The Tales of SUSY

Dark Magic of Integrable Systems

elliot kienzle

UMD RIT Geometry phys

Chapter 1

a Cult of Symmetry

Once upon a time in gauge grove...

a Coven of Quantum field theorists

A canvas...

a manifold
dimension of 4

a bundle
of vectors,
rank of 2

& A field... a connection

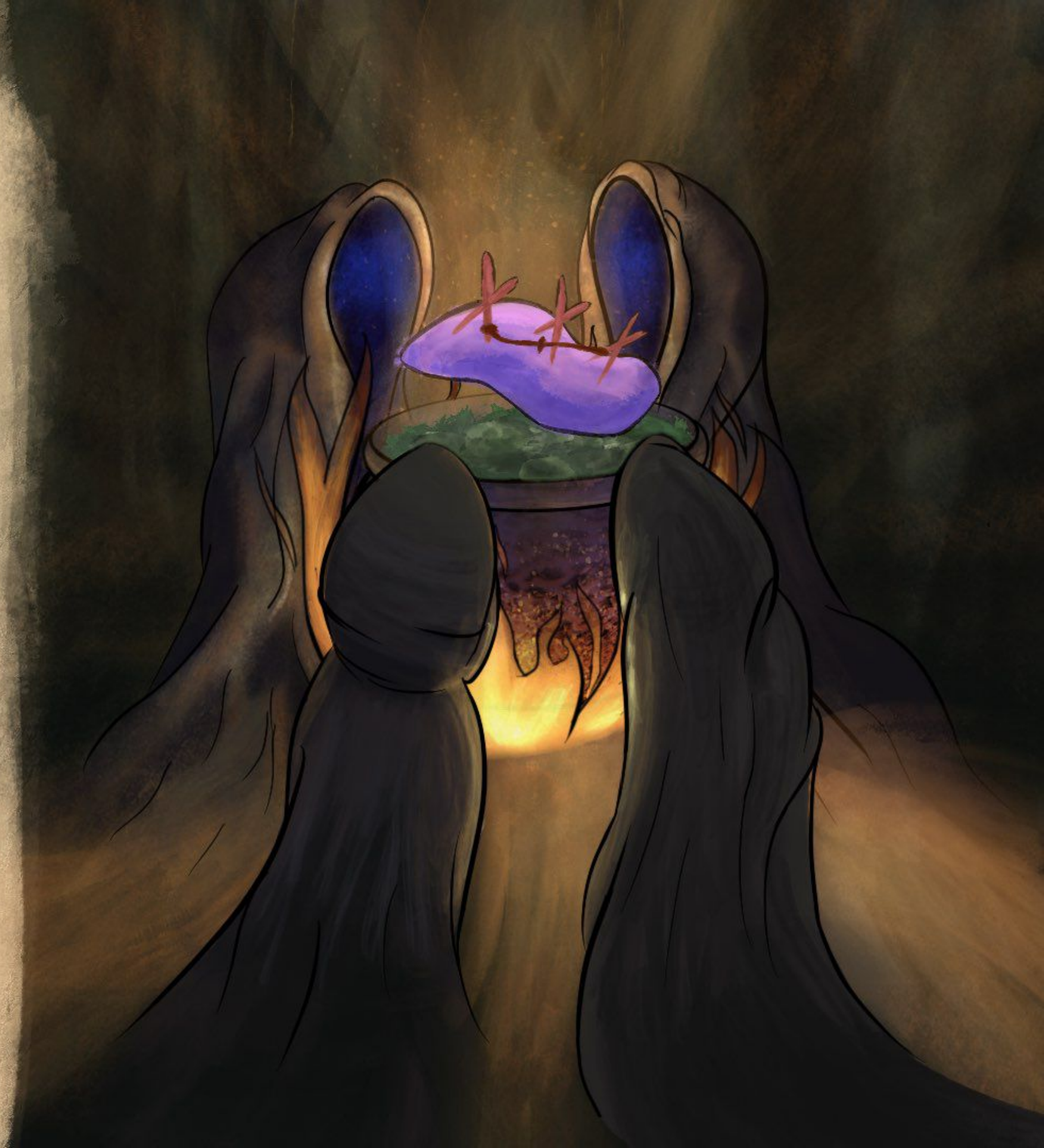
Give a #: The Action

Total square curvature: $\|F\|^2 = \int F \wedge * F$

Topological curvature: $\int F \wedge F$

Yang-Mills

U(1) E&M
SU(2) weak
SU(3) strong



But they were too greedy

Supersymmetry!

anticommuting partners to fields

$\mathcal{N}=2$ Supersymmetry Generators

thus evoking...

The **Dark Magic** of *integrable systems*

and the theory was nowhere to be seen...

it must be found!

fields $\longleftrightarrow$ diffential forms Ω^k

Boson $\longleftrightarrow$ Ω^{2n} $a\wedge b = b\wedge a$

fermion $\longleftrightarrow$ Ω^{2n+1} $a\wedge b = -b\wedge a$

$\mathcal{N}=1$: $d: \Omega^{2n} \to \Omega^{2n+1}$

Riemanian geo

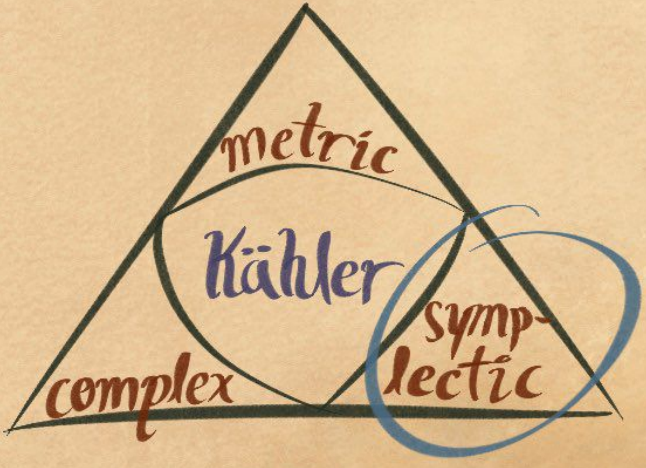
holomorphic

$$d = \partial + \bar{\partial}$$

anti holomorphic

$$\mathcal{N}=2: \begin{array}{l} \partial: \Omega^{p,q} \to \Omega^{p+1,q} \\ \bar{\partial}: \Omega^{p,q} \to \Omega^{p,q+1} \end{array}$$

Kähler geo



Chapter 2

a fishy situation

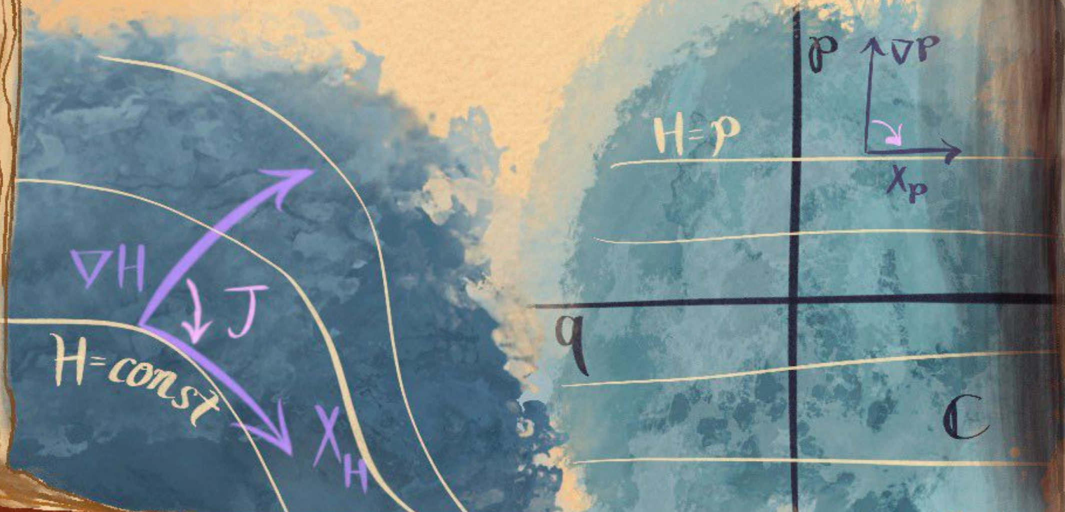
Hamiltonian mechanics

phase space: Kähler mfld M

Hamiltonian $H: M \to \mathbb{R}$ "$\times\, i$"

flow under H is $X_H = J \cdot \nabla H$

eg: 1D dynamics, $X_p = J \cdot (0,1) = (1,0)$



Symmetries

Y symmetry vect. field:

preserves H: $Y(H) = 0$ preserves M: $Y = X_{H'}$

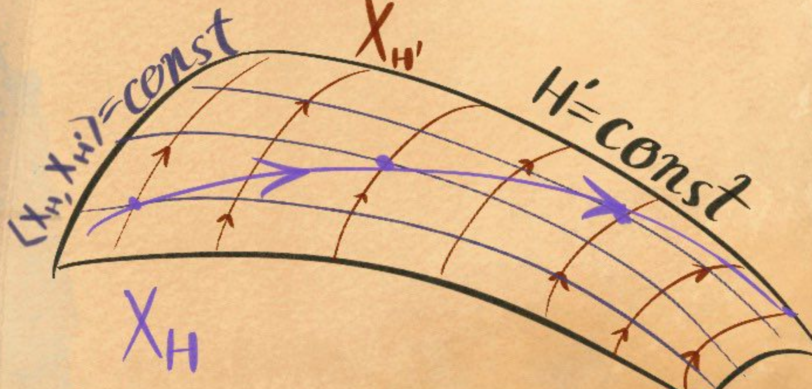
Noether's Thm:

H' conserved!

$$0 = X_{H'}(H) = \langle \nabla H, J \nabla H' \rangle = \langle -J \nabla H, \nabla H' \rangle = -X_H(H') = 0$$

symmetry acts double!

1) H' const 2) $\langle X_H, X_{H'} \rangle = \text{const}$



Louiville integrabillity

Maximal symmetry

N mutually commuting symmetries

$H_1, \dots, H_n$ w/ $[X_{H_i}, X_{H_j}] = 0$

$X_{H_i}(H) = 0 \ \forall i \Rightarrow X_H$ lives on $\bigcap_i \{H_i = \text{const}\}$

$\parallel$

n-fold $\mathcal{L}$

$\{X_{H_i}\}$ span $T\mathcal{L} \Rightarrow \langle X_H, X_{H_i} \rangle = \text{const}$

<u>fixes</u> X_H!

e.g on $\mathbb{R}^3$, p_x, p_y, p_z conserved $\Rightarrow X_H$ const. velocity.

$\mathcal{L}$ lie grp w/ abelian Lie alg $\{X_{H_i}\}$

$\Rightarrow \mathcal{L} = U(1)^p \times \mathbb{R}^q$

compactness $\Rightarrow$

$\mathcal{L} = U(1)^n$

X_H <u>linear</u> flow

symplectic form $\omega(X,Y)=\langle X, JY\rangle$

$[X_{H_i}, X_{H_j}] = 0 \Leftrightarrow \underline{\omega|_{\mathcal{L}} = 0}$

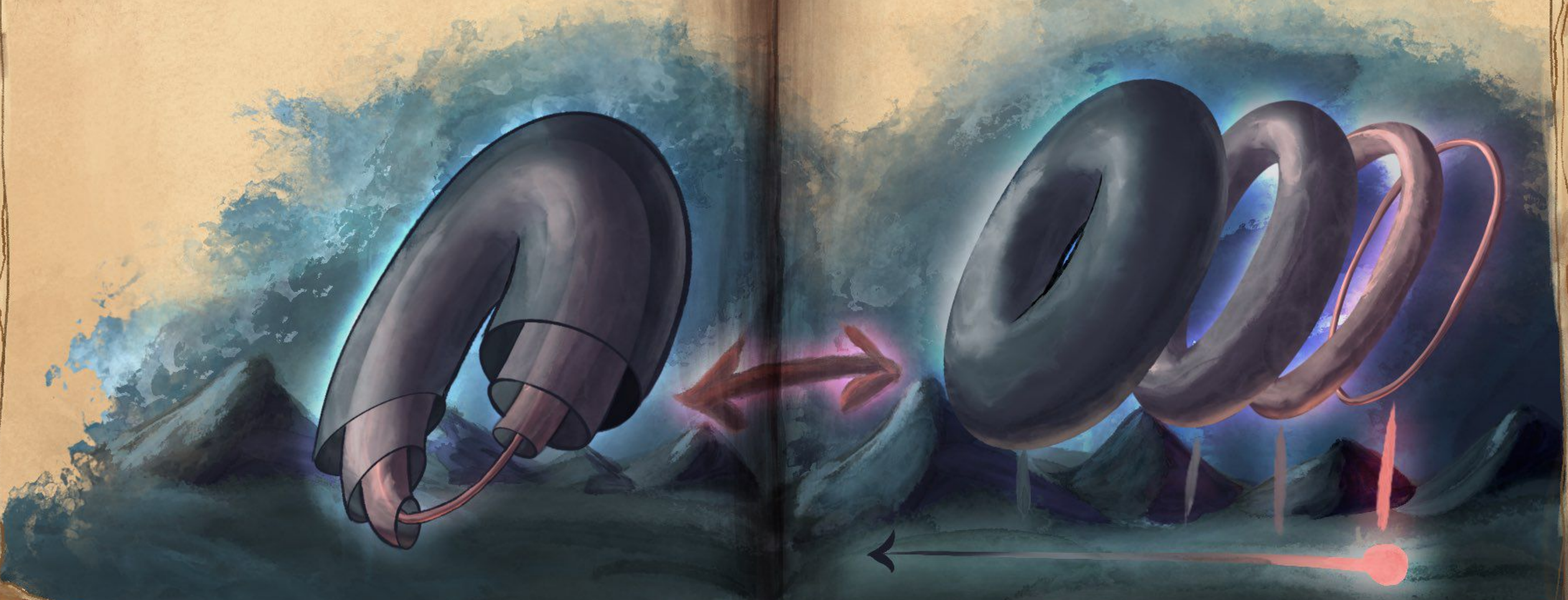
$\mathcal{L} = U(1)^n$ Lagranian torus

Phase space is a

Lagrangian Torus fibration

Base = {possible $\vec{H_i}$} fiber = $\vec{H}^{-1}(\vec{b})$

(degenerates in places)



Chapter 3

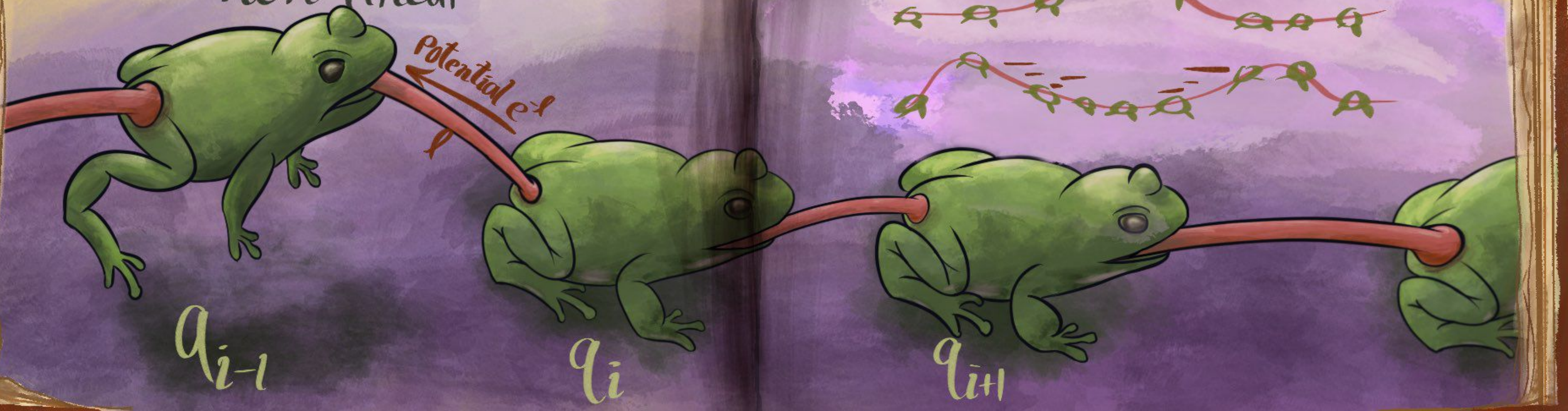
frog :(

Toda Lattice

N toads on a Line

$$H = \sum p_i^2 + e^{q_{i+1} - q_i}$$

non-linear

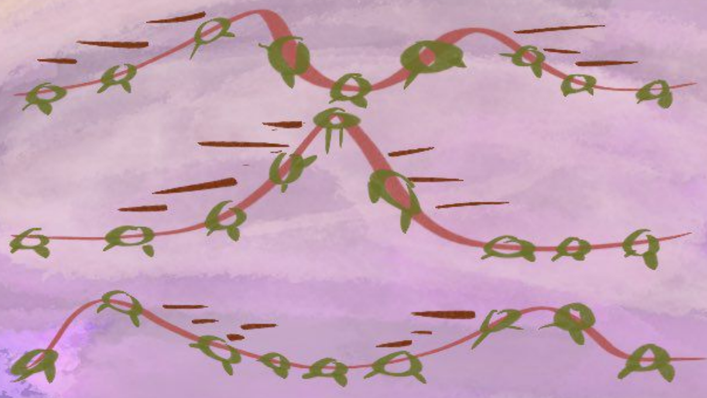


has <u>Solitons</u> which cross w/o interacting

splits into sum of solitons

Completly integrable : conserved quantity

= ammount of n^{th} soliton



Lax pair

write toda lattice E.O.M as

$$\dot{L} = [A, L]$$

then $L(t) = U L U^{-1}$, $U = \exp(At)$

so, spectrum constant!

Eigenvalues = conserved quantities

Eigenvectors = `angles'

Important example:

N=2 periodic

natural 1-parameter family

$$L(z) = \begin{bmatrix} p_1 & e^{\bar{q}} + \frac{1}{z} e^{-\bar{q}} \\ e^{\bar{q}} + z e^{-\bar{q}} & p_2 \end{bmatrix} \qquad \bar{q} = \frac{q_1 - q_2}{2}$$

$$A(z) = \begin{bmatrix} 0 & e^{\bar{q}} - \frac{1}{z} e^{-\bar{q}} \\ -e^{\bar{q}} + z e^{-\bar{q}} & 0 \end{bmatrix} \qquad z \in \mathbb{C}$$

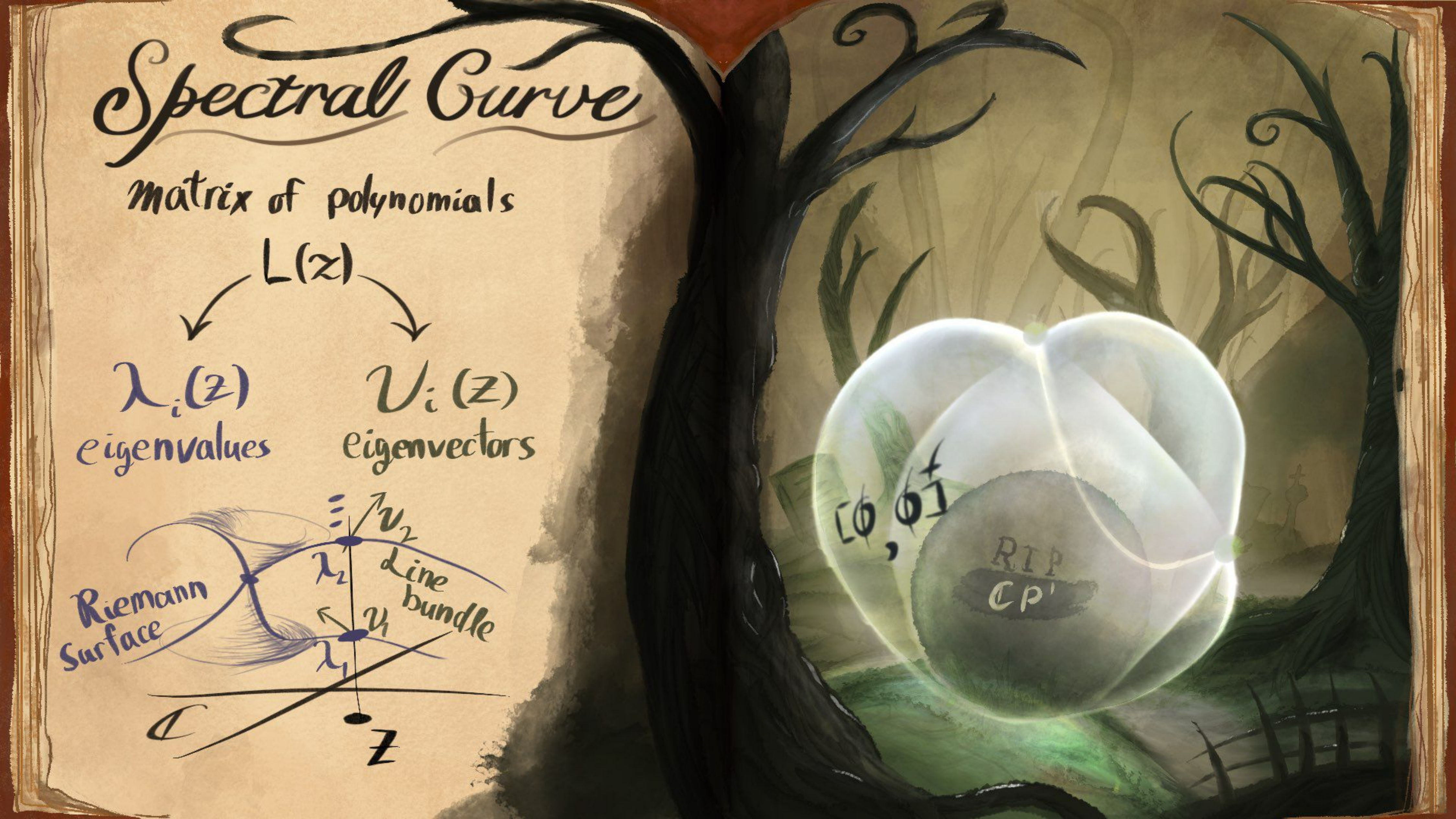
Solitons = e.vecs of L (2 of them)

classified by e.vals, determined by I.C.s

evolution: travel around circle linearly

$\Rightarrow$ linear flow around $U(1)^2$:)


Chapter 4
Spectral cemetary
Gauge grove
Lagrangian Tori
Liou-ville
Toda lattice
Soliton Swamp
Spectral cemetary
Spectral cave



Spectral Curve
matrix of polynomials
$L(z)$
$\lambda_i(z)$
eigenvalues
$v_i(z)$
eigenvectors
v_2
λ_2
Line bundle
Riemann Surface
v_1
λ_1
$\mathbb{C}$
z
$[\phi, \phi^\dagger]$
RIP
$\mathbb{C}P^1$

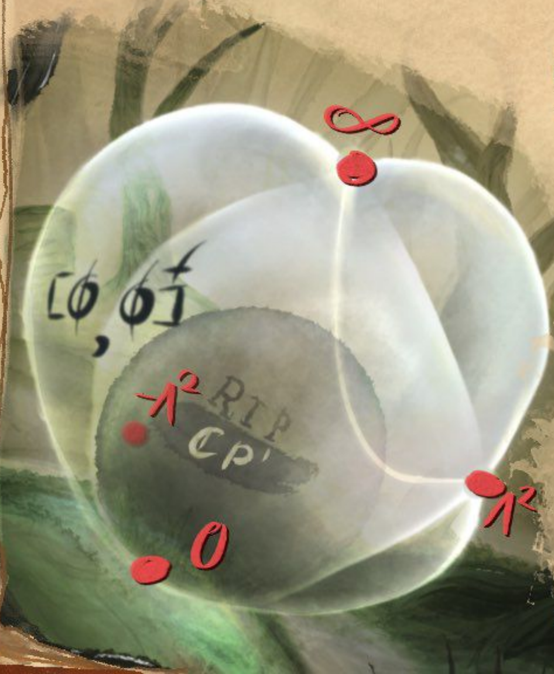
$N=2$ toda $\Rightarrow L(z)$ 2×2

spectral curve $\det(\lambda - L(z)) = 0 \Rightarrow \lambda^2 = f(z)$

Hyperelliptic curve

toda: $f(z) = z(z-\Lambda)(z+\Lambda)$ $\quad \Lambda \in \mathbb{C}$

spectral curve = conserved quantity



genus g curve

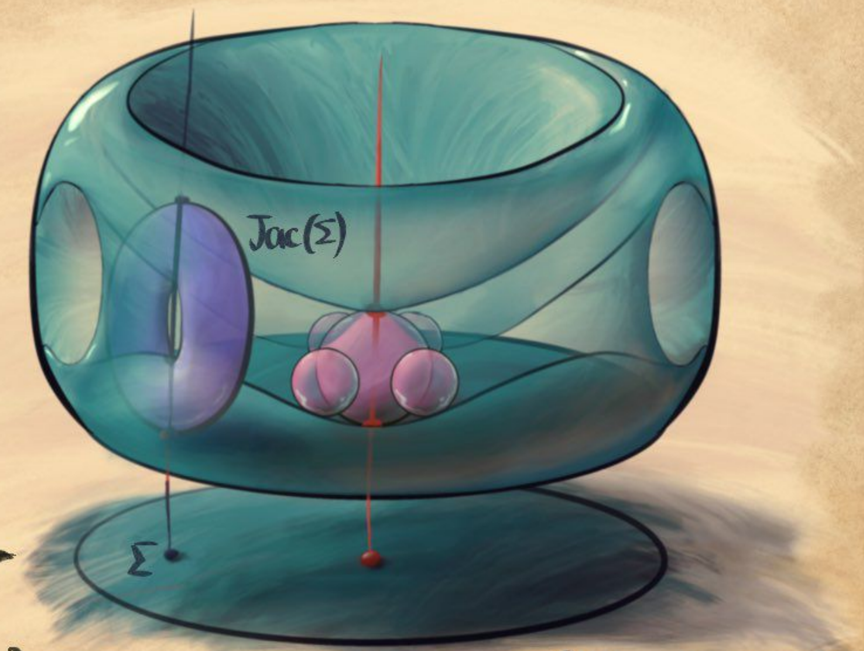
Defined by $2g\text{-}1$ roots

fiber for spectral curve Σ:
space of line bundles $\mathrm{Jac}(\Sigma)$

line bundle $\approx$ choice of phase around loops in Σ

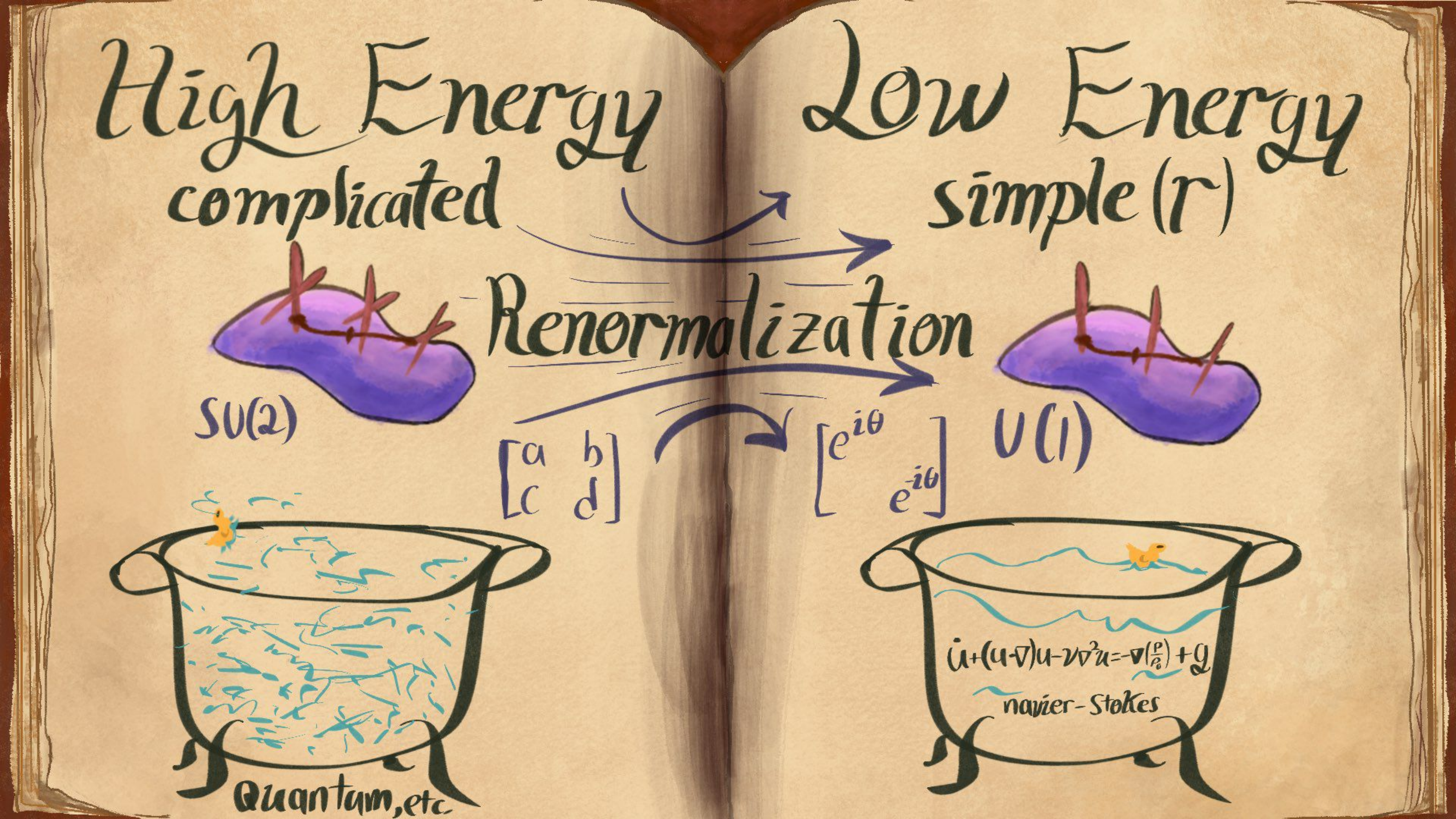
$\mathrm{Jac}(\Sigma) \simeq \mathrm{Hom}(\pi_1(\Sigma), U(1)) \simeq U(1)^{2g}$, a torus!

flow linear on $\mathrm{Jac}(\Sigma)$!



Lagrangian torus fibration!


Gauge grove
Lagrangian Tori
Liou-ville
Sieberg
-Witten
Summit
Nilpotent
cone
Toda lattice
Soliton
Swamp
Spectral
cemetary
$\mathbb{CP}^1$
Spectral
curve


High Energy
Low Energy
complicated
simple(r)
Renormalization
SU(2)
U(1)
$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$
$\begin{bmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{bmatrix}$
$\dot{u}+(u\cdot\nabla)u-\nu\nabla^2u=-\nabla\left(\frac{p}{\rho_0}\right)+g$
navier-Stokes
Quantum, etc

Result of renormalization:

$\|dA\|^2$ topological

U(1) gauge theory: $\mathcal{L} = \frac{1}{g(u)^2}\int dA \wedge * dA + \theta(u) \int dA \wedge dA + \dots$

1-form A

combine into $\tau(u) = \theta(u) + \frac{i}{g^2(u)}$ "electric"

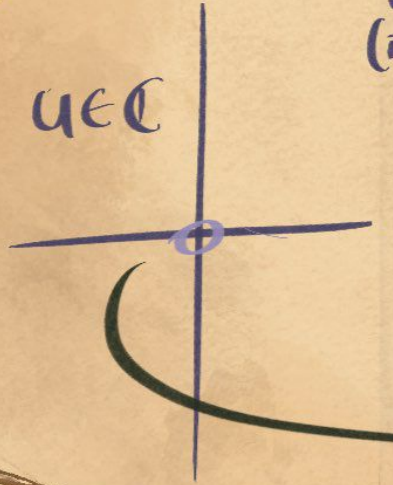
"magnetic"

strengths $g(u), \theta(u)$ depend on a parameter:

"space of vacua" coloumb branch

possible **zero-energy** configurations

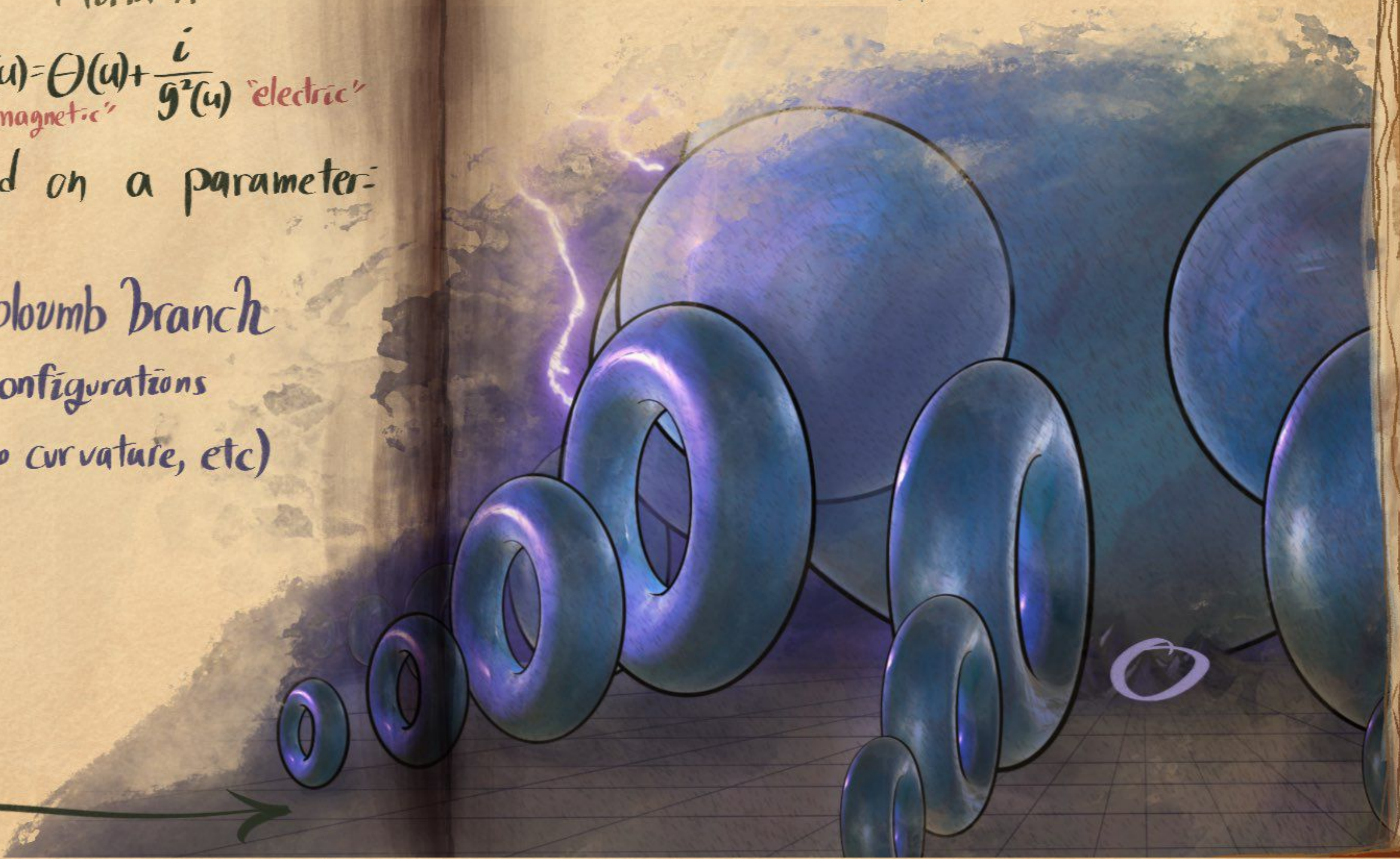
(i.e zero curvature, etc)



Symmetries: $\tau \mapsto \tau + 1$

$\tau \mapsto -1/\tau$ electric-Magnetic duality

$(\theta = 0 \Rightarrow g \mapsto 1/g)$



$\tau \mapsto \tau+1$
$\tau \mapsto -1/\tau$ generate $SL(2,\mathbb{Z})$ action

$\{1, \tau\}$ forms a lattice!!

τ naturally

Paramertizes genus 1 Riemann surface (elliptic curve)

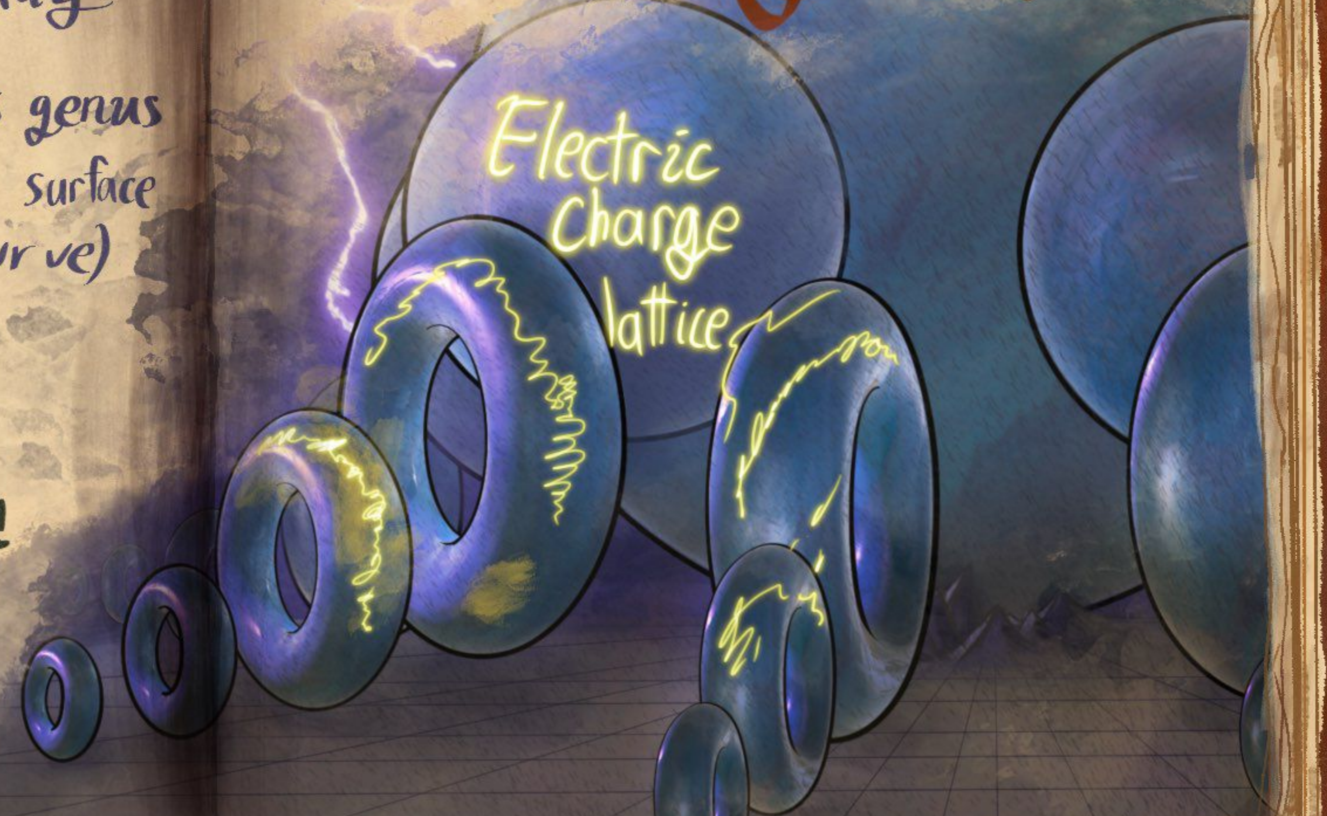
$\Rightarrow \tau(u)$ gives elliptic curve fibration over $\mathbb{C}$!

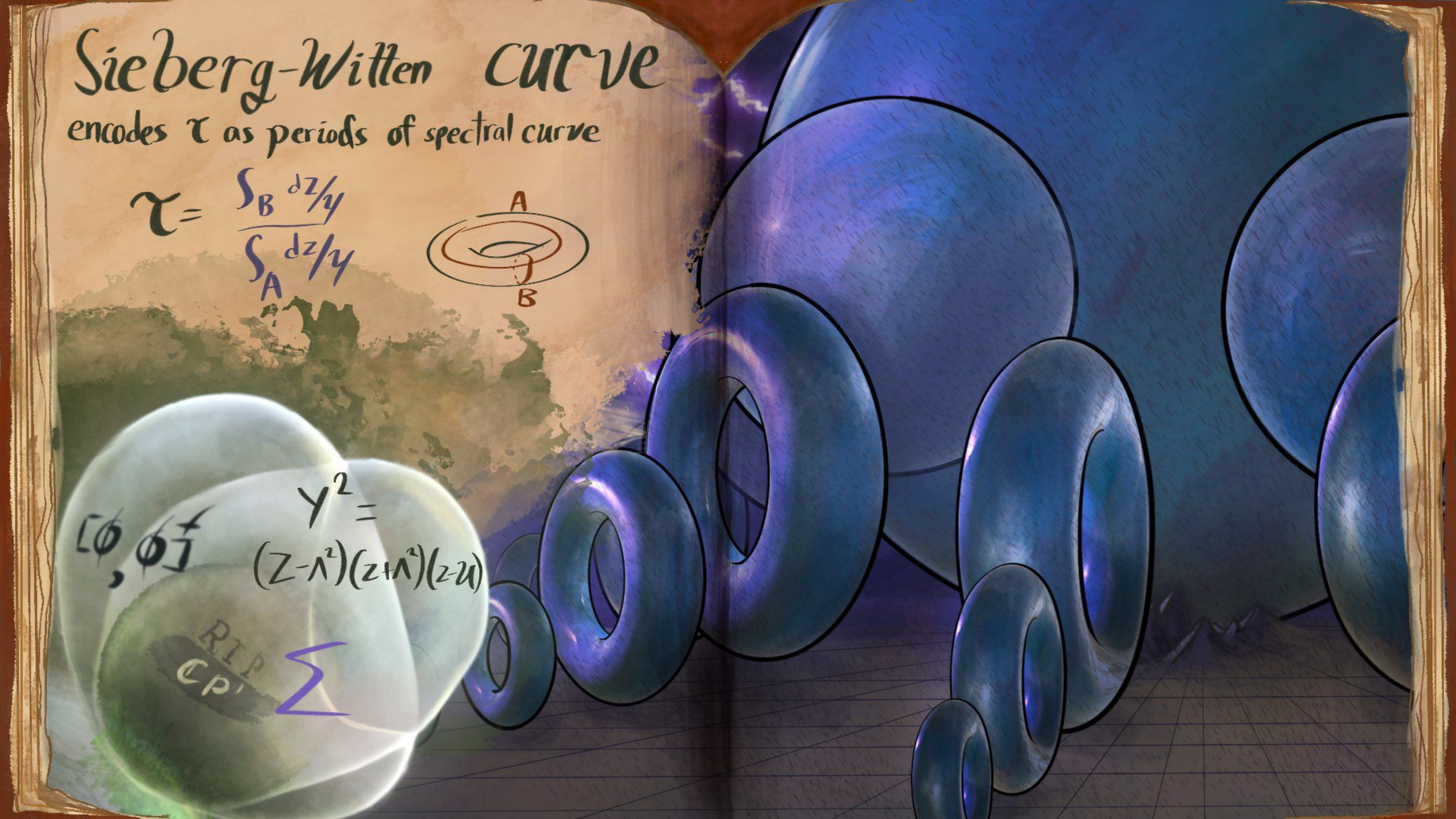
Over zero: renormalization breaks $\Rightarrow$ fibers degerate

carries natural Kähler structure

fiber tori Lagrangian

Integrable!





Sieberg-Witten curve
encodes τ as periods of spectral curve
$\tau = \frac{\oint_B dz/y}{\oint_A dz/y}$
A
B
$[\phi, \phi^\dagger]$
$y^2 = (z-\Lambda^2)(z+\Lambda^2)(z-u)$
RIP $\mathbb{C}P^1$
Σ