Vignettes on Caustics

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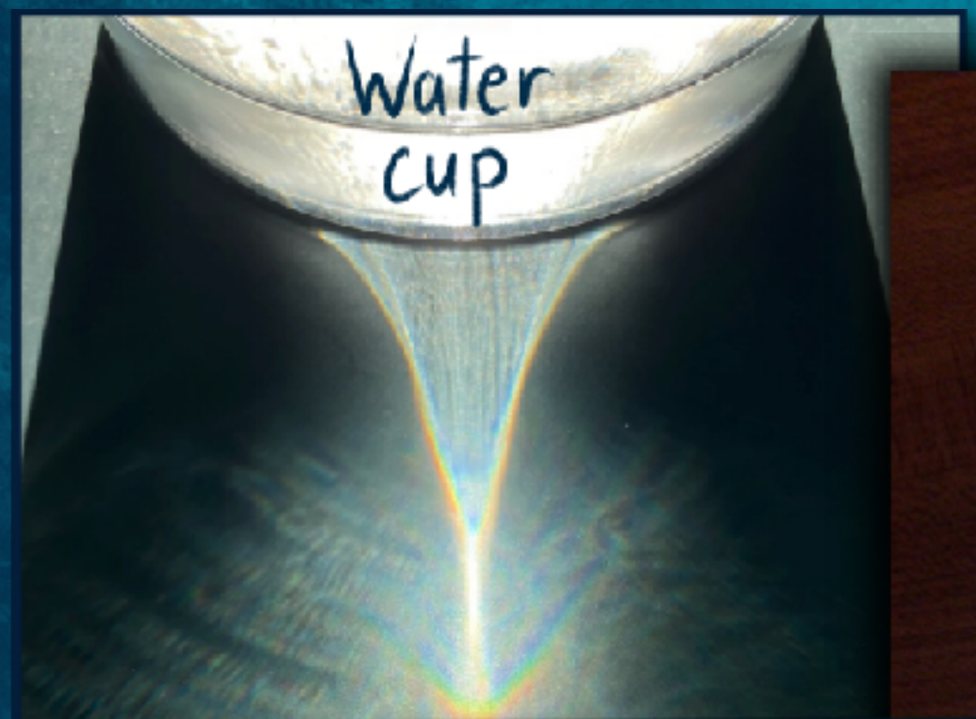
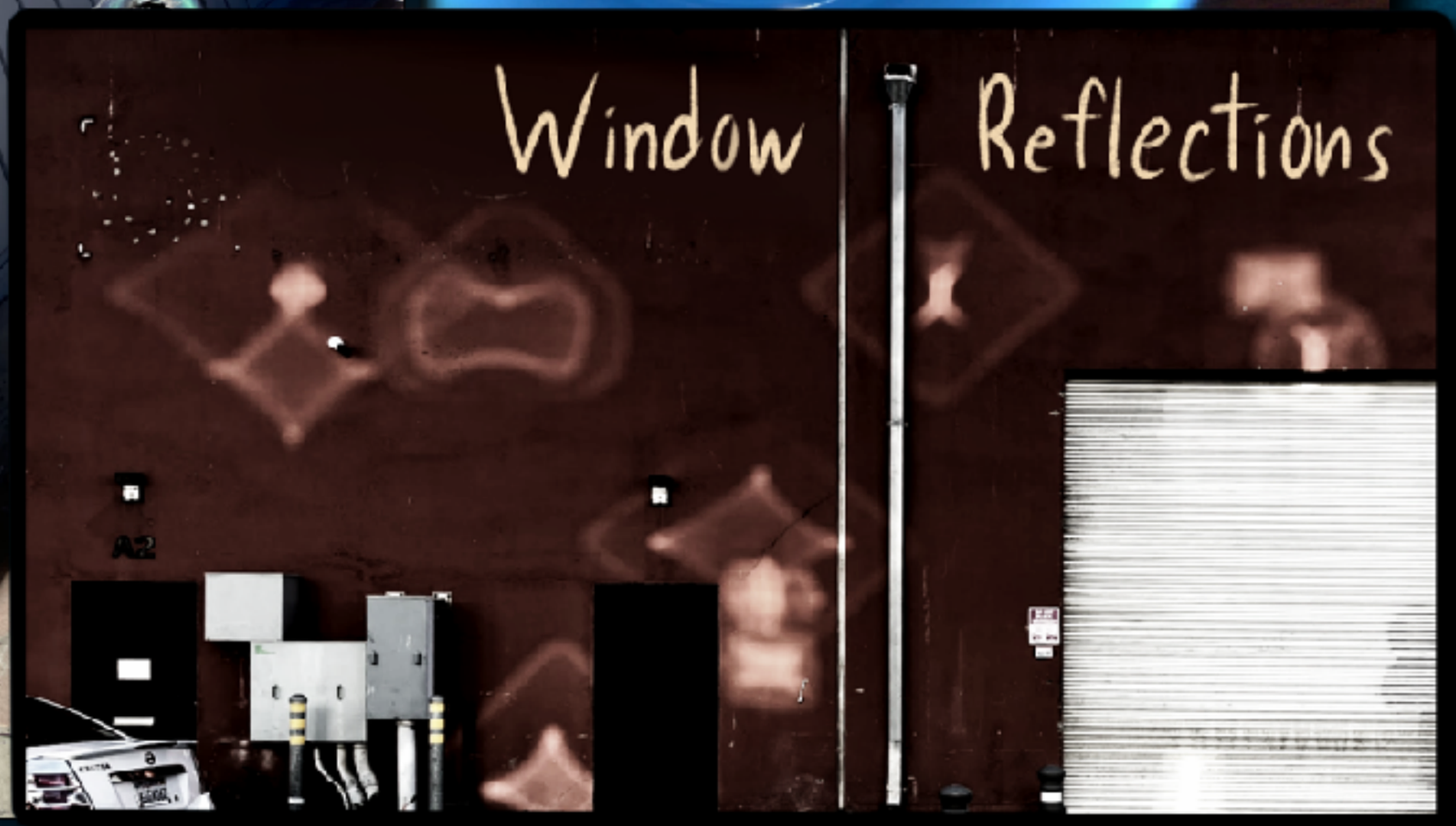
Catastrophes


Water
cup


Coffee
mug


Bike Wheel


Window
Reflections
A2

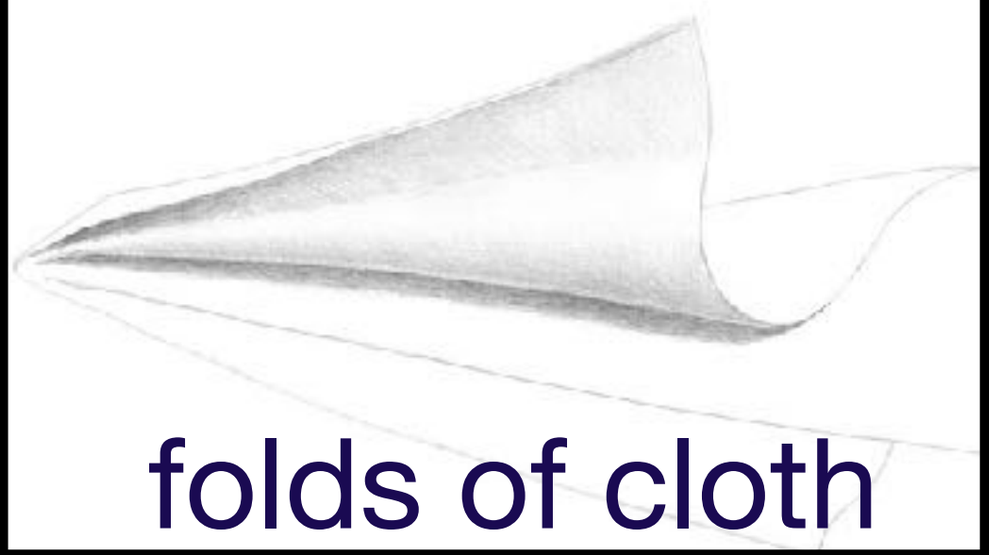
Caustics

appears where many rays of light pass through the same point

from Greek καυστός To Burn

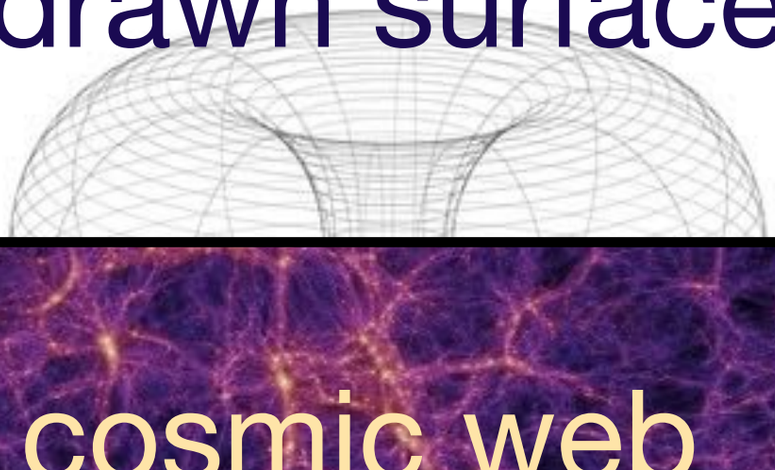
Real life caustics

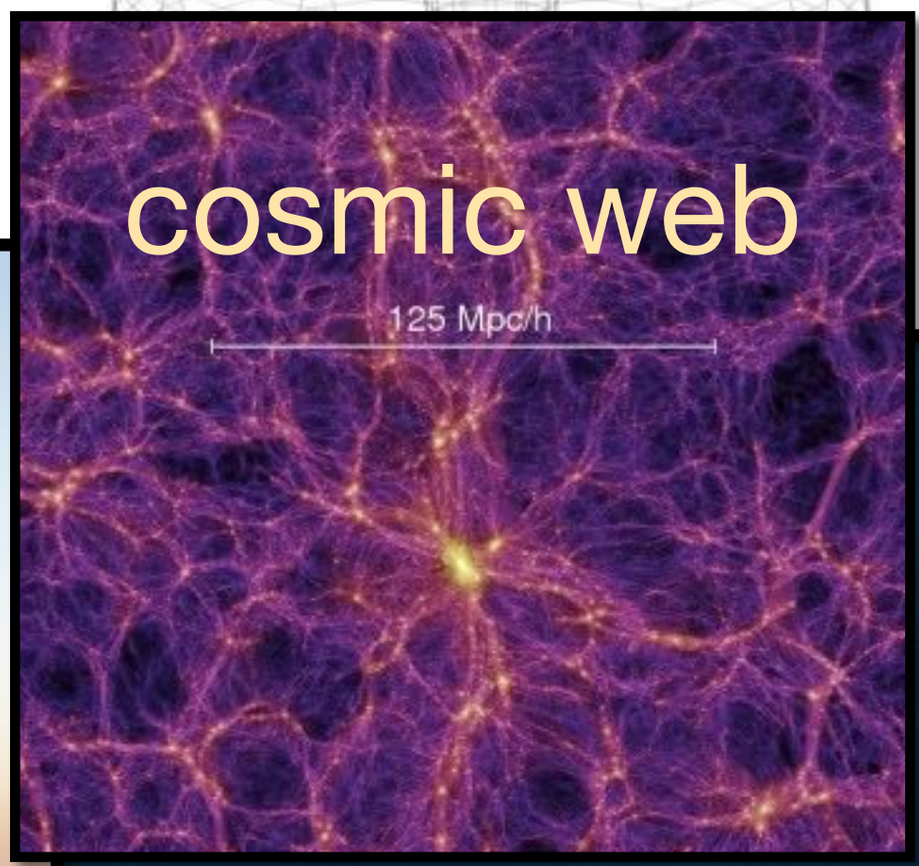
Caustics appear in many guises...



folds of cloth

drawn surfaces



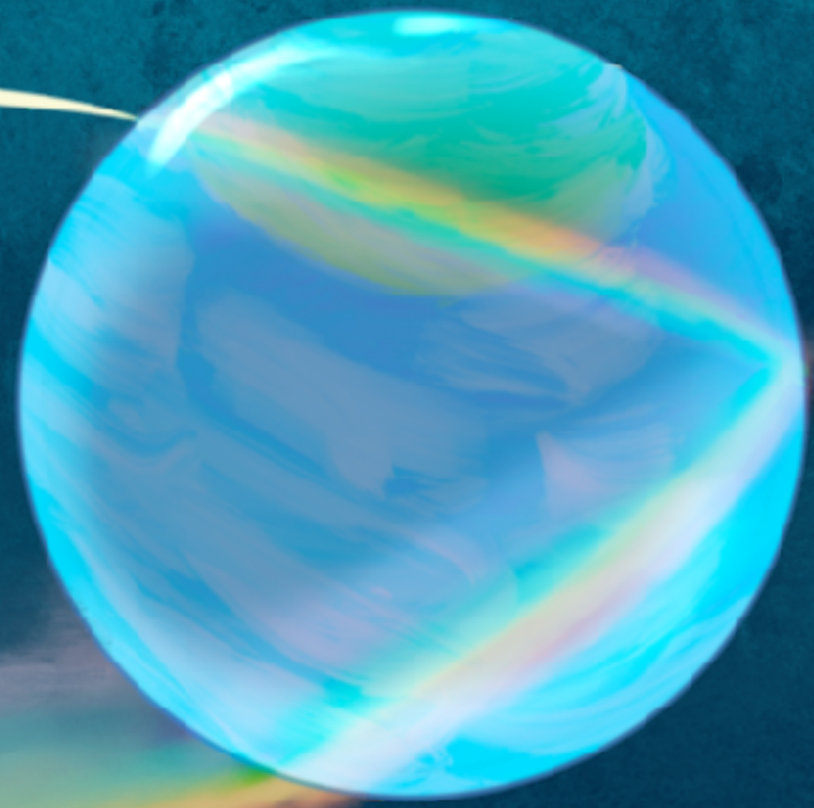


bird flocks

Today: a unifying Geometric explanation

Vignette 1:

Rainbows are folded light



Caustics in folded cloth

Density of a fold:

① lift to graph

② project to y-coord

$f(x) = 4x(x-1)$

0 1

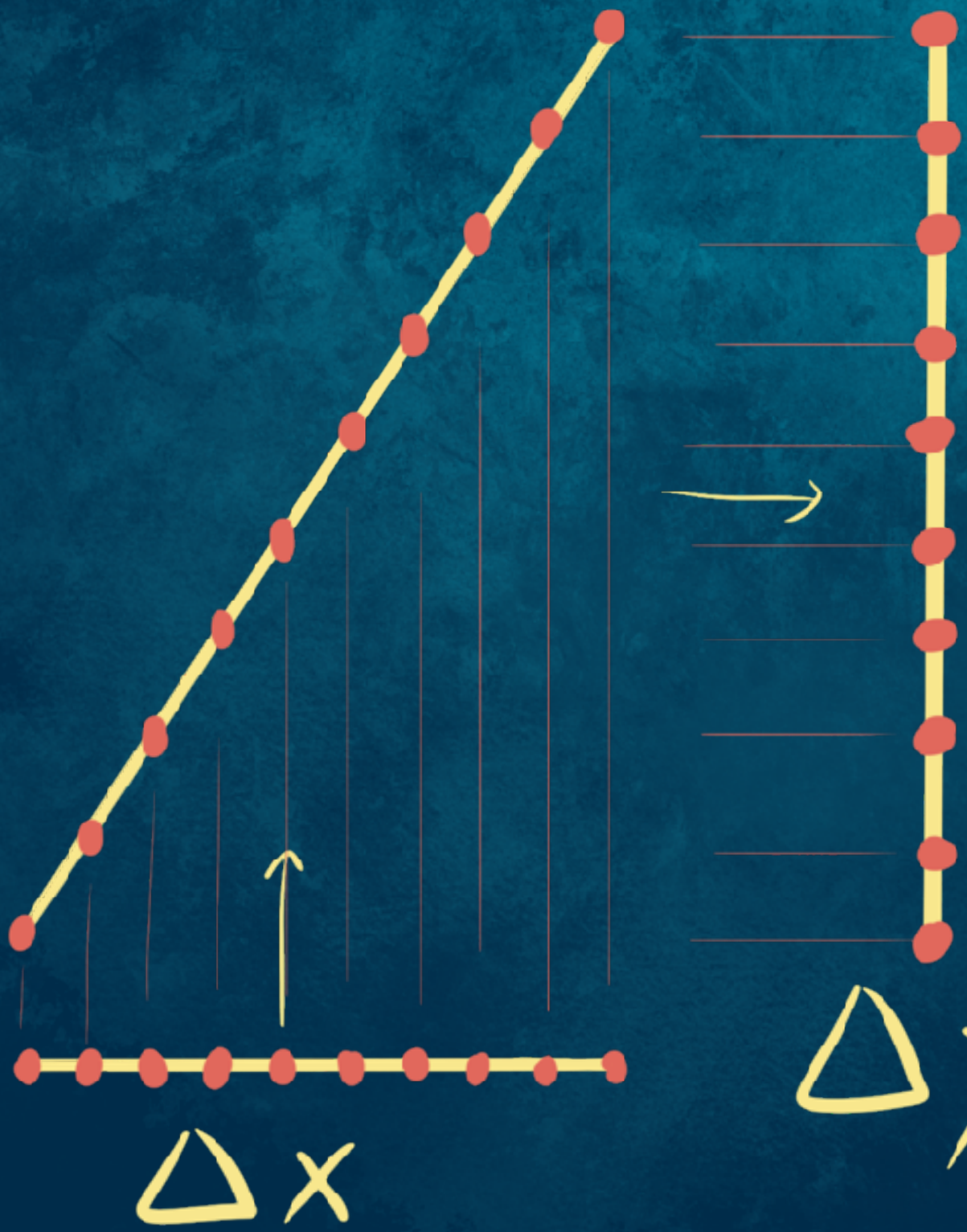
0 density

large density

small density

0

Local picture: approximate graph by line



Amount of "stuff" = Δx

spread over distance Δy

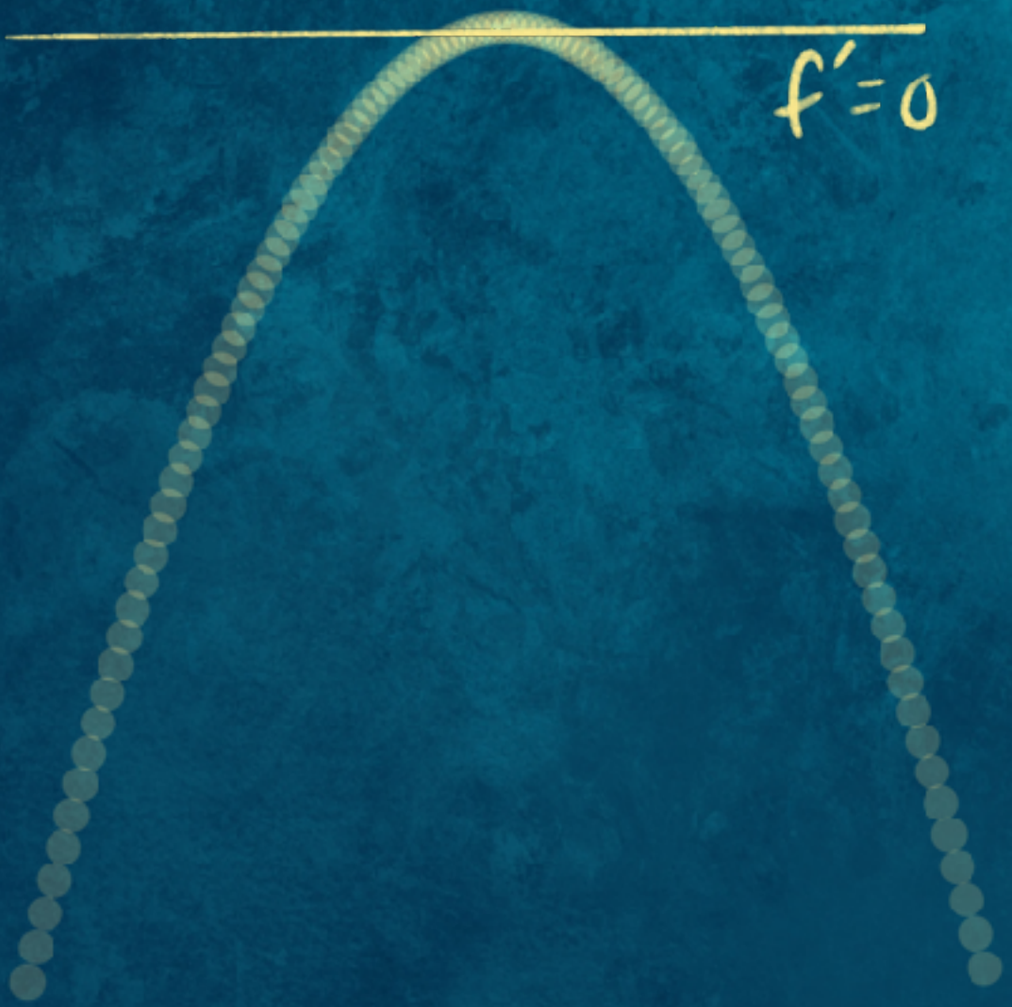
$$\text{Density} = \frac{\text{Stuff}}{\text{distance}} = \frac{\Delta x}{\Delta y}$$

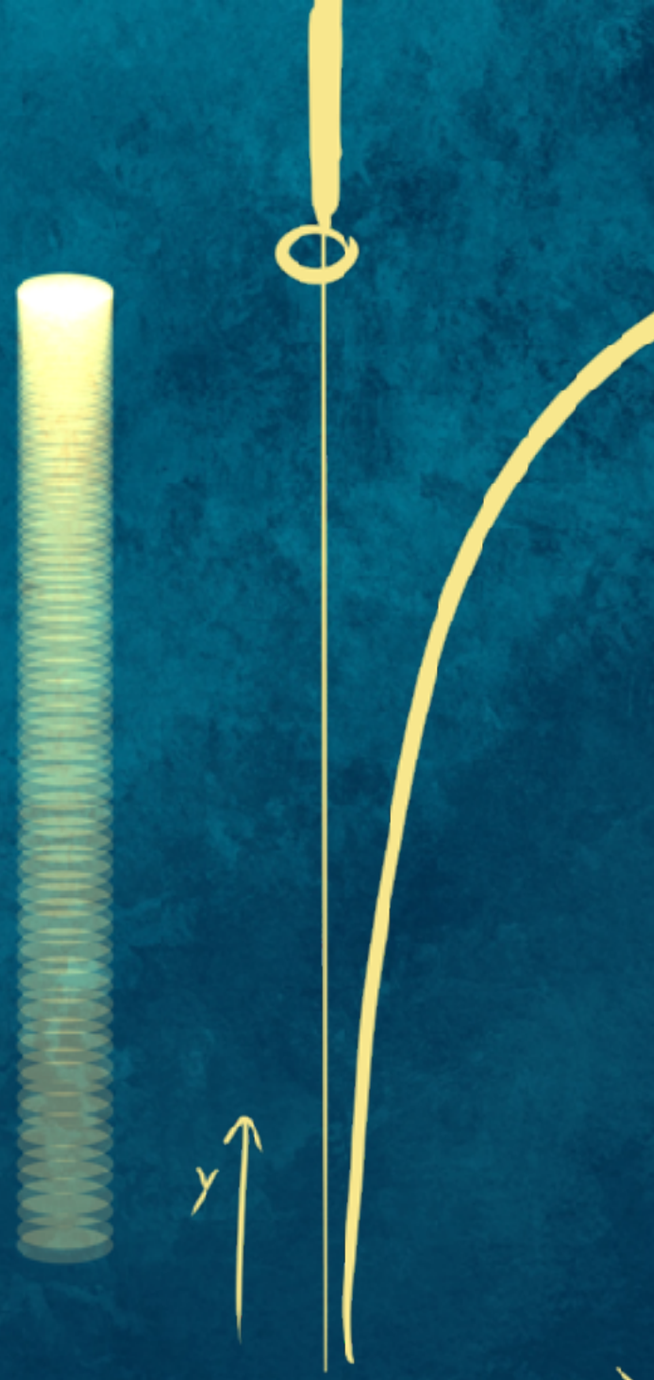
$$= 1/\text{Slope}$$

$$\text{Density} = \frac{1}{f'(x)}$$

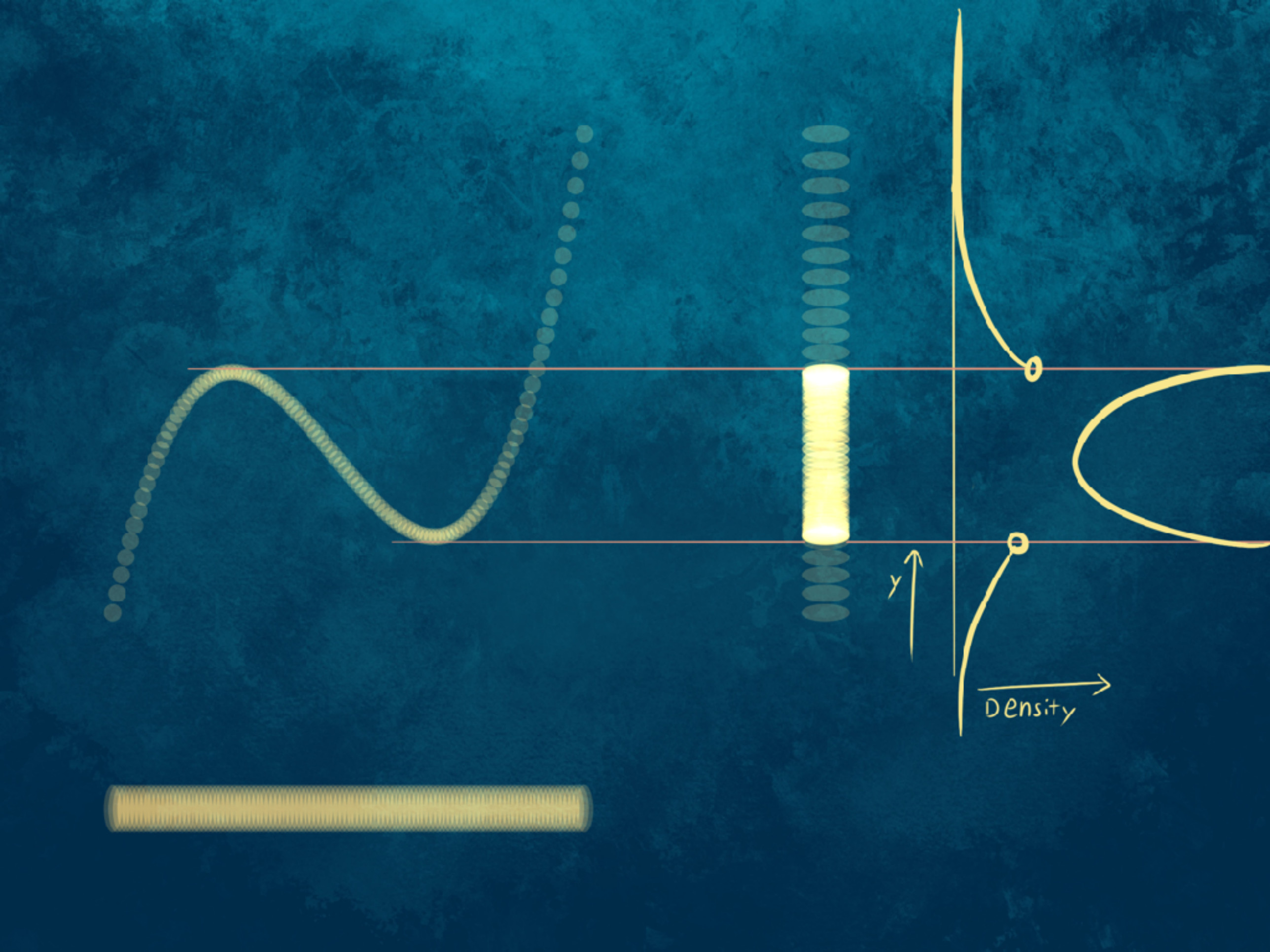
Density $\sim 1/f'(x)$

$f' = 0$



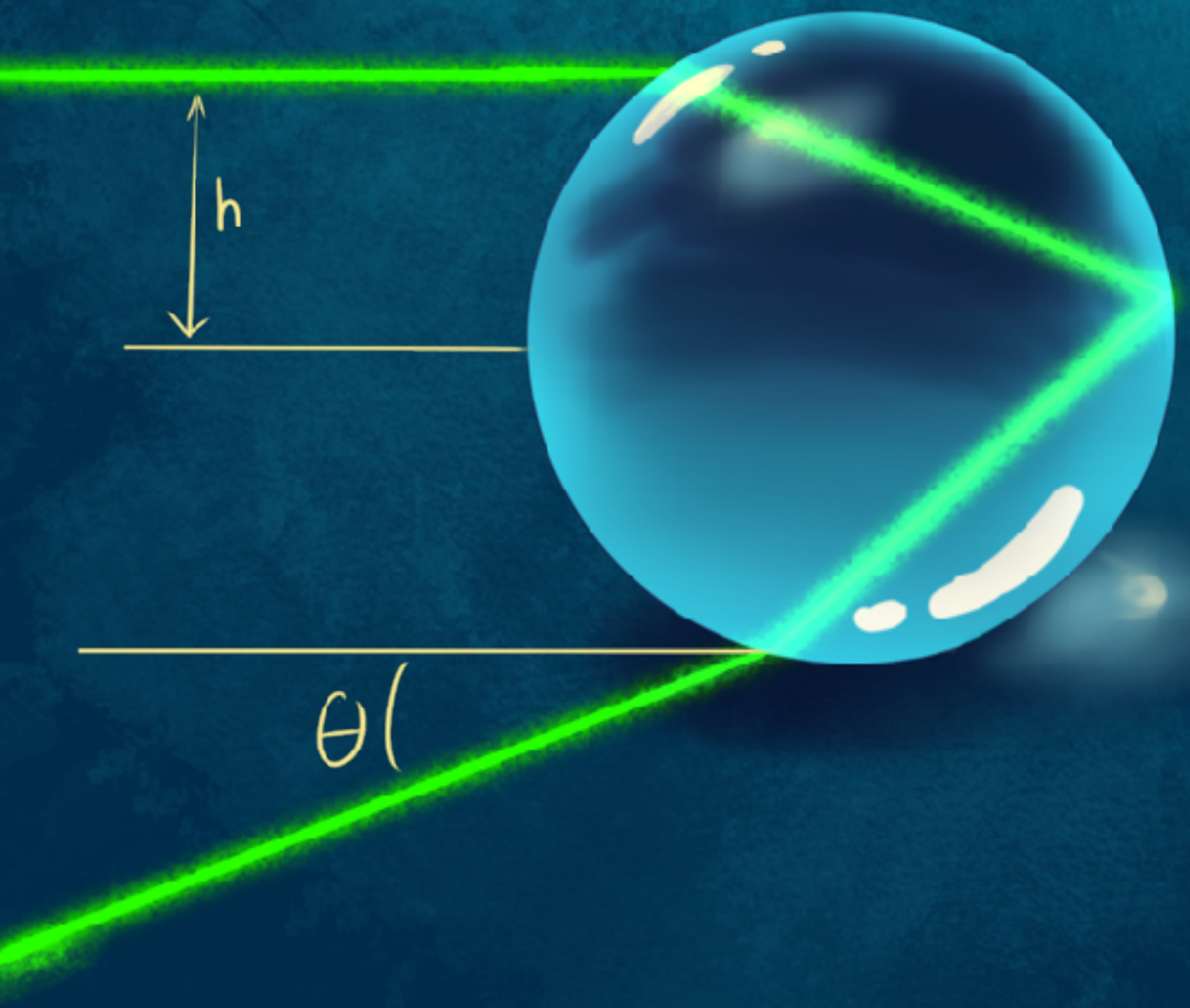


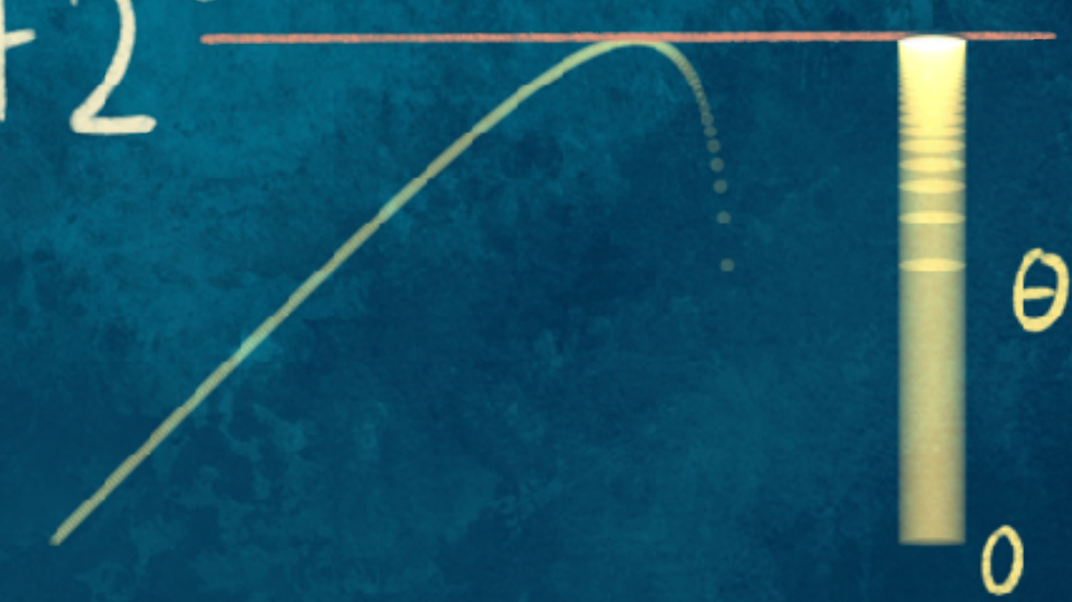
Density diverges whenever $f'(x) = 0$

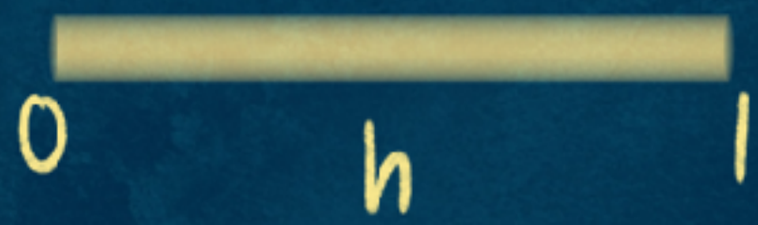

y
Density

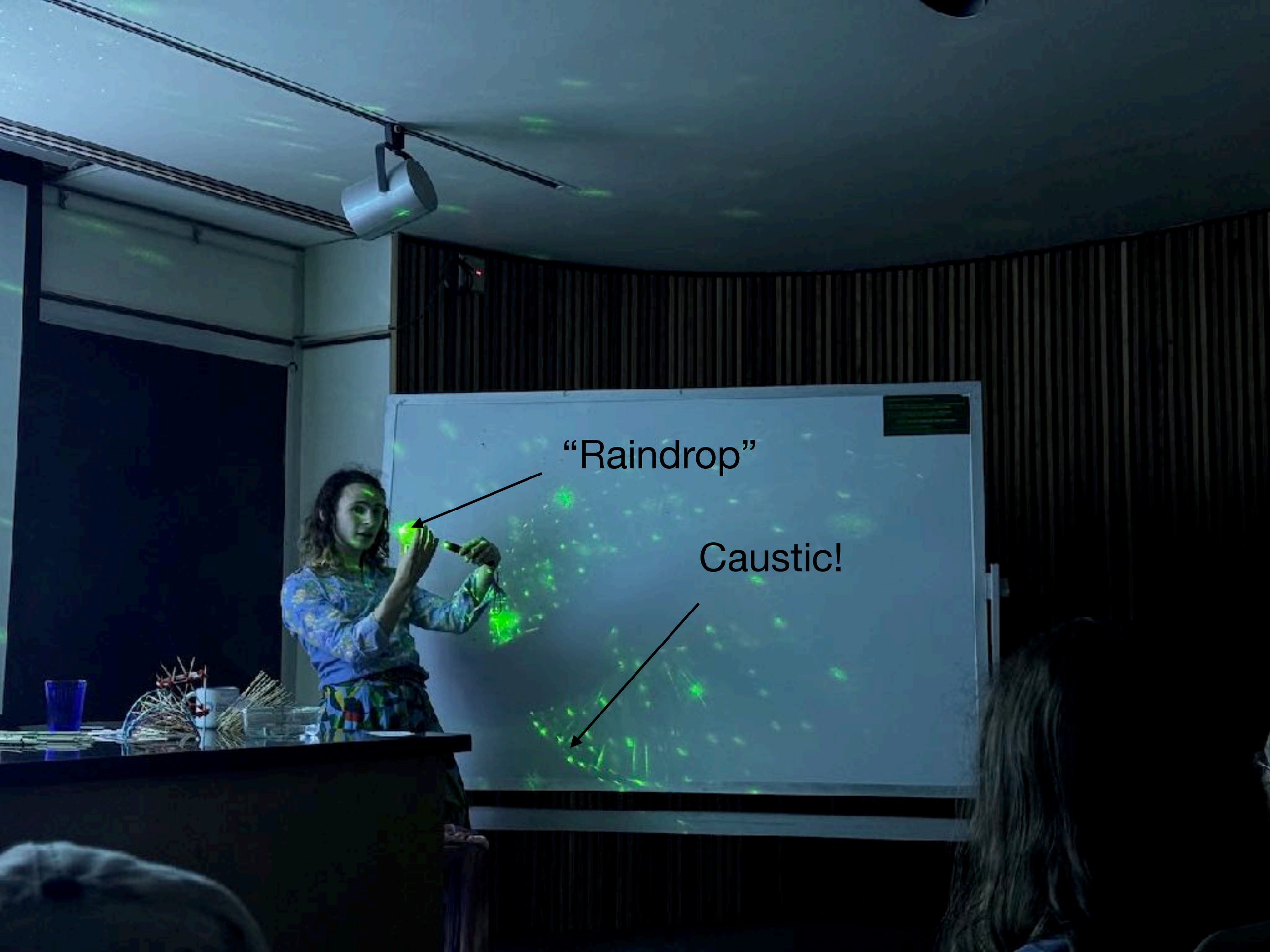
Rainbows are...

Caustics from raindrops





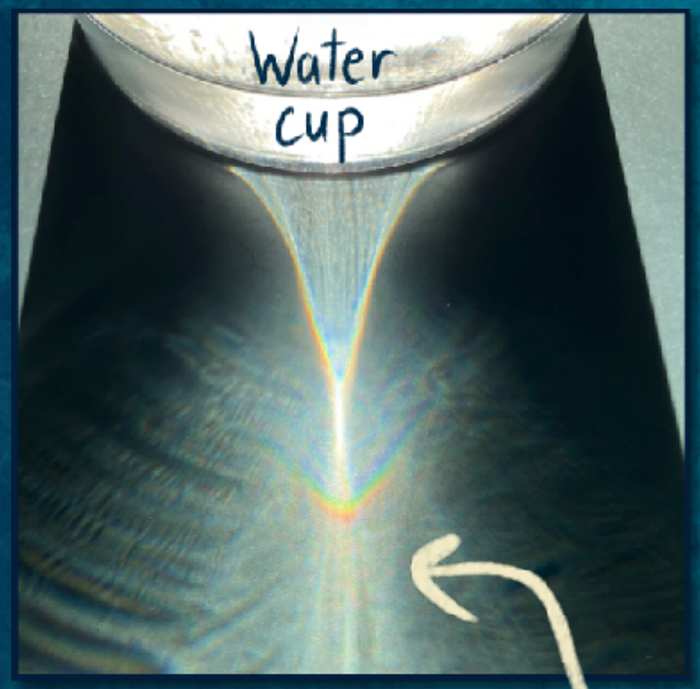



“Raindrop”
Caustic!

Vignette 2:

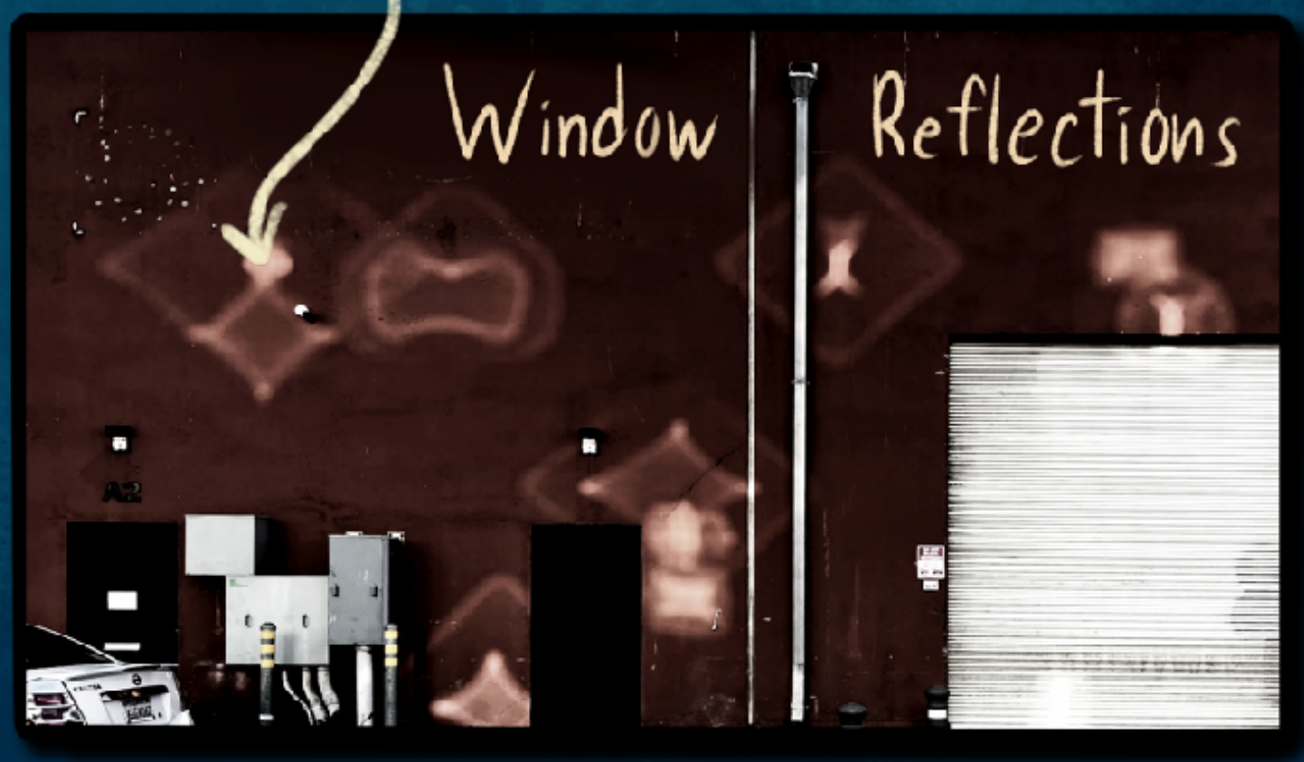
Classifying caustics

Cusps, Butterflies & Swallowtails

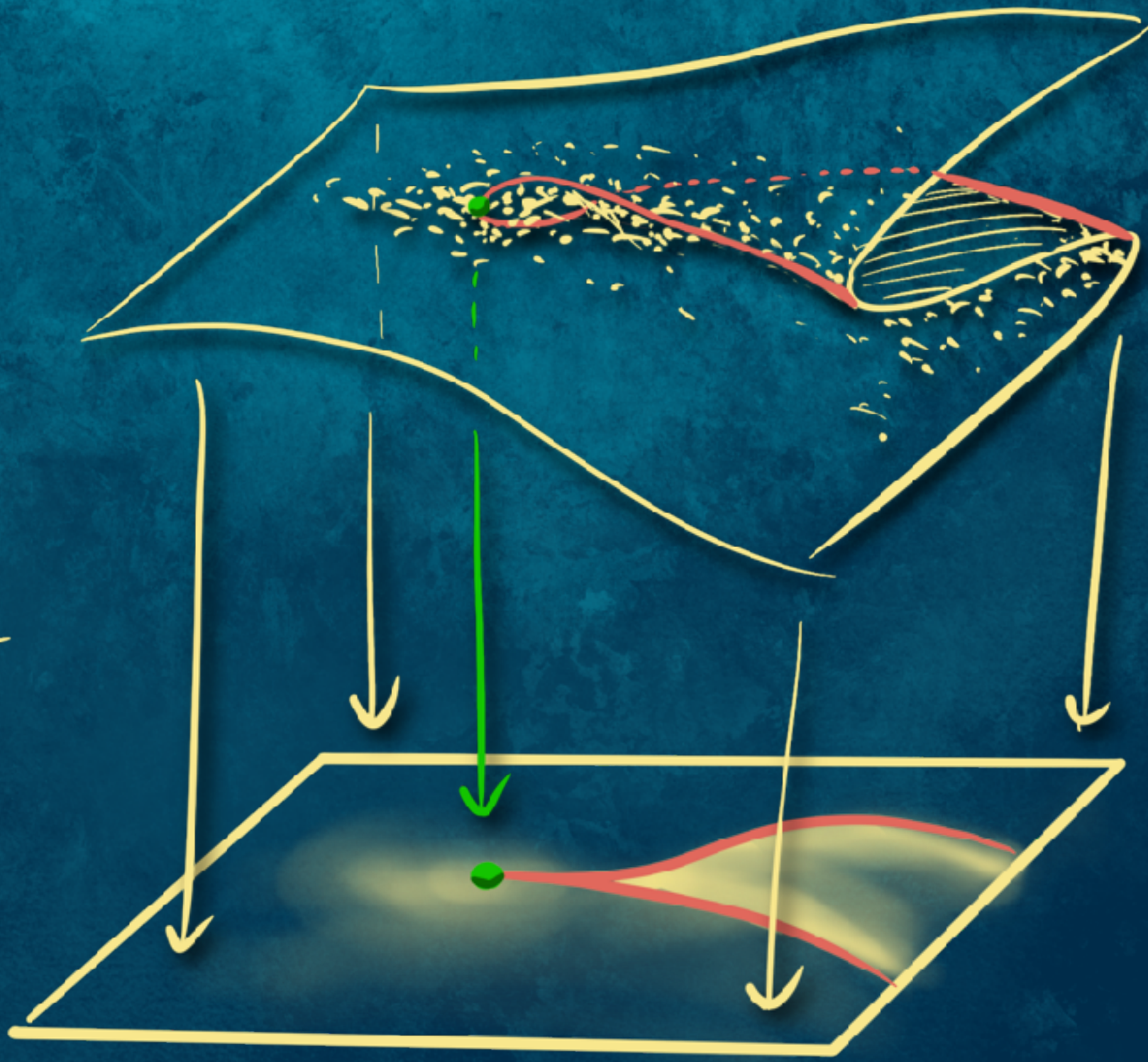


These are

Cusps

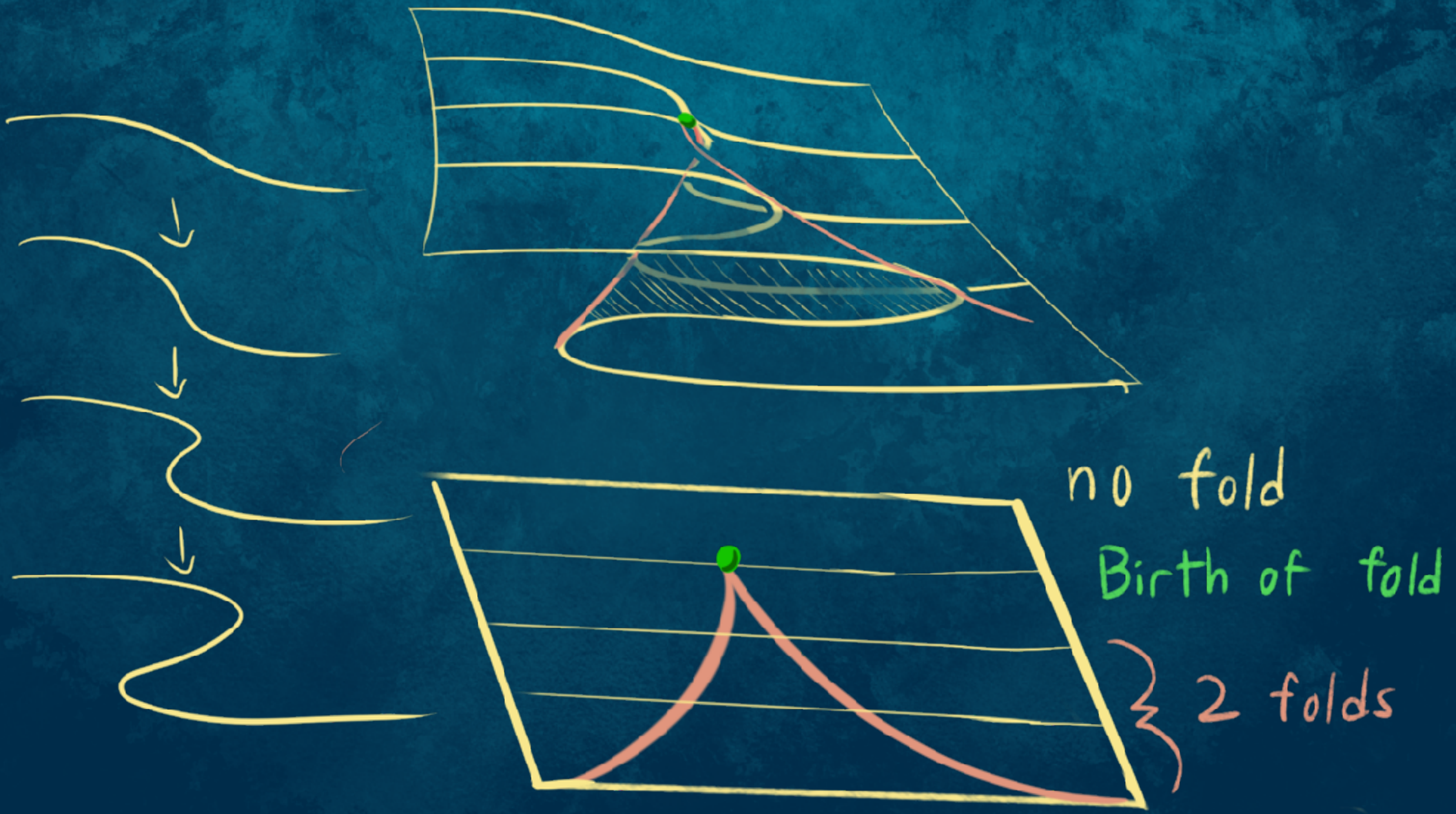


Cusp Caustic

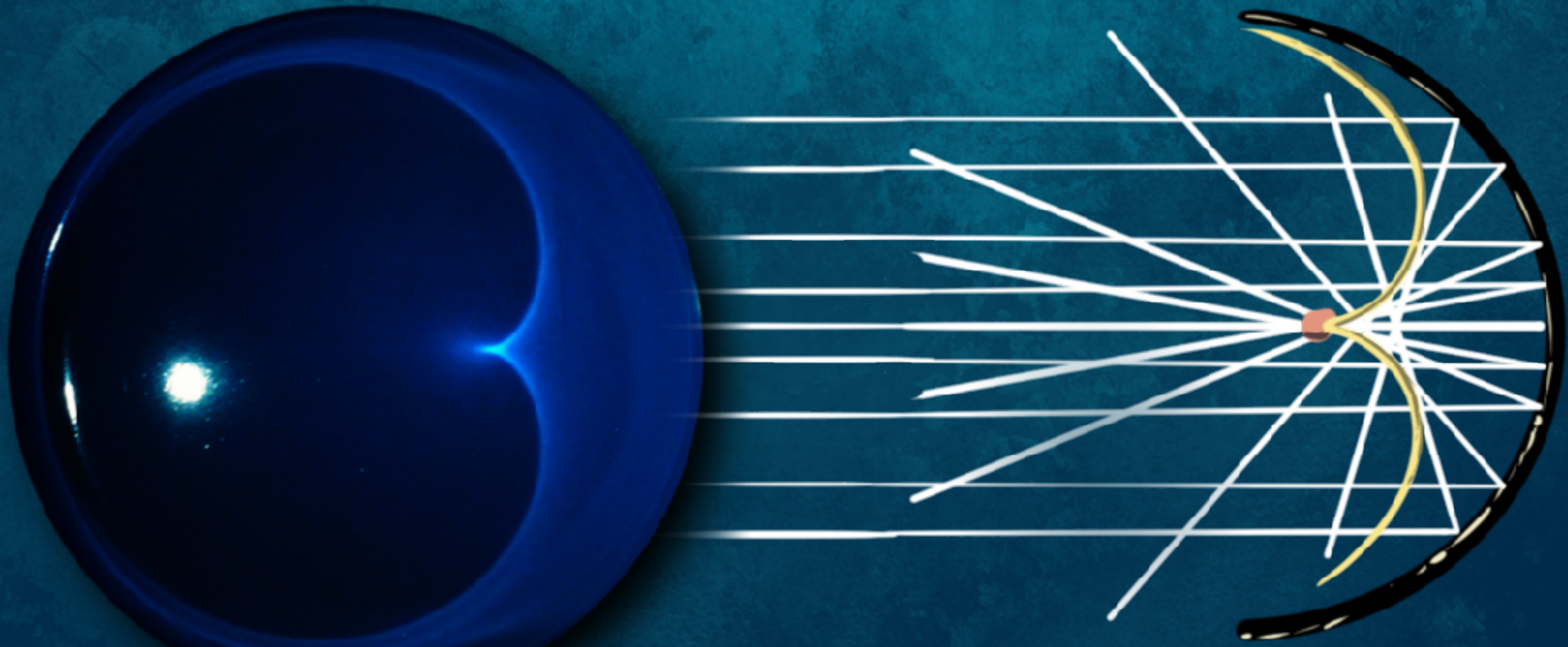


Cusps arise from projections of 2D sheets

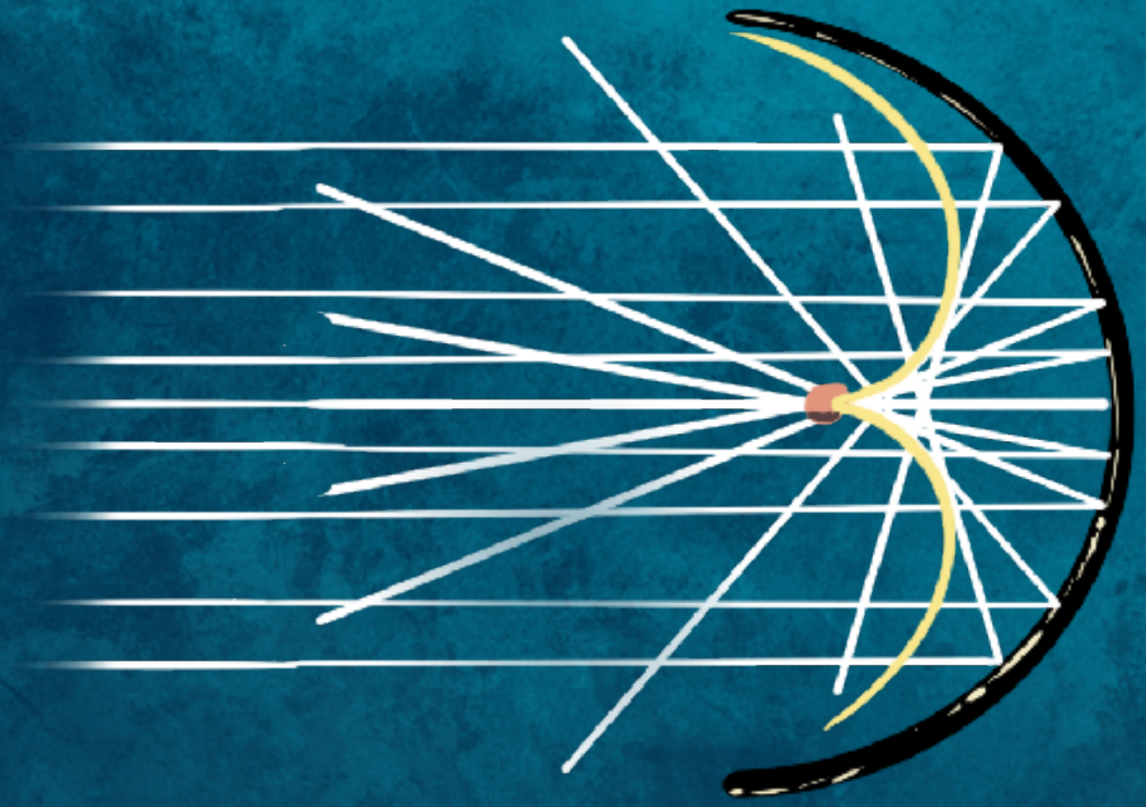
Cusp as Birth of 2 folds



Cusps of light



Definition: The Caustic of a family of lines is a curve tangent to each line



Where is the Sheet??

Keep track of position & direction of light

$$\mathcal{S} = \{(P, v) \mid P, v \in \mathbb{R}^2,\ \|v\| = 1\}$$

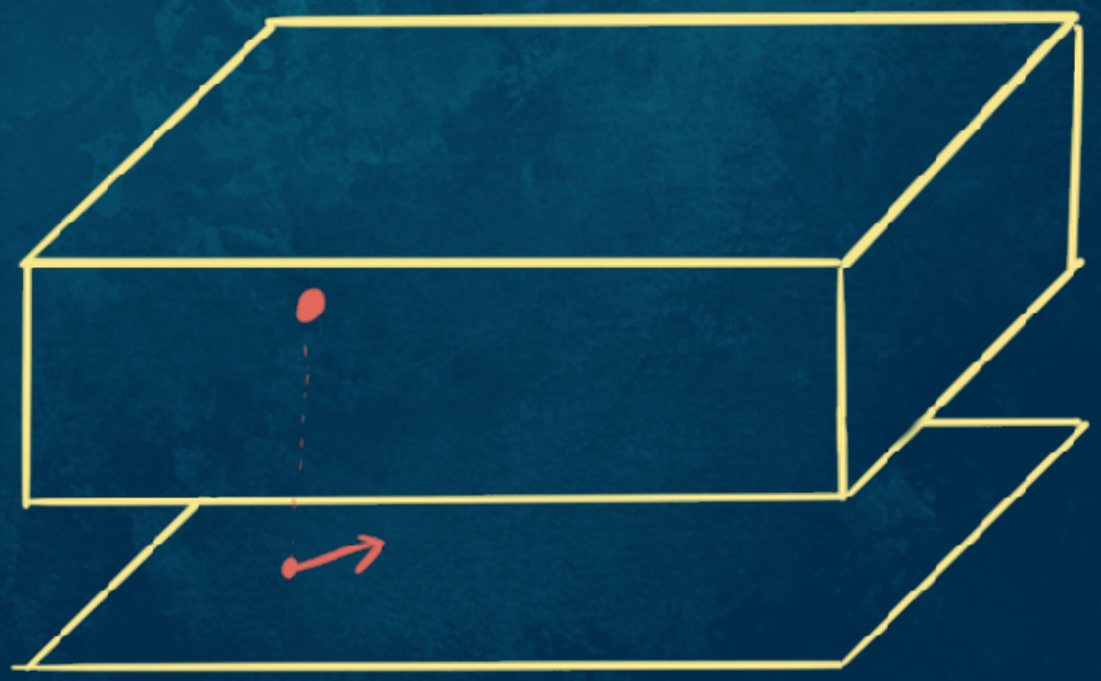
unit vector @ P

"phase space" $= \mathbb{R}^2 \times S^1$

draw circle as interval

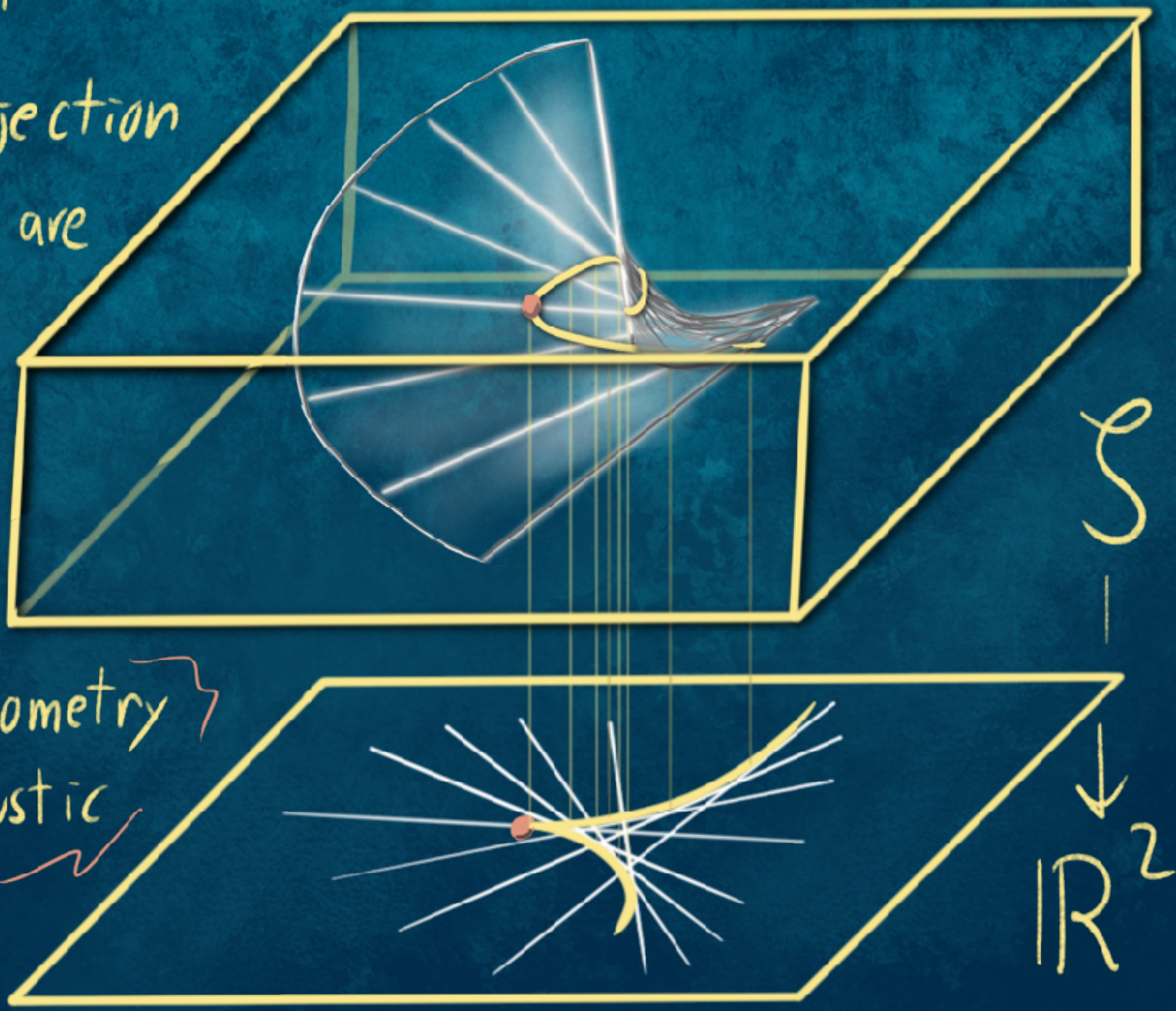
$[0,1]/0\sim1$ $\quad [0,1]$ $\quad \mathbb{R}^2$

$\mathcal{S}$ ↓



Light rays define Surface in $\mathcal{S}$

folds of projection from $\mathcal{S} \to \mathbb{R}^2$ are Caustics

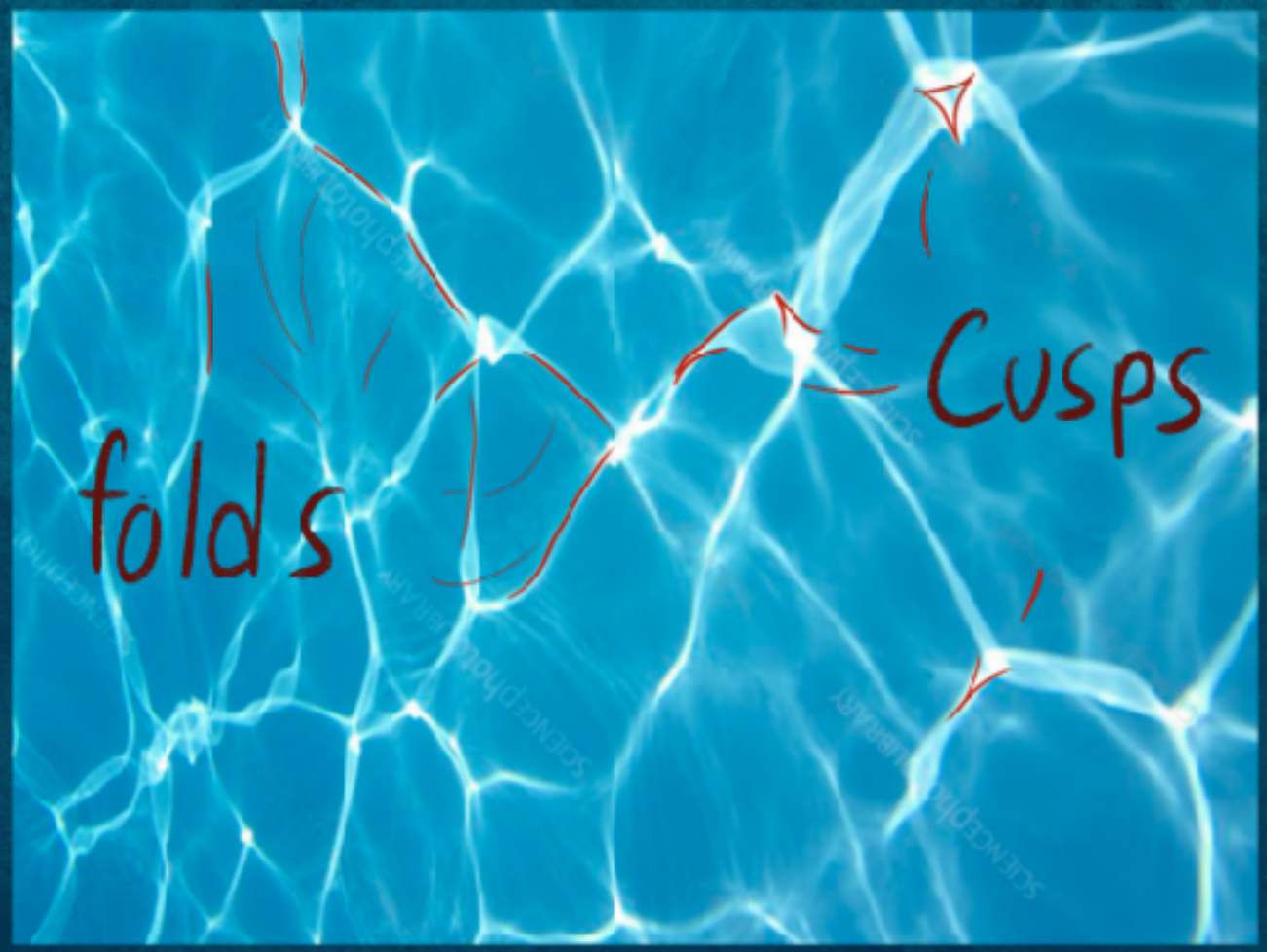


Hidden Geometry Governs Caustic


Cusp from shadow

Theorem (Whitney): folds & cusps are only stable 2 dimensional caustics

preserved under small changes



all cusps are the Same!!!

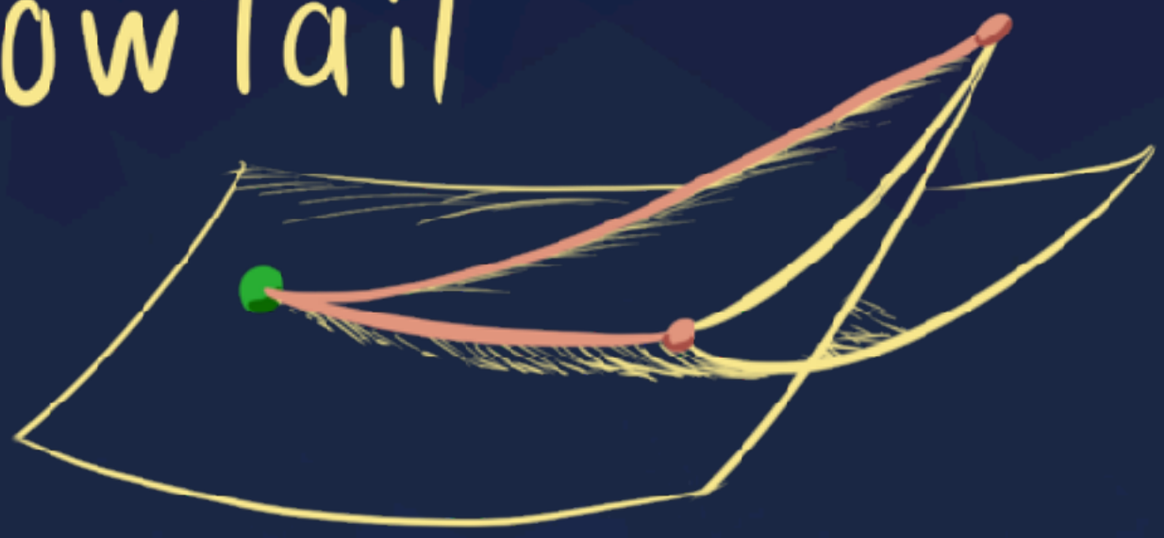
(up to change of coordinates)

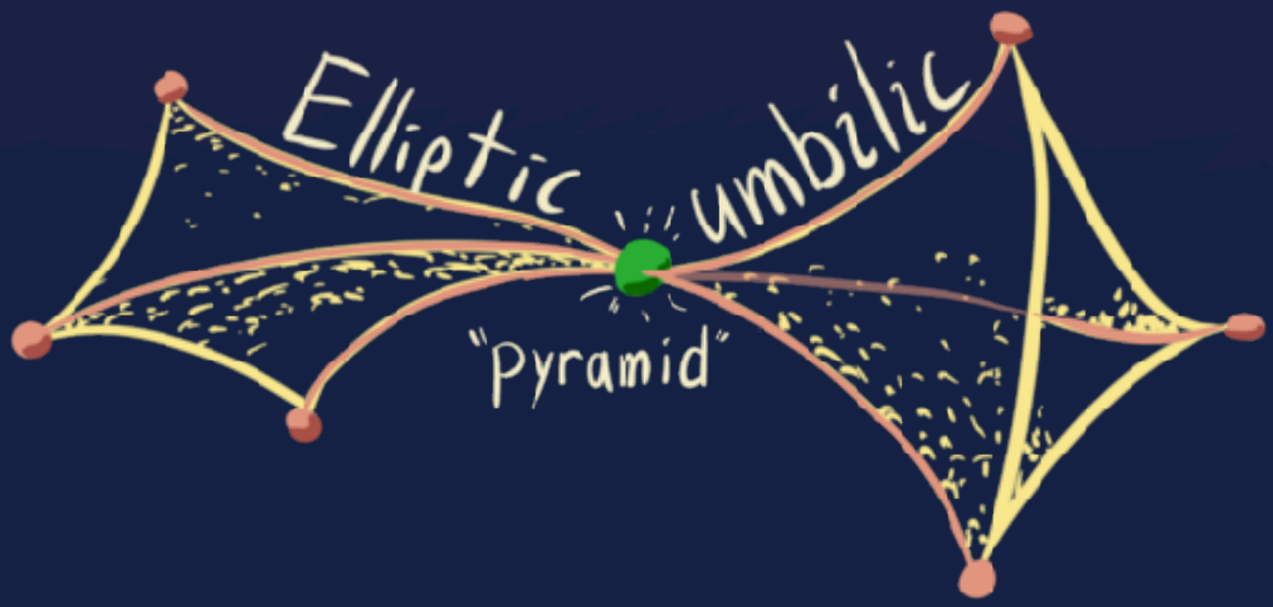
We understand 2D caustics...

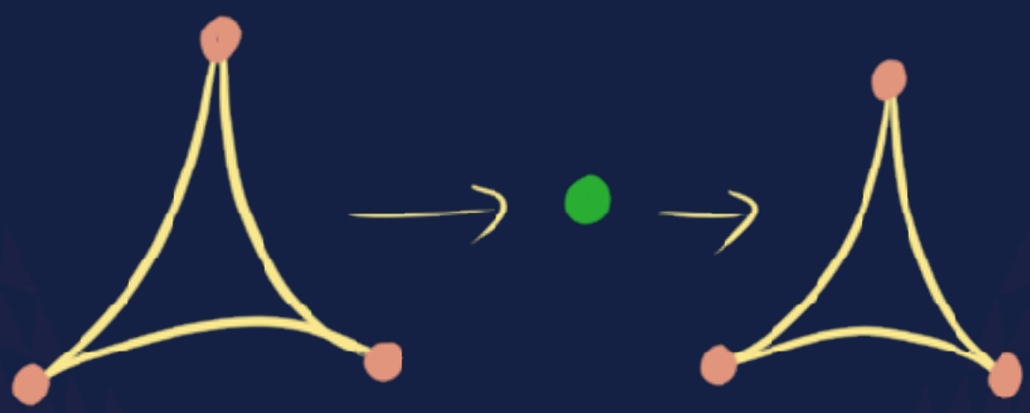
What about 3D?

3D cusp

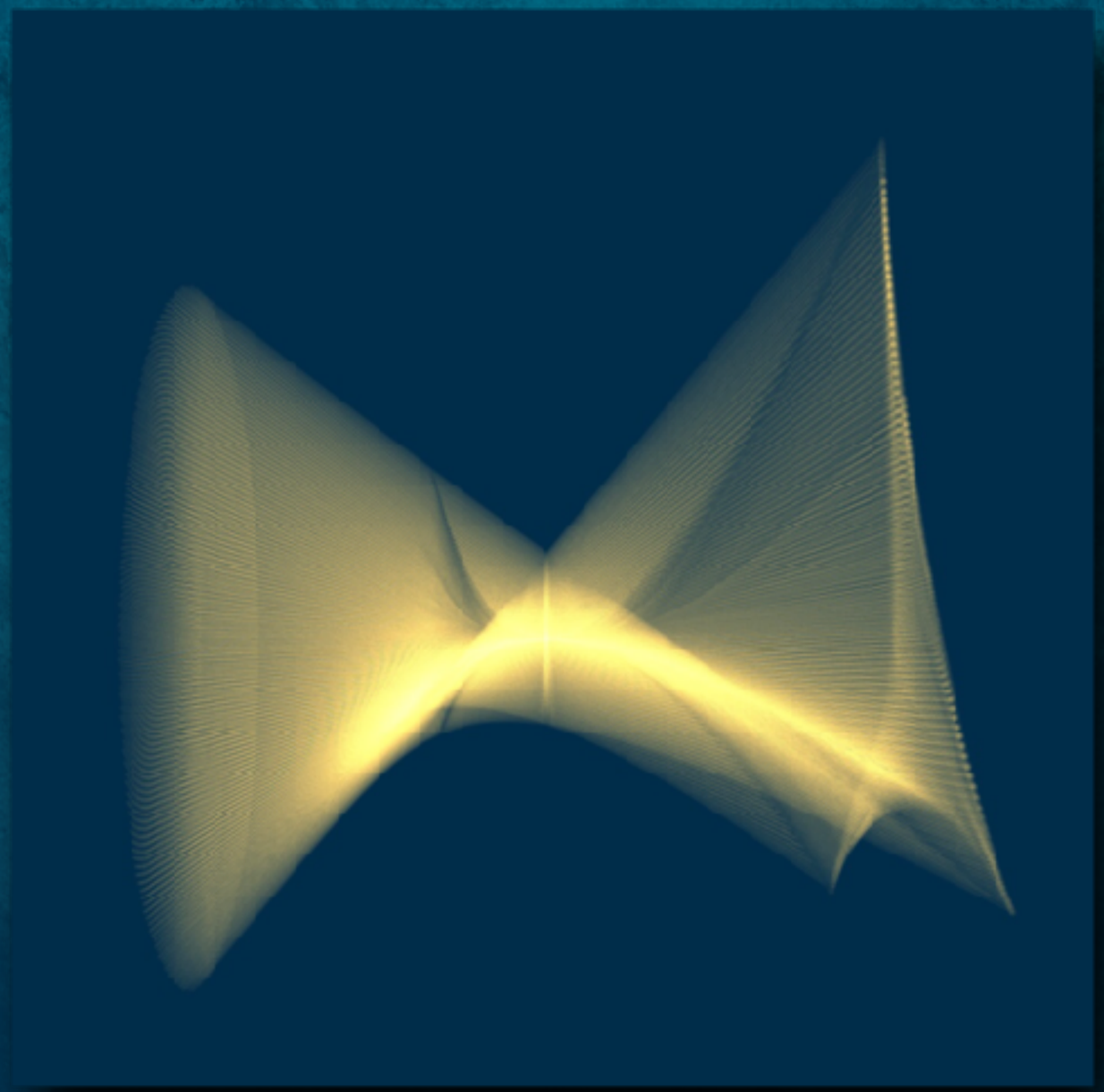
Swallowtail




Elliptic
umbilic
"Pyramid"

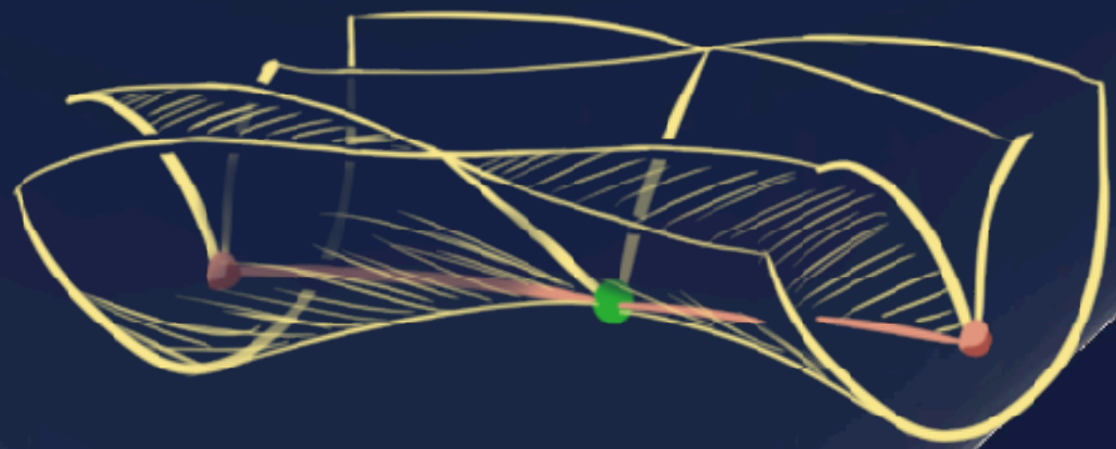


3D elliptic umbilic



https://openprocessing.org/sketch/2353684

Hyperbolic umbilic



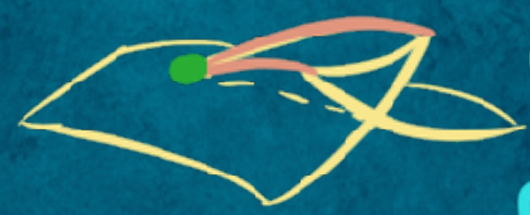
"Purse"

1D:

2D:

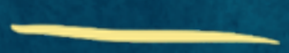
3D:

4D:



swallowtail

Butterfly



fold

cusp

Pyramid

Parabolic umbilic

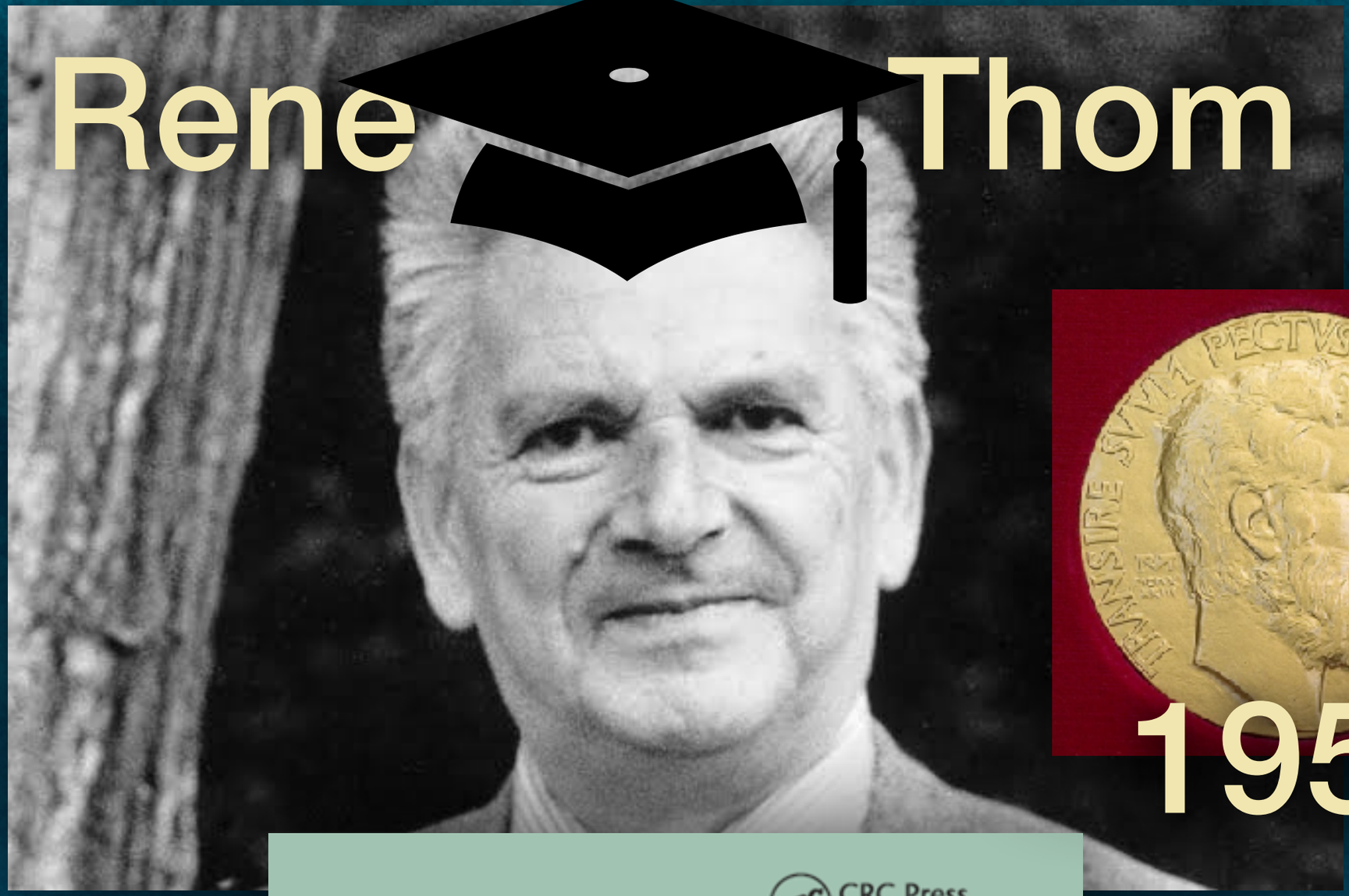
Purse

The Caustic Managerie

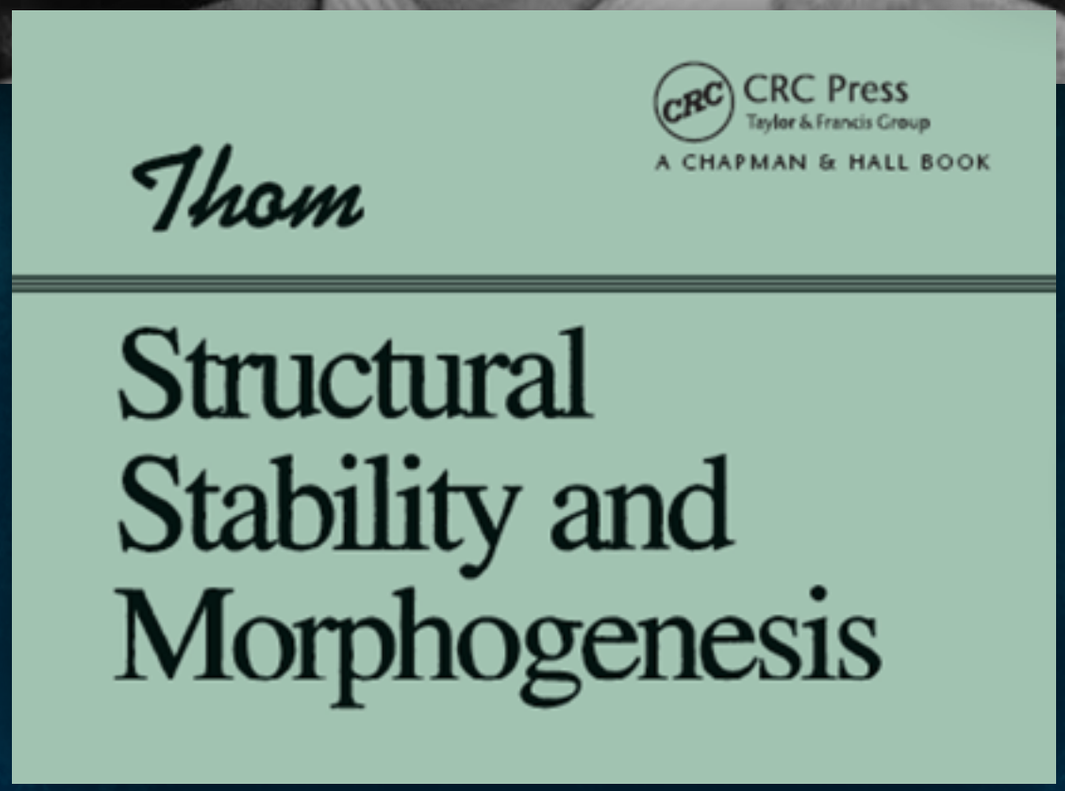
Vignette 3:

Catastrophe theory's Great catrastophe

Rene Thom



1958

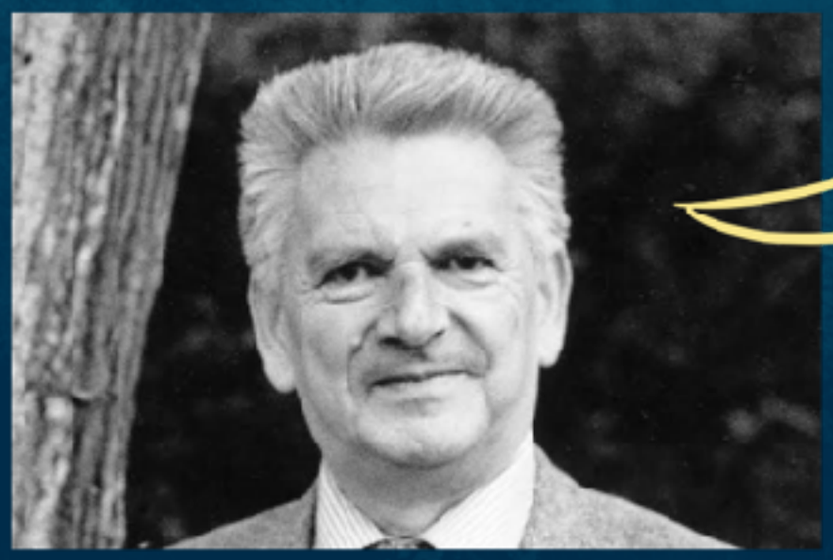


~1960s

Nature Has Discontinuities:

example: morphogenesis

how do we model abrupt changes in form??



use caustics

stable form

cusp

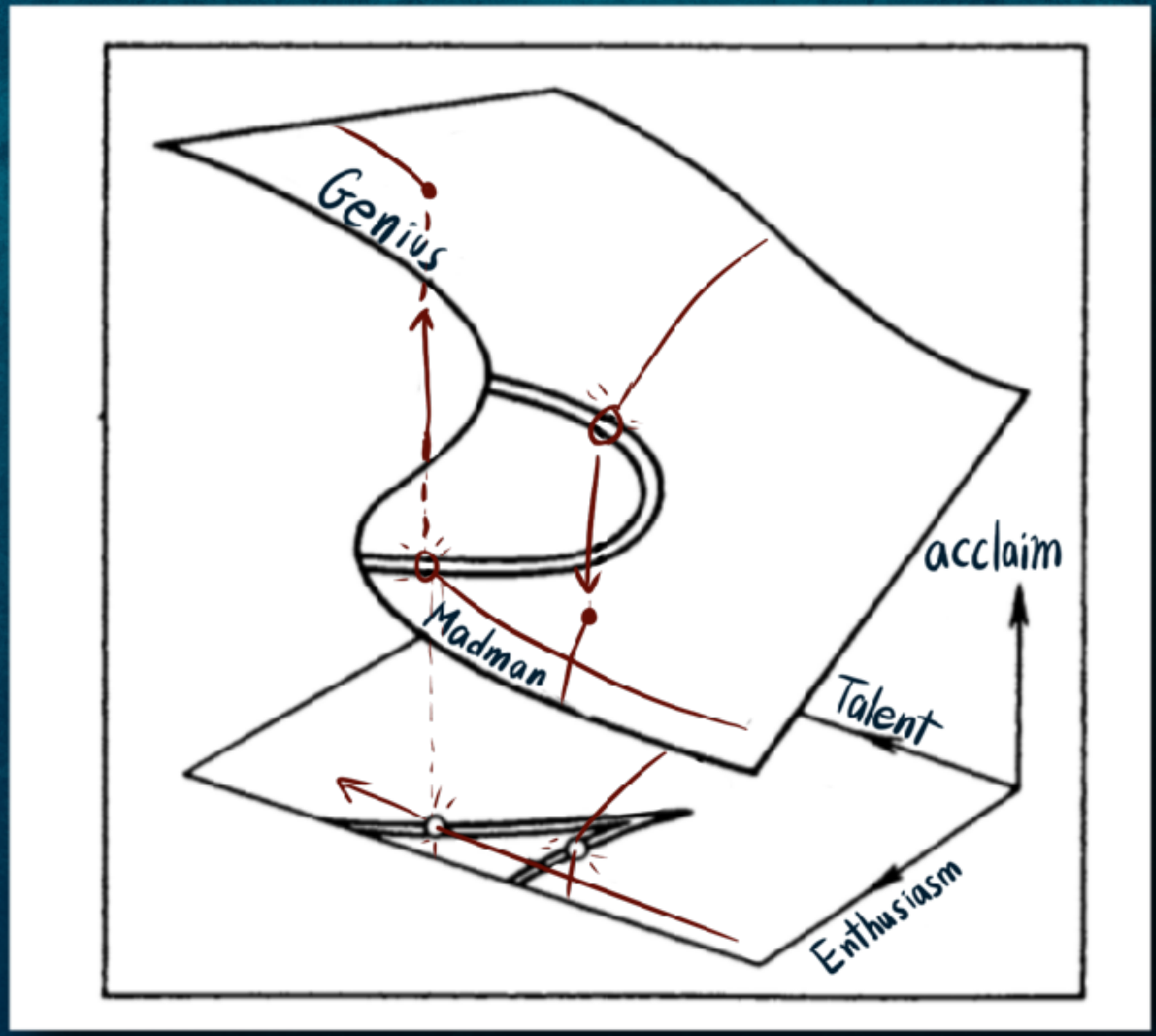
“There is little doubt that the epigenesis of a living being as a metabolic form resembles successive fibrations in Eilenberg-MacLane complexes more than cellular subdivision”

Thom, Structural stability and morphogenesis, p.201

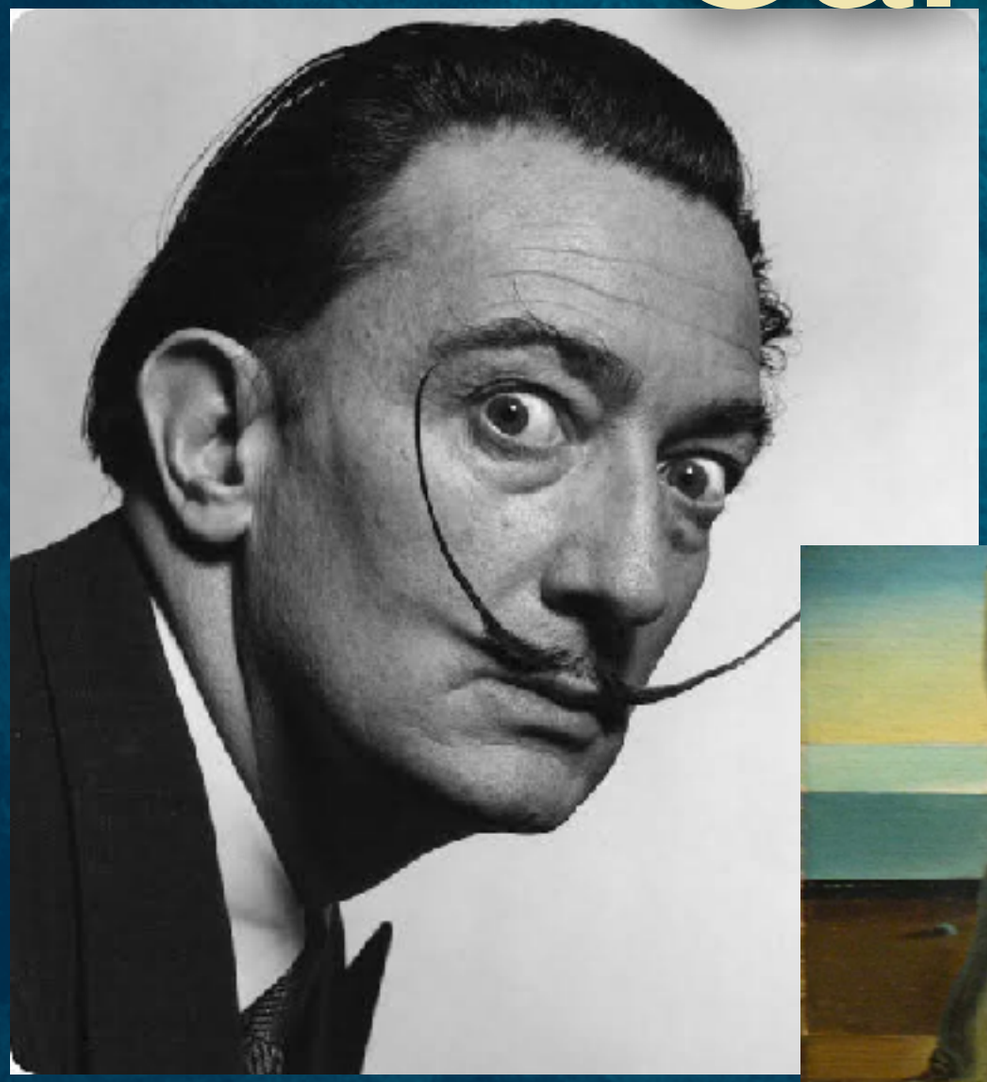
Christofer Zeeman

Catastrophe theory

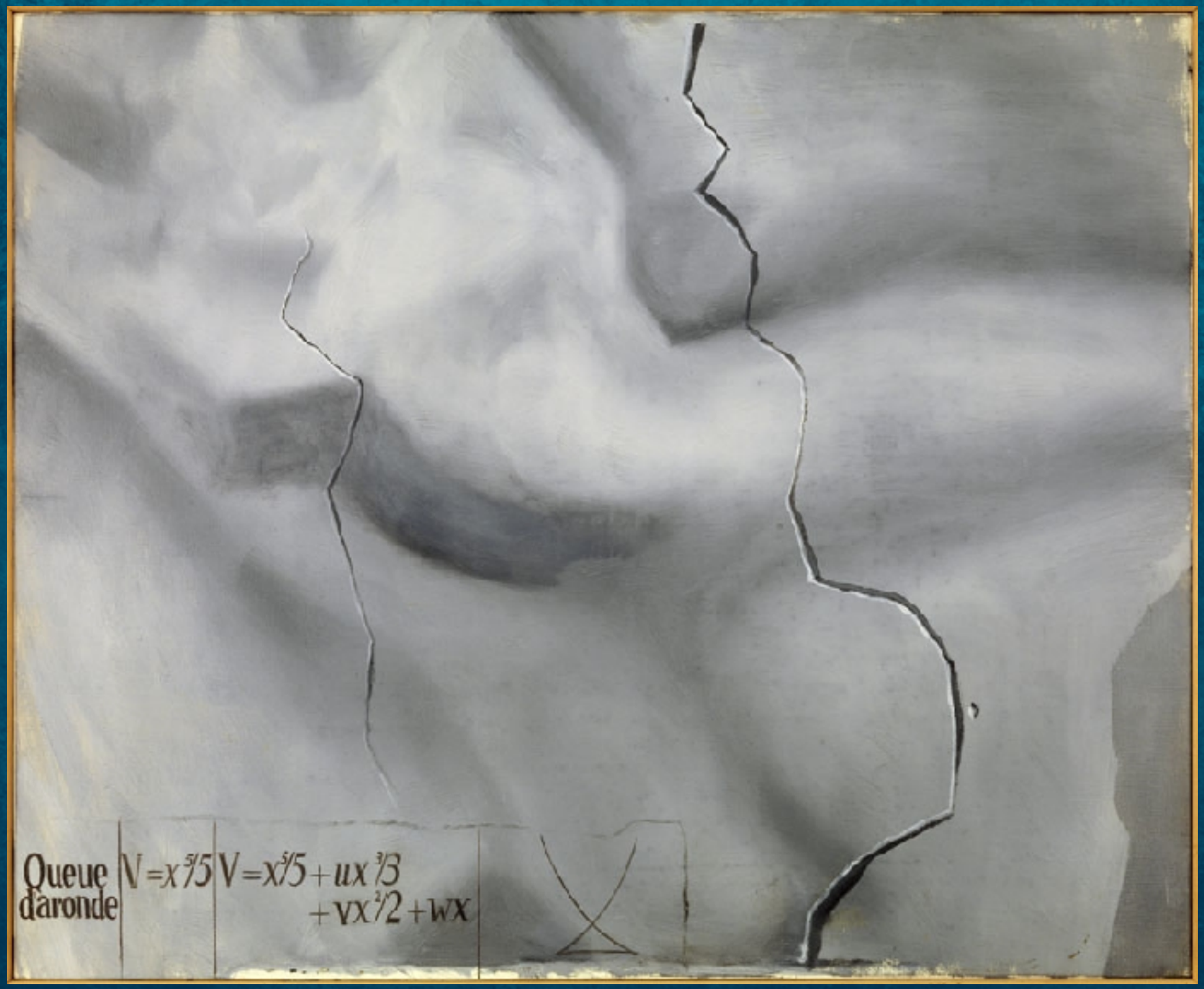
Crank cusp catastrophe



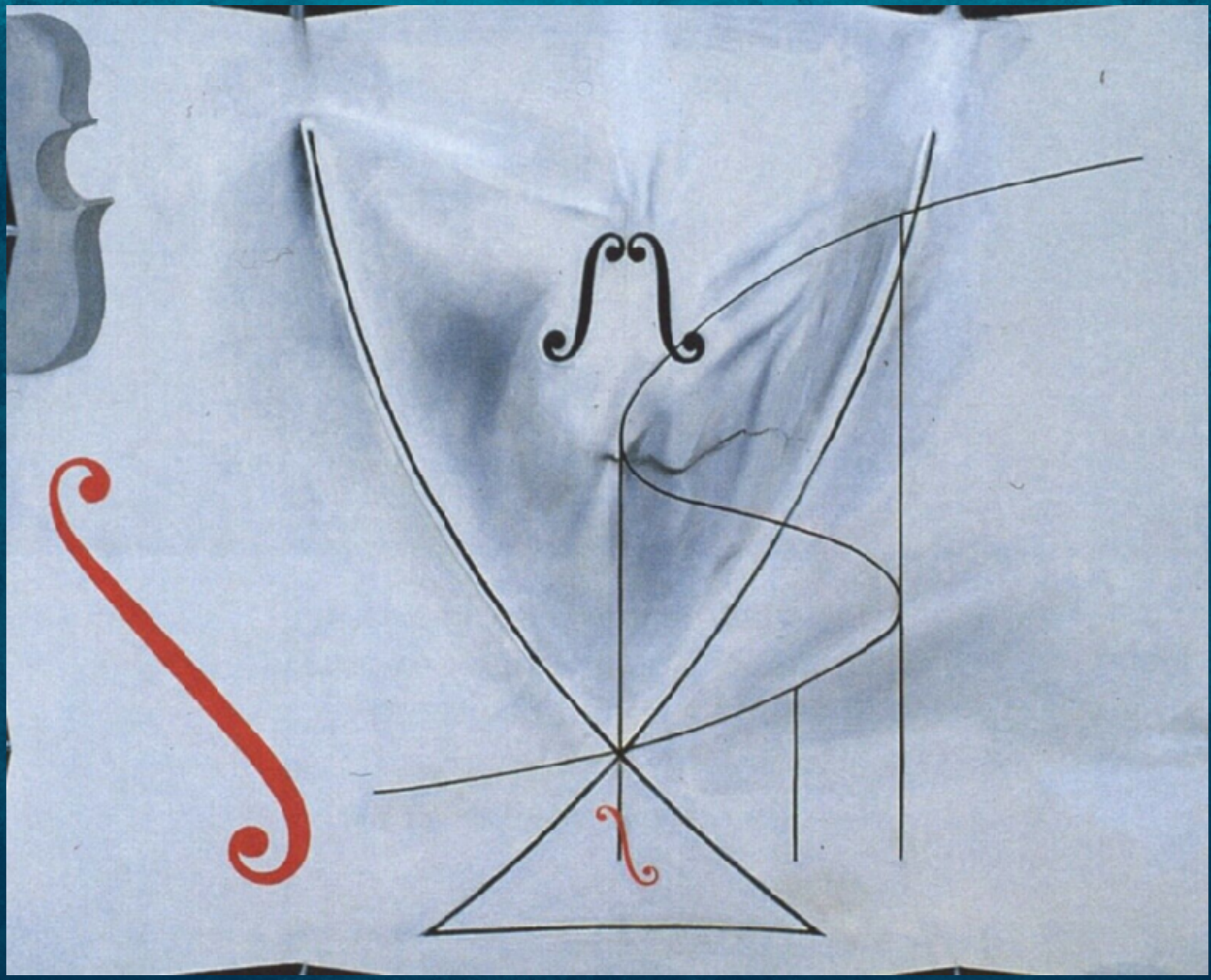
Salvador Dali



Cosmogenic ecstasy in the train station



Topological abduction of Europe. Homage to Rene Thom



The Swallow's tail

Dali's last painting

On the plane of philosophy properly speaking, of metaphysics, catastrophe theory cannot, to be sure, supply any answer to the great problems which torment mankind. But it favorizes a dialectical, Heraclitian view of the universe … It is a fundamentally polytheistic outlook to which it leads us: in all things one must learn to recognize the hand of the Gods. …. Just as the hero of the Iliad could go against the will of a God, such as Poseidon, only by invoking the power of an opposed divinity, such as Athena…so shall we be able to restrain the action of an archetype only by opposing to it an antagonistic archetype, in an ambiguous contest of uncertain outcome.

One will perhaps be able to demonstrate the inevitable nature of certain catastrophes, such as illness or death. Knowledge will no longer necessarily be a promise of success or of survival; it might just as well mean the certainty of our failure, of our end.

Thom, on catastrophe theory.

The emperor has no clothes

"I remark only that articles on catastrophe theory are distinguished by a sharp and catastrophic lowering of the level of demands of rigour and also of novelty of published results.”

- Vladimir Arnold

Abstract

Several representative attempts to apply catastrophe theory to biological and social science problems turn out on close analysis to be characterised by incorrect reasoning, far-fetched assumptions, erroneous consequences, and exaggerated claims. Catastrophe theory seems to have made no significant contributions to biology and the social sciences, and to have no advantage over other better-established mathematical tools which have been used to better effect.