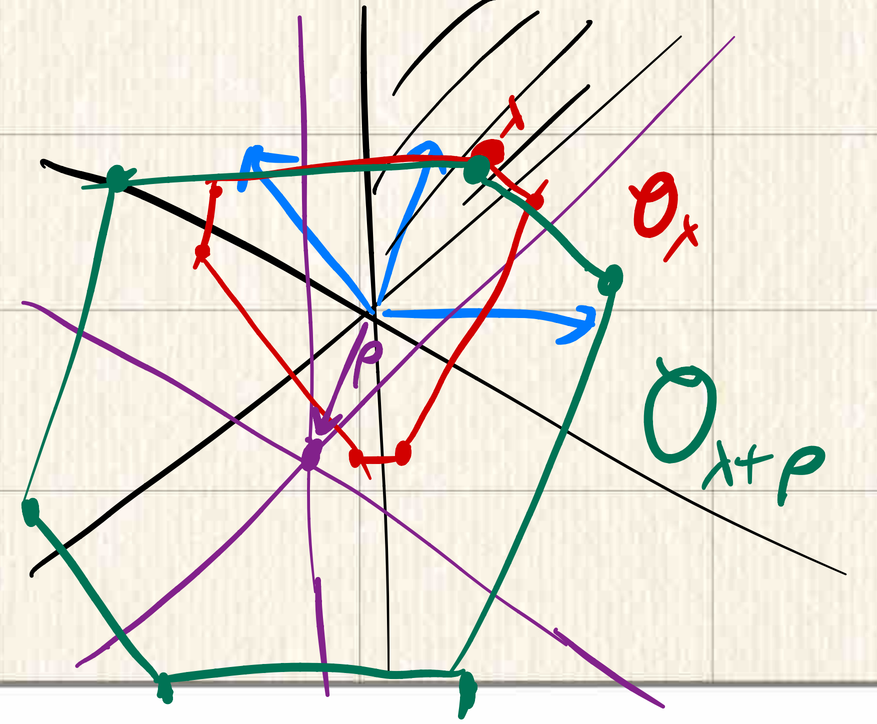
idea: introduce canonical bundle $\Omega^n M$, & twist $\mathcal{L} \mapsto \mathcal{L} \otimes \sqrt{\Omega^n}$

then, for sections s_1, s_2, $\langle s_1, s_2 \rangle_{\mathcal{L}} \in \sqrt{\Omega^n} \otimes \sqrt{\Omega^n} = \Omega^n$, so $\int_M \langle s_1, s_2 \rangle_{\mathcal{L}}$ is well defined

for $\mathcal{O}_\lambda$, choice of $\sqrt{\Omega}$ is choice of positive roots Δ^+

Define $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$

Half-form correction $\mathcal{O}_\lambda \Rightarrow \mathcal{O}_{\lambda+\rho}$



Charecter formulae

Weyl charecter formula

$$\mathrm{tr}_\alpha(e^H) = \sum_{w\in W} \frac{(-1)^{\overset{\text{length of } w}{\ell(w)}} e^{w(\lambda+\rho)(H)}}{\prod_{\alpha\in\Delta^+}\left(e^{\alpha(H)/2} - e^{-\alpha(H)/2}\right)}$$

Weight space decomposition $\mathcal{H}(\mathcal{O}_\lambda)$ provided by Atiyah-Bott fixed point formula.

$$\mathrm{tr}\left(G \circlearrowleft H^0(\mathcal{O}_{\lambda+\rho}, \mathcal{L})\right) = \sum_{x\in \mathrm{fix}} \frac{\mathrm{trace}(g \circlearrowleft \mathcal{L}_x)}{\det(1-g\,|\,T^*_x)}$$

w/ $X = \{w(\lambda+\rho)-\rho\}$

Kirillov charactor formula

$$\sqrt{j(e^H)}\,\chi(e^H) = \int_{\mathcal{O}_{\lambda+\rho}} e^{i\overbrace{\langle\xi,H\rangle}^{=\langle\mu(\xi),H\rangle}}\, d\xi$$

since μ generates $U(1)^n$ action this also has a fixed pt formula!