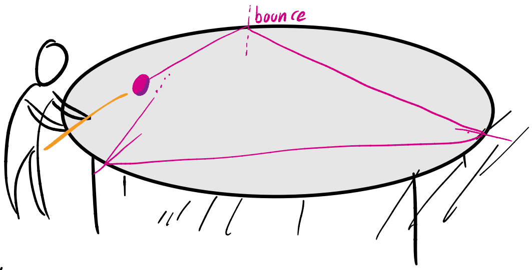
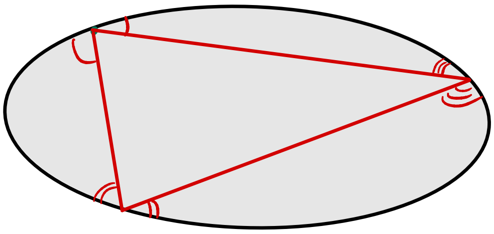
Ellipses in TRIANGLES in Ellipses

Imagine playing billiards on an elliptical table



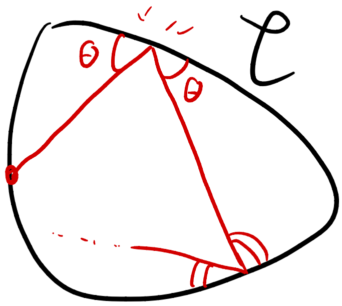
we want to understand the dynamics of the bouncing ball

Theorem: Every Elliptical table has a triangular billiard orbit



Thm (2024): Conversely, every triangle occurs as an orbit in a unique elliptical table.

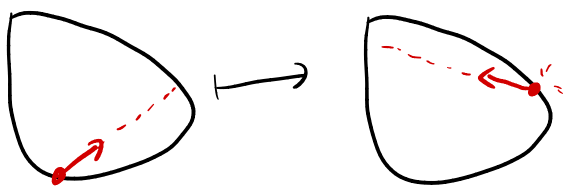
Mathematical idealization:



for a billiards table described by a curve $\mathcal{C}$ bounding a convex set, assume

- ball moves along straight lines on interior of table
- ball bounces elastically: on hitting a wall incident angle and reflected angle are equal

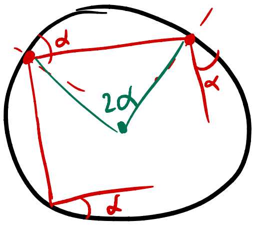
We describe the balls location & velocity discretly, only recording the point $p \in \mathcal{C}$ of each hit, & the velocity $v \in S^1$ after the hit.



The map extrapolating one hit to the next is the Billiards map B

$$B : \mathcal{C} \times [-\pi/2, \pi/2] \longrightarrow \mathcal{C} \times [-\pi/2, \pi/2]$$

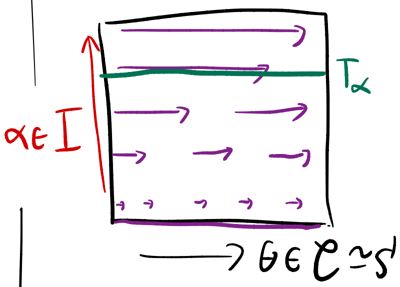
Example: Circular billiards table



the ball will bounce predictably around the outside
the bounce angle $\alpha \in [-\pi/2, \pi/2] =: I$ is preserved under B, &
each bounce moves location around by π.

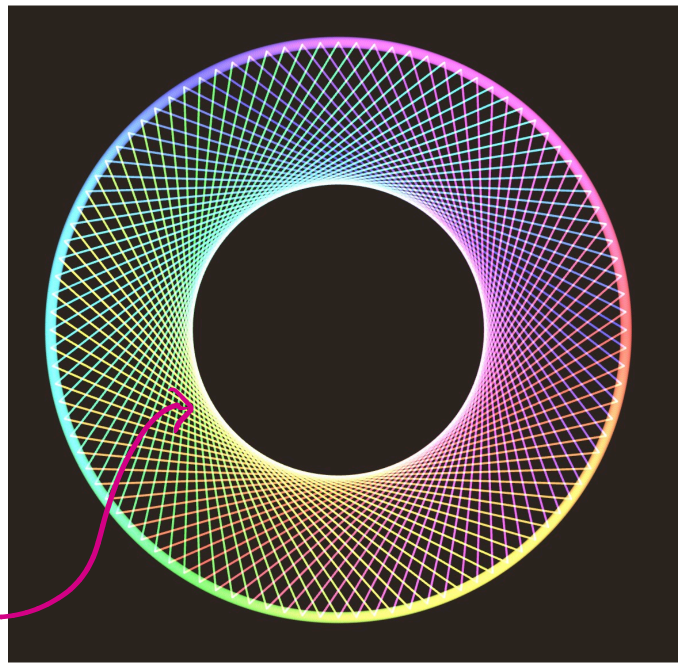
the location of the hit advances by 2α.

or, $B(\theta, \alpha) \longmapsto B(\theta + 2\alpha, \alpha)$



The space $\mathcal{C} \times I$ is <u>foliated</u> by circles
$\alpha = \text{const}$, each preserved by B.
call these the <u>invariant</u> <u>tori</u> $T_\alpha \subset \mathcal{C} \times I$.
on each invariant torus, B acts by a fixed rotation

Typical Billiards orbit
of a circle ⟶



Each point in $\mathcal{C} \times I$ defines
a line in $\mathbb{R}^2$

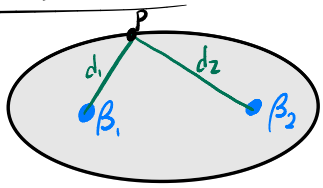
every line in T_α is tangent
to a circle of radius determined
by α — The Caustic of that invariant torus

Definition: The billiards map for a table $\mathcal{C}$ is called <u>integrable</u>
if $\mathcal{C} \times I$ is foliated by invariant tori.

Fact: Every invariant torus has a caustic.

Elliptical Billiards: To play elliptical billiards, we need to build a table

Construction: fix 2 foci $\beta_1, \beta_2 \in \mathbb{R}^2$. An ellipse w/ foci β_1, β_2 is the locus of points p s.t

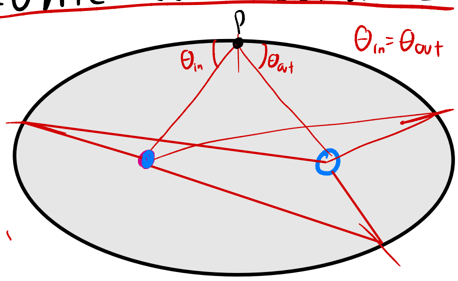


$$\boxed{d_1 + d_2 = \text{const.}}$$

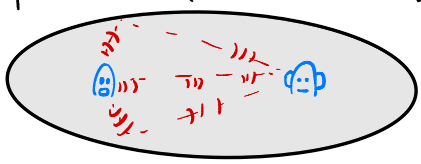
Now we play billiards. place the cue ball at one focus and the pocket at the other, & the game becomes very easy: Every direction sends the ball into the pocket!

Conic construction 2:

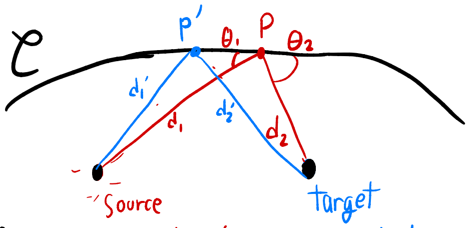
for any $p \in \mathcal{C}$, the lines $p\beta_1$ & $p\beta_2$ form the same angle with the tangent of the ellipse. i.e, these two lines are part of a billiards trajectory



$\Rightarrow$ ellipses form "whispering galleries"



take a point $p \in \mathcal{C}$ & draw lines to two fixed points, the source & target. measure the total length of the lines, $\ell(p) = d_1 + d_2$, as a function of p.



$\theta_1 < \theta_2 \Rightarrow d_1 + d_2 < d_1' + d_2'$

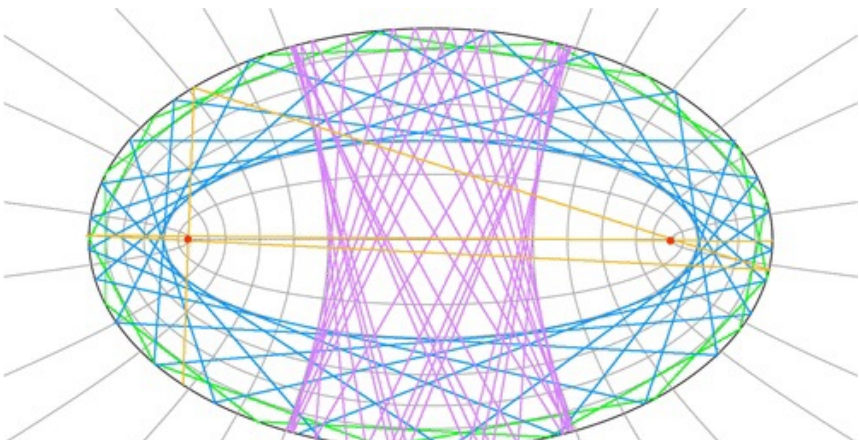
Fact: suppose the angles with $T\mathcal{C}$ have $\theta_1 < \theta_2$. Then, $\ell(p)$ can be reduced by moving p towards θ_1

But, for $\mathcal{C}$ an ellipse, $\ell(p)$ is constant (by definition)

so $\theta_1 = \theta_2$ everywhere on an ellipse

Remark: this is why light reflects when $\theta_1 = \theta_2$: Principle of least action.

Here are some representative billiards orbits on an ellipse:



on the integrability of birkhoff billiards, kalhosin and sorrentino

The billiard orbits bouncing around the outside look like the circle!

Theorem: (Birkhoff) Billiards on an elliptical table are integrable

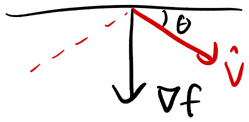
the "Joachimstal Integral" (not an integral :))

recall $\mathcal{C} = f^{-1}(1)$, for $f(x,y) = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$. for $(x, \hat{v}) \in \mathcal{C} \times I$, $J(x,\hat{v}) = \hat{v} \cdot \nabla f$

Bounce location

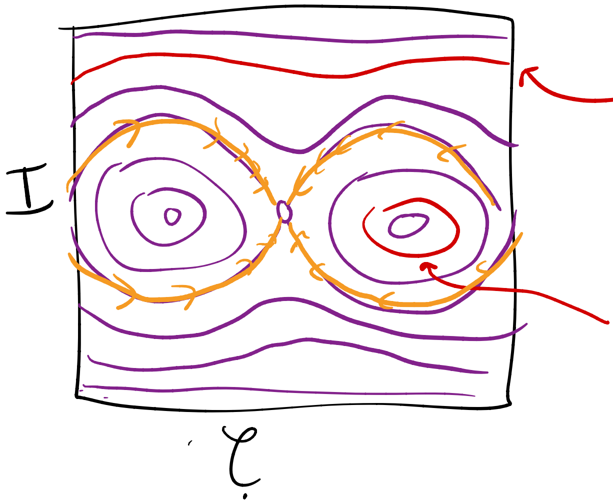
Bounce direction

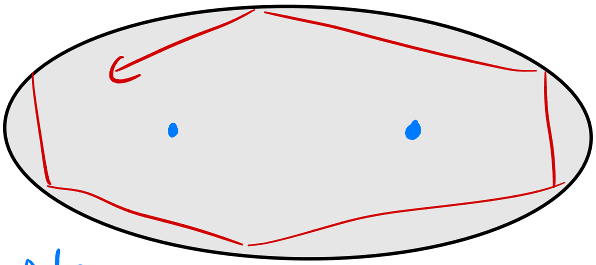
$= \cos\theta \, |\nabla f|$



Fact: J is conserved under B

Level sets of J = invariant tori T_J

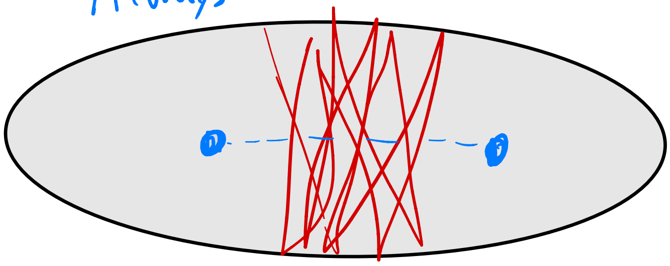




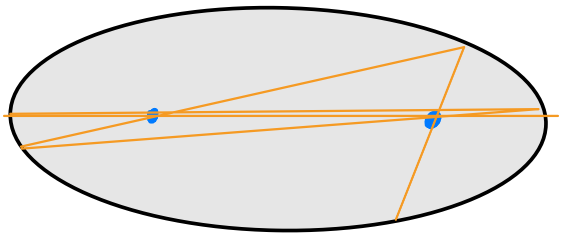
Never

Always

intersects line between foci



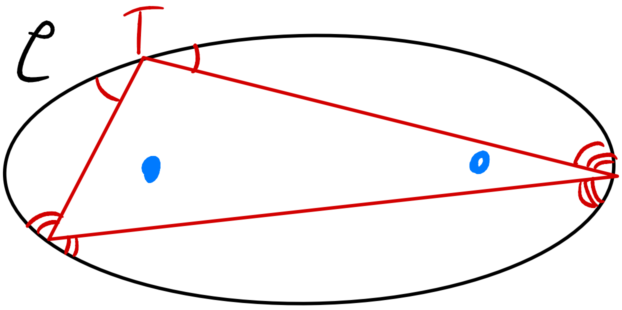
• If the shot passes a focus, it will asymptotically approach the semimajor axis



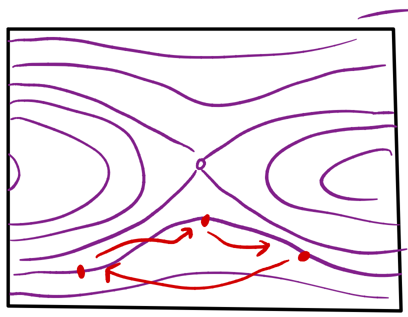
Thm: There are action-angle coordinates (α, z) on $\mathcal{C} \times I$, defined by:

- $\exists\ \alpha(J)$ s.t $\{(\alpha(J), z) \mid z \in I\} = T_J$. the level sets of α are invariant tori
- $B(\alpha, z) = (\alpha, z + \alpha)$

Correllary: every elliptical table in finitly many triangular billiards orbits (i.e a 3-periodic point of $B : \mathcal{C} \times I \circlearrowleft$

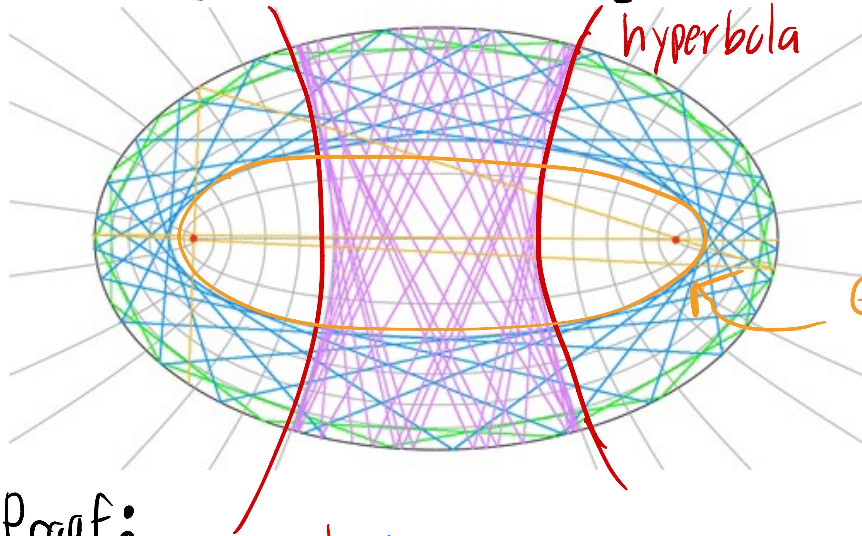


Proof: Take $\alpha = 2\pi/3$. every point on invariant tori T_α is 3-periodic

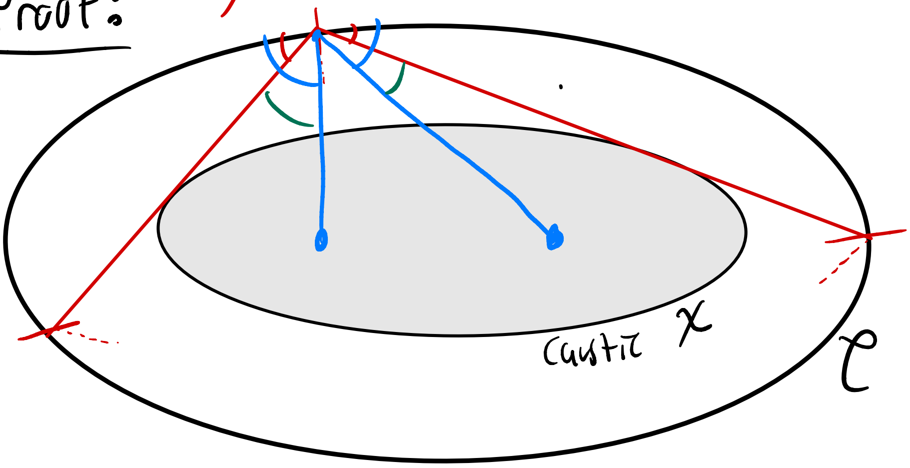


$\alpha = 2\pi/3$

Fact: The caustic of any invariant torus is a conic w/ the same foci as our table



Proof:



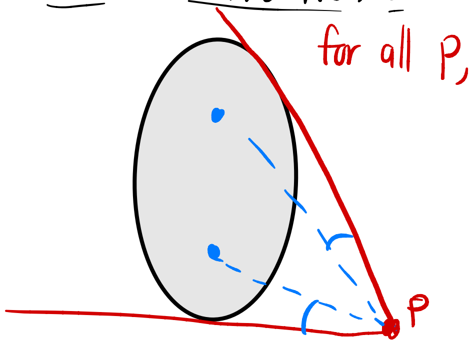
red is billiards trajectory ⇒ ∠ are equal

blue passes thru foci ⇒ billiard trajectory ⇒ ∠ equal

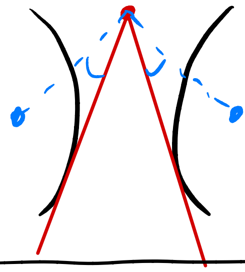
∠ = ∠ − ∠ are equal

if the tangents to $\mathcal{X}$ are billiard trajectories, $\mathcal{X}$ satisfies conic definition 3 for all $p \in \mathcal{E}$ ⇒ $\mathcal{X}$ is a conic w/ same foci as $\mathcal{E}$

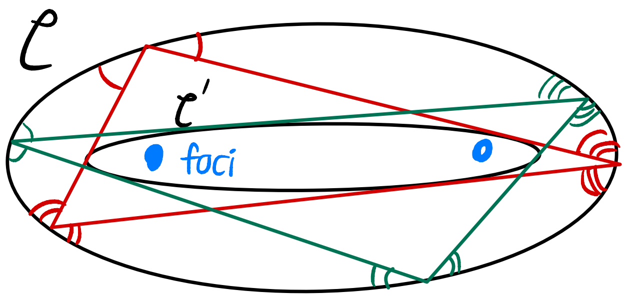
Conic construction 3



equal angles between tangents & foci



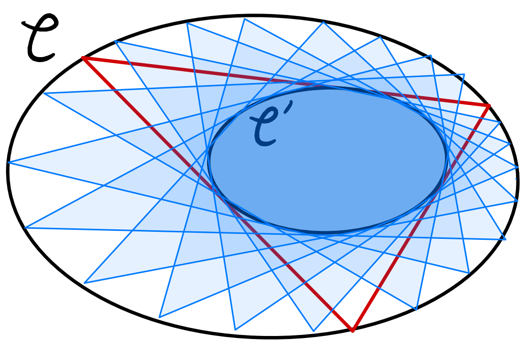
Corollary: There is an ellipse $\mathcal{E}'$, w/ same foci as $\mathcal{E}$, circumscribed by every triangular billiards orbit & , every triangle circumscribing $\mathcal{E}'$ & inscribed in $\mathcal{E}$, is a billiards orbit



https://observablehq.com/@dan-reznik/elliptic-billiard-triangle

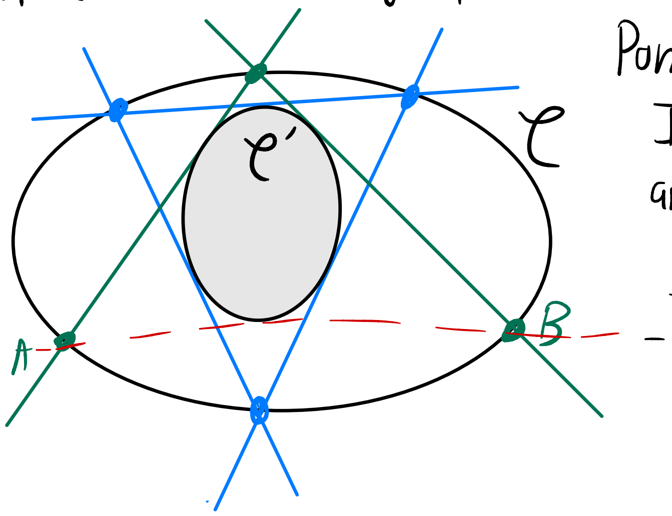
this is a special case of something more general...

Fact: Poncelet's porism



if there exists a triangle circumscribing a conic $\mathcal{C}'$ & inscribing a conic $\mathcal{C}$, then there are infinitly many such triangles

A conic is uniquely defined by 5 line tangencies

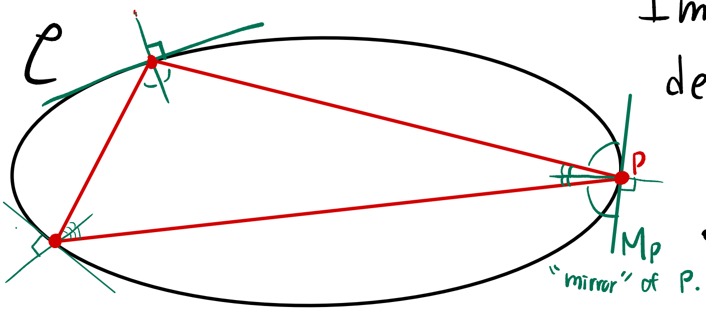


Poncelet's porism implies:

If $\mathcal{C}'$ is a conic tangent to the 5 lines, arranged relative to $\mathcal{C}$ as shown, it must Also be tangent to a 6th line AB

Every Triangle is the billiards orbit of some ellipse

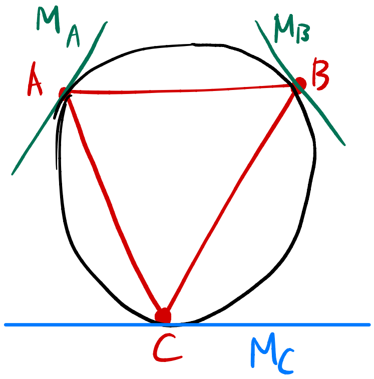
following https://arxiv.org/abs/2405.08922



Imaging building a ellipse around a triangle designed so that the triangle is a billiards orbit.

- $\mathcal{E}$ goes through 3 points of the triangle
- $\mathcal{E}$ is tangent to the 3 perpendiculars of the angle bisectors of each vertex (the mirrors of the vertex)

This is 6 conditions! But a conic is determined by only 5... Rephrased,



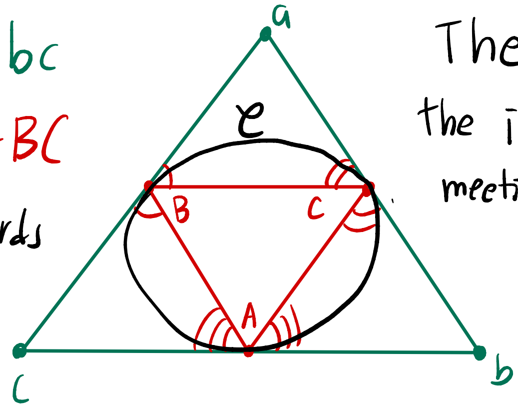
for a triangle ABC, there is a unique ellipse which passes through A, B, C & is tangent to 2 of the 3 mirrors M_A, M_B

Thm: This ellipse is Also tangent to the mirror M_C

We will construct the ellipse with ABC as its billiards

build the triangle abc with edges the mirrors of ABC

By construction, ABC is a billiards trajectory for the table abc



The ellipse $\mathcal{E}$ is the inscribed ellipse of abc, meeting abc at points A, B, C

To construct ellipses in triangles, we turn to complex anylisis

consider $a, b, c \in \mathbb{C}$, & construct the polynomial $P(z) = (z-a)(z-b)(z-c)$

Gauss-Lucas Theorem:

the roots of $P'(z)$ are contained in the convex hull of the roots $P = \prod (z-a_i)$

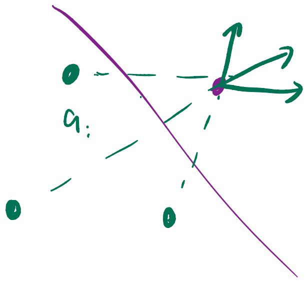
Proof: the roots of P' are the zeros of $\nabla |P(z)|$

noting $\nabla \log P = \frac{\nabla P}{P}$, these coincide with the zeros of $\nabla |\log P|$

$|\log P| = \sum_{a_i} \log|z-a_i|$ (electric potential produced by charges +1 placed at each root a_i)

so $\nabla \log|P| = \sum_{a_i} \frac{z-a_i}{|z-a_i|^2}$ (electric field produced by these charges)

if $P \in \mathbb{C}$ is not in the convex hull of a_i, then there is a line



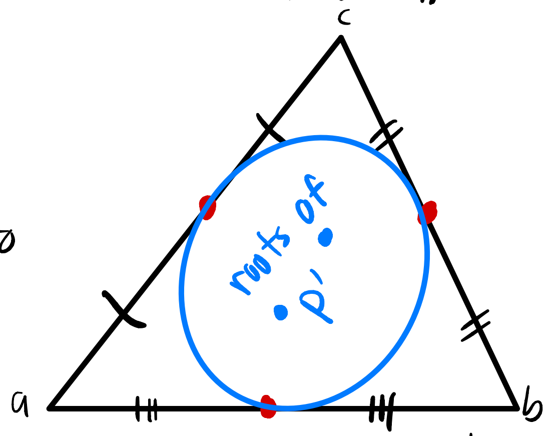
seperating p from all a_i

The gradient of each term of $\log P(z)$ points outwards from the seperating line, so their sum cannot be zero.

Thus, if $\nabla \log|P(z)| = 0$, p must lie in the convex hull of a_i.

Theorem (Marden's thm, proved by Siebeck, 1864)

for $P(z) = (z-a)(z-b)(z-c)$, the roots of $p'(z)$ are the foci of an ellipse which is tangent to abc @ the midpoints of the sides.
(The Steiner inellipse)



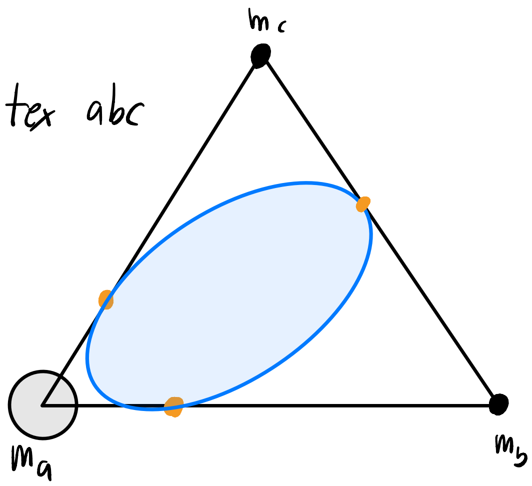
We can upgrade this construction to give different inscribed ellipses!

Thm: (Marden 1945)

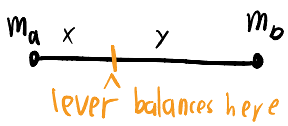
Choose weights m_a, m_b, m_c for each vertex abc

consider $f(z) = \sum m_{a_i} \log|z-a_i|$

the critical points of f are the foci of an ellipse, inscribed in abc

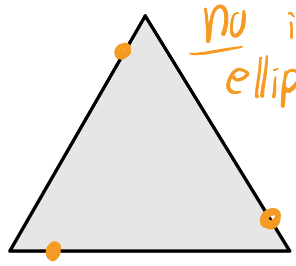


the intersections of the ellipse w/ the triangle satisfy



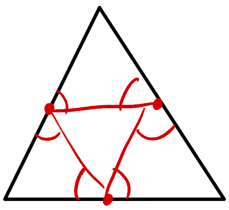
$x m_a = y m_b$

furthermore, every inscribed ellipse is of this form



no inscribed ellipses!

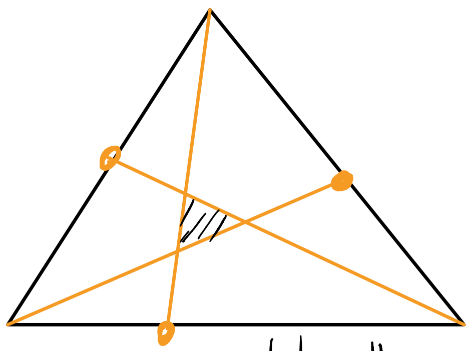
we want an ellipse inscibed in a triangle, intersecting @ points



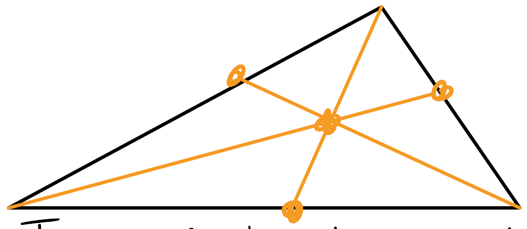
so we need to construct weights m_a, m_b, m_c

Ceva's Theorem:

Such weights exist iff the lines from each vertex to the oppisite point meet at one points

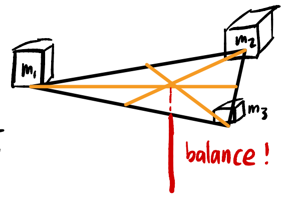


no weights !!

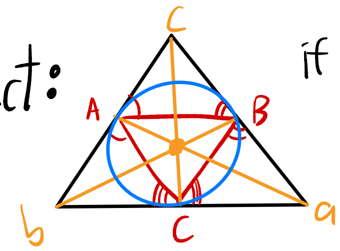


I can find the weights !

"Proof": Place weights at the verticies of the triangle if the weights on each side balance at the orange point, then the constructed lines converge at the center of mass of the triangle



Fact:



if a triangle ABC is a billiards of abc, then the lines Aa, Bb, Cc meet at a point

By Cevas theorem, we can define weights m_a, m_b, m_c

By Marsden's thm, we can find an ellipse inscribed in abc, meeting points A, B, C.

The triangle ABC is a billiards orbit of the ellipse ∎

Hartshorne's ellipse

connecting gauge theory & the Poncelet Porism!!

The construction of ellipses using electromagnetism is deeper than it appears

Consider $SU(2)$ Yang mills theory on S^4. choose an $SU(2)$ principle bundle $\begin{matrix} P \\ \downarrow \\ S^4 \end{matrix}$, represented by a rank 2 complex bundle V; with $\int_{S^4} C_2(V) = k \in \mathbb{Z}$

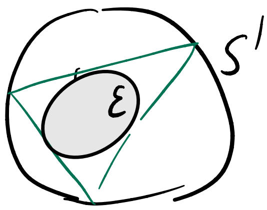
a charge k ASD instanton is a unitary connection on V w/ curvature F, satisfying $F = -\star F$. These are the minimal energy gauge configurations.

we are interested in moduli spaces of instantons

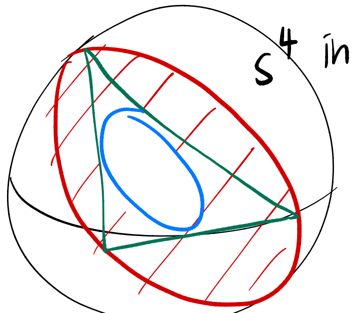
Thm (Hartshorne, "Stable vector bundles and instantons", 1978):

Pick an ellipse $\mathcal{E}$ in the unit ball $B^5 \subset \mathbb{R}^5$. in the ellipses plane, we have an ellipse contained within a circle

$\mathcal{E}$ satisfies the poncelet condition if $\exists$ a triangle circumscribing $\mathcal{E}$, inscribed in S^1



Thm: The moduli space of $k=2$ instantons is the moduli space of $\mathcal{E}$ satisfying the poncelet condition



Each instanton is determined by a potential

These are only determined up to gauge transforms. We can always choose a representive potential (a JNR potential) which looks like

$$\rho(x) = \sum_{i=1}^{k+1} \frac{m_i}{|x - X_i|^2}$$

X_i are points in $\mathbb{R}^4$, m_i is the "charge" (looks like the electric potentials from our proof of gauss-lucas)

(See "Geometry & Kinematics of Two Skyrmions" by Atiyah & Manton)

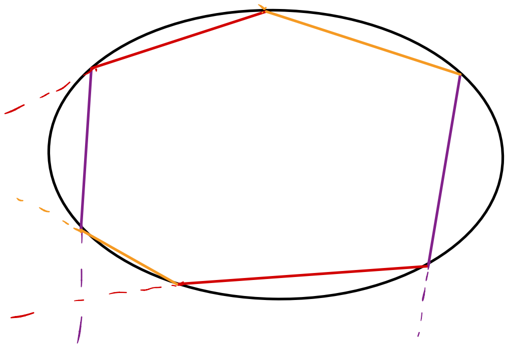
BUT the representation X_i, m_i are not unique up to gauge transform!

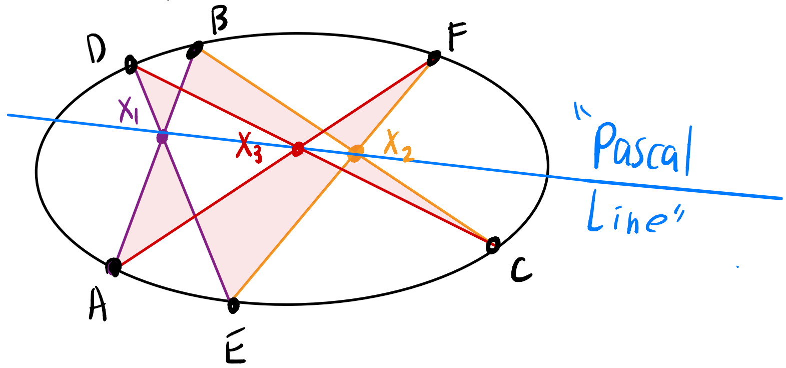
Hartshorne conjecture says: as $\{X_i\}$ rotate along their poncelet family of triangles circumscribing $\mathcal{E}$ & m_i are the weights defining $\mathcal{E}$ inside of T, the potentials are gauge equivelent

$\Rightarrow$ instantons are uniquely determined BY Their Ellipse! wow!

Pascal's Hexagon theorem

Thm (Pascal, 1639. He was 16 years old): consider a hexagon inscribed in a conic. extend the oppisite sides until they intersect. Then, these 3 intersections are colinear.

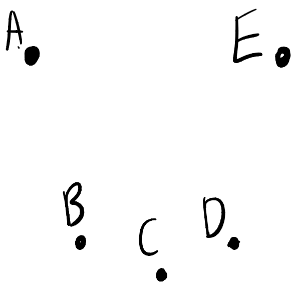




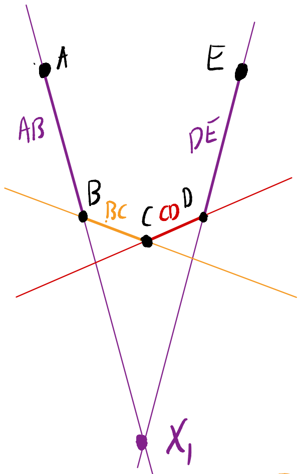
Thm: There is a unique conic passing through 5 fixed points

PF: Construct possible hexagons satisfying condition in pascal's theorem

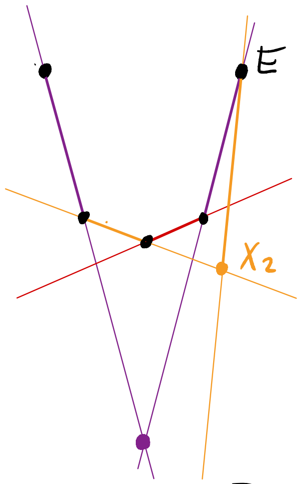
1. pick 5 points, labled A-E



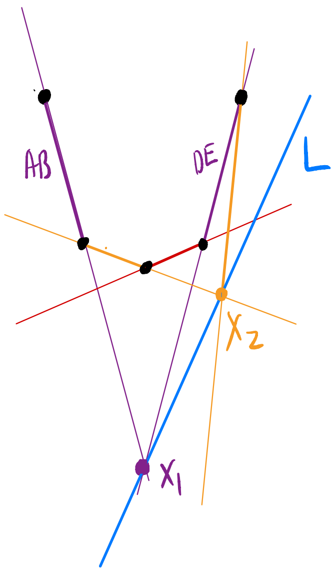
2. Extend lines AB BC CD DE (4 edges of hexagon) mark $AB \cap DE = X_1$



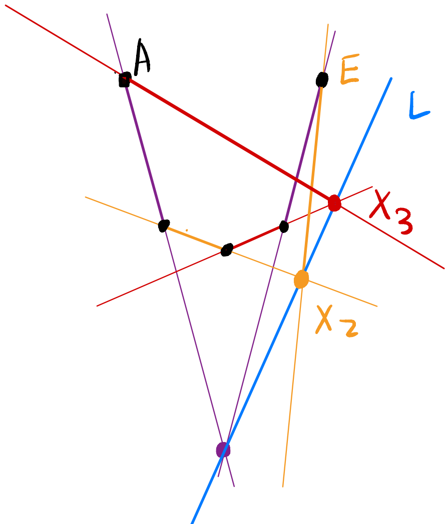
3. choose point X_2 on line 23 construct Line EX_2 (edge 5 of hexagon)



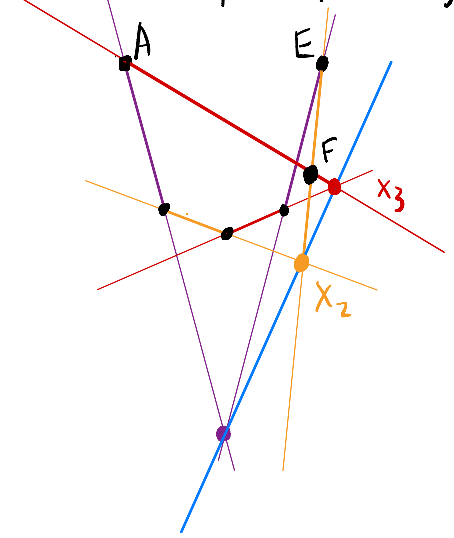
4. construct Line thru X_1 and X_2 (Pascal line)



5. construct $X_3 = L \cap EX_2$ Construct line AX_3 (edge 6)



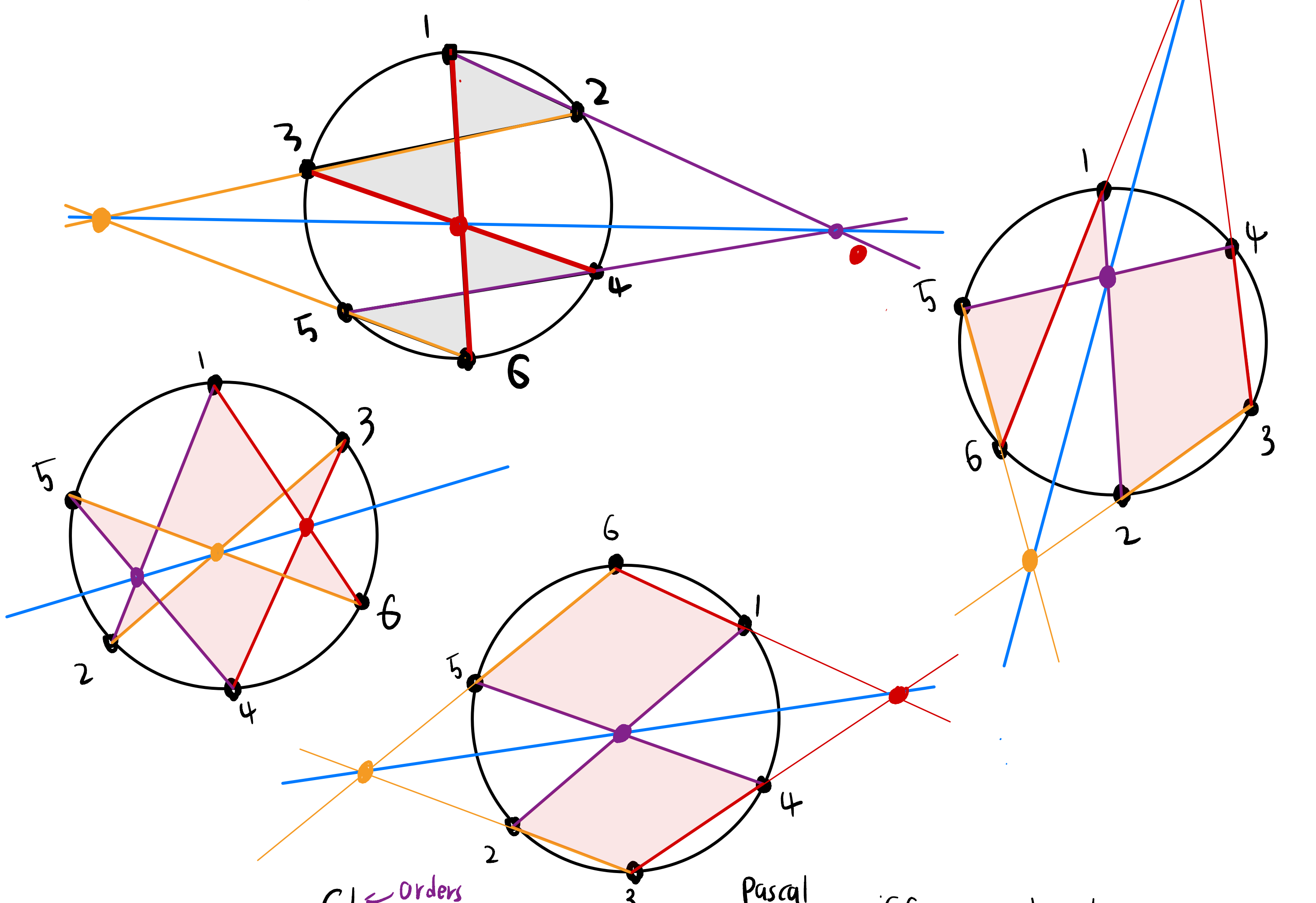
6. construct $F = EX_2 \cap AX_3$ F is 6th pt on pascals hexagon



Braikenridge–Maclaurin theorem: Converse to Pascal's thm

As chosen point X_2 varies on its line, constructed pt F traces entire conic.

There are many hexagons we can build out 6 points



6 points on conic $\Rightarrow$ $\dfrac{\overset{\leftarrow \text{orders}}{6!}}{\underset{\text{cyclic permutation}}{6} \cdot \underset{\text{reversing order}}{2}} = 60$ hexagons $\xRightarrow{\text{Pascal}}$ 60 pascal lines

This collection of lines is called the

Hexagrammum mysticum

As Thomas Kirkman proved in 1849, these 60 lines can be associated with 60 points in such a way that each point is on three lines and each line contains three points. The 60 points formed in this way are now known as the **Kirkman points**.[5] The Pascal lines also pass, three at a time, through 20 **Steiner points**. There are 20 **Cayley lines** which consist of a Steiner point and three Kirkman points. The Steiner points also lie, four at a time, on 15 **Plücker lines**. Furthermore, the 20 Cayley lines pass four at a time through 15 points known as the **Salmon points**.[6]