Classical geometry w/ Symplectic Geometry

~ the Caratheodory conjecture ~

Carathéodory's conjecture (~1920)

any convex embedded surface $\Sigma \hookrightarrow \mathbb{R}^3$ has ≥ 2 umbilic points

Symplectic Reformulation (2008)

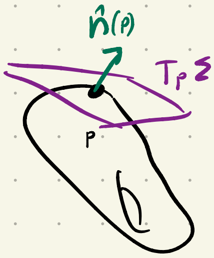
indefinite Kahler structure

any lagrangian section $\Sigma \hookrightarrow (TS^2, \Omega, \mathbb{J}, \mathbb{G})$ has ≥ 2 complex pts

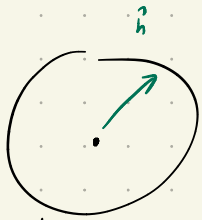
"As of 2024, none of this work has been published" wikipedia

Classic Differential Geometry:

Gauss map:

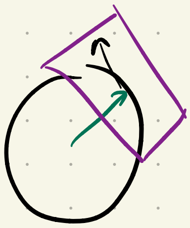


$$\hat{n} : \Sigma \longrightarrow S^2$$



$p \longmapsto$ unit normal at p

TS^2:



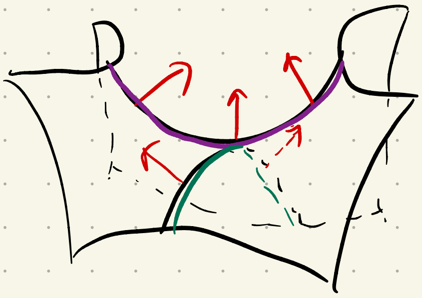
$$\{ (\hat{n}, \vec{v}) \mid \|\hat{n}\| = 1, \langle \hat{n}, \vec{v} \rangle = 0 \}$$

$$T_{\hat{n}} S^2 = \{ v \mid \langle v, \hat{n} \rangle = 0 \}$$

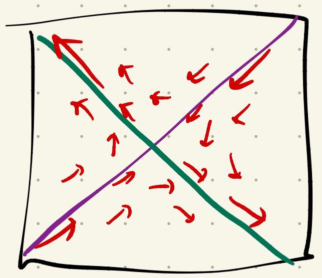
Second fundamental form:

$$\mathrm{II} = DG : T_p\Sigma \to T_{\hat{n}(p)} S^2 \simeq T_p\Sigma$$

symmetric matrix



top down view:



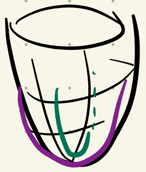
$\kappa_1 > 0$ $\quad \kappa_2 < 0$ curvature

$\nearrow$ $\quad \searrow$ principle directions

linearization of $1 - \langle \hat{N}(p), \hat{n}(p) \rangle$ is second fundamental form

Def: the **principle curvatures** at p are the eigenvalues κ_1, κ_2 of II
the **principle directions** are the e.vecs v_1, v_2
the **gaussian curvature** is $\kappa_1 \cdot \kappa_2$ (= curvature of induced metric)

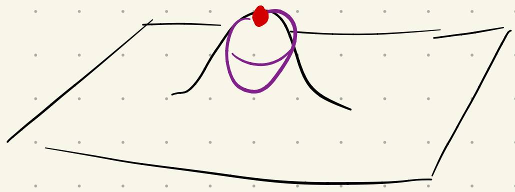
Def: p is an **umbilic point** if $\kappa_1 = \kappa_2$



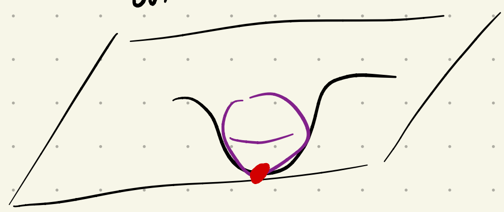
- Quadradic approximate is $z = \kappa (x^2 + y^2)$
- tangent sphere is tangent to **higher order**

Types of umbilics:

"innie"

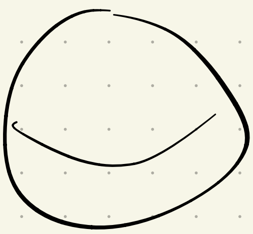


"outie"

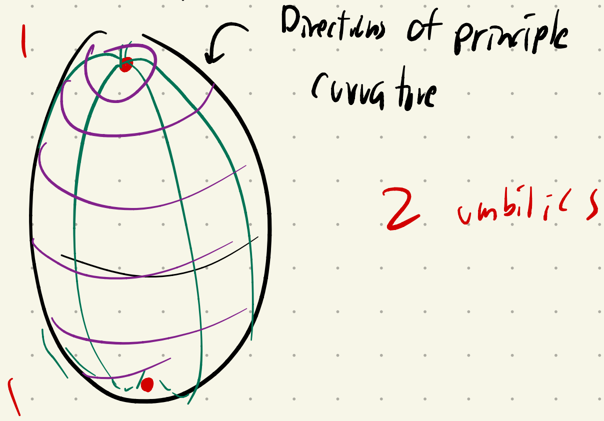


examples of umbilical points:

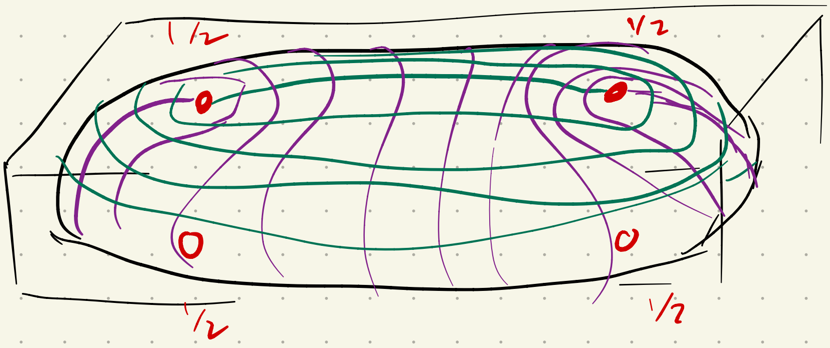
Round S^2 $\Rightarrow$ all umbilical



rotationally symmetric ellipsoid

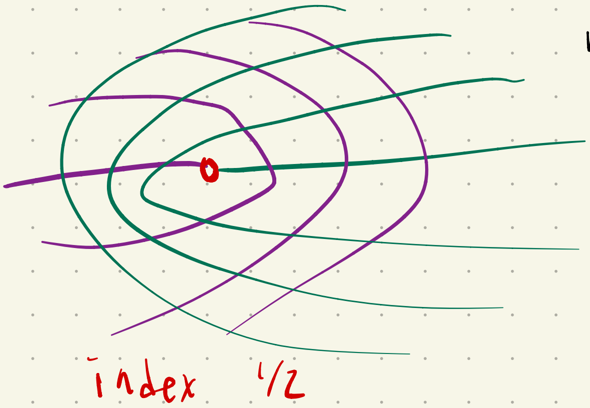


triaxial ellipsoid:



4 umbilics

Hamburger index: index of principle foliation around umbilic



measures "rotation #" of foliation around umbilic

Setting: Σ is convex $\iff$ Σ has positive gaussian curvature everywhere

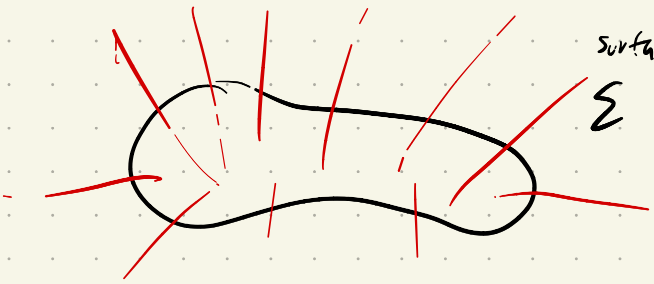
$\implies \Sigma \simeq S^2$

Poincare - Hopf thm $\implies \sum_{p \text{ umbilic}} \text{index} = \chi(\Sigma) = 2$

Caratheodory conjecture: Σ convex has ≥ 2 umbilic points

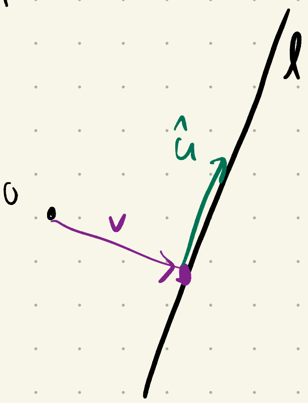
index 2??

Symplectic Geometry of Surfaces in $\mathbb{E}^3$



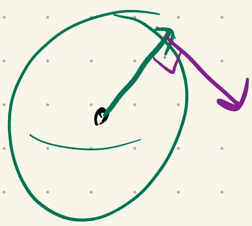
surface $\Sigma \Leftrightarrow$ family of lines normal to Σ

Space of oriented lines in $\mathbb{E}^3$:



a line ℓ is defined by:

- a direction $\vec{u}$ (w/ $\|u\|=1$)
- a displacement $\vec{v}$: the point on ℓ closest to origin ($\langle \vec{v}, \hat{u}\rangle = 1$)



$u \in S^2$

$\vec{v} \in T_u S^2$

$\Rightarrow$ moduli space of lines is $\mathbb{T} = TS^2$

Def: a "congruence" is a 2 parameter family of lines $L: \Sigma \hookrightarrow TS^2$

for $i: \Sigma \to \mathbb{E}^3$, the <u>normal congruence</u> is

$N: \Sigma \to \mathbb{T}$ $\quad N(p)$ is normal line passing thru $i(p)$

Remark: N is a souped-up Gauss map.

For $\pi: \mathbb{T} \cong TS^2 \to S^2$ the usual projection, then $\pi(N(p)) = \hat{n}(p)$

Question: when is a line congruence a normal congruence?

for line congruence $L: \Sigma \hookrightarrow \mathbb{T}$, try to construct "normal surface" $i: \Sigma \to \mathbb{E}^3$ (i.e. $L(\Sigma) = N(\Sigma)$)

for path $\gamma : [0,1] \hookrightarrow \Sigma$, define the "normal lift"

as $\tilde{\gamma} : [0,1] \hookrightarrow \mathbb{E}^3$ s.t

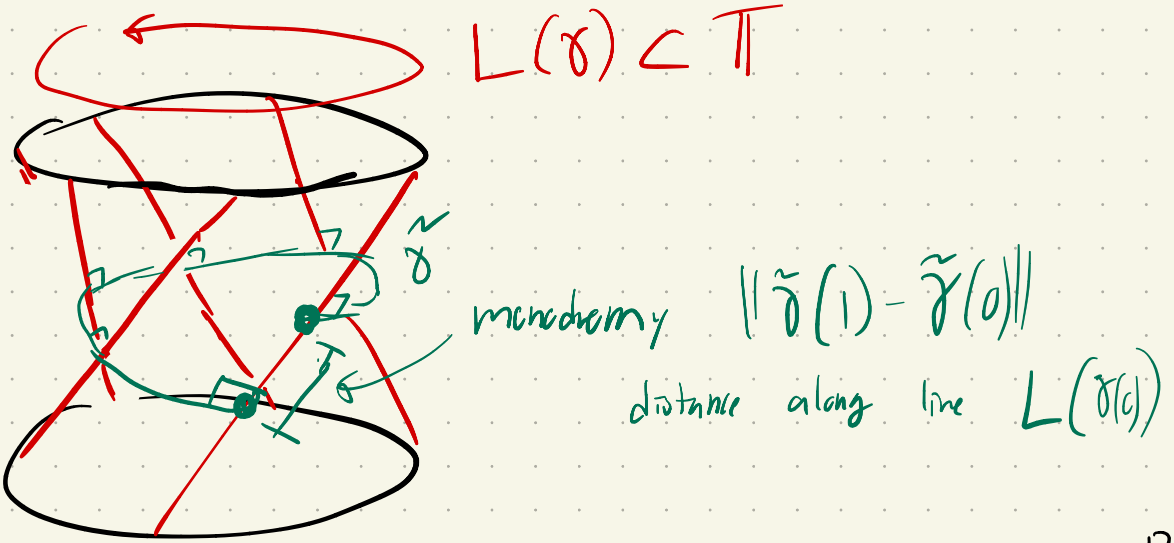
1. $\tilde{\gamma}(t)$ is on line $L(\gamma(t))$

2. $\dot{\tilde{\gamma}}(t)$ is normal to $L(\gamma(t))$

if $L(\Sigma)$ had a normal surface $i : \Sigma \hookrightarrow \mathbb{E}^3$, then $\tilde{\gamma} = i \circ \gamma$

hence, if $\gamma(0) = \gamma(1)$ but $\tilde{\gamma}(0) \neq \tilde{\gamma}(1)$, no normal surface exists

"Monodromy obstruction to normal surfaces"



$\mathbb{T}$ has a tautological principle $\mathbb{R}$-bundle

$$\begin{tikzcd} \mathbb{R} \arrow[r, hook] & P \arrow[d] \\ & \mathbb{T} \end{tikzcd}$$

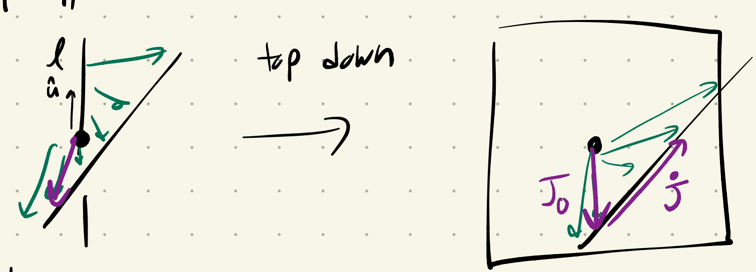
"Normal" condition gives a natural connection!

$\exists$ principle $\mathbb{R}$-connection $\alpha \in \Omega^1(\mathbb{T})$ s.t $\displaystyle\int_{L(\gamma)} \alpha = \|\tilde{\gamma}(1) - \tilde{\gamma}(0)\|$

goal: write α down:

coordinates on $T\mathbb{T}$

fix line ℓ w/ direction $\hat{u}$

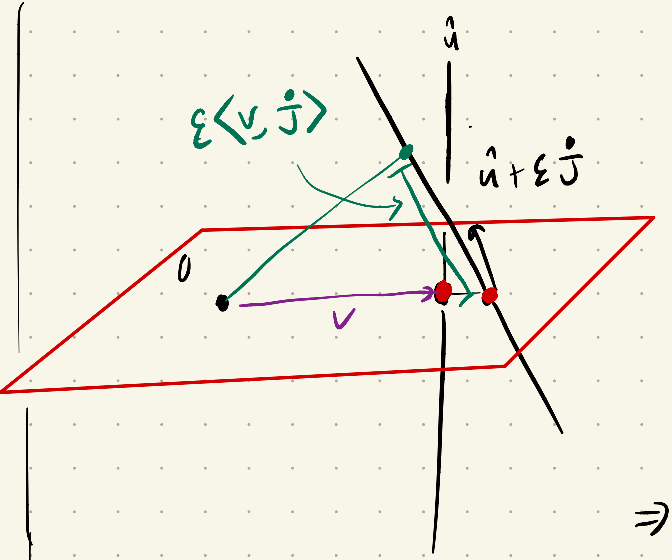


J_0 gives starting displacement

$\dot{J}$ gives "slope"

$J_0, \dot{J} \in \hat{u}^\perp$

$T_\ell \mathbb{T} = \{ J_0, \dot{J} \in \hat{u}^\perp \} = T_{\alpha(\ell)} S^2 \oplus T_{\alpha(\ell)} S^2$



as a line moves in direction $(J_0, \dot{J})$,

if $\tilde\gamma(0) = \ell_0(0)$ (the path starts @ the closest pt to the origin)

then $\lim_{s\to 0} \tilde\gamma(s) = \ell_s(-s\langle v, \dot J\rangle)$

as ℓ tilts, the point on ℓ closest to O moves distance $\sim s\langle v, \dot J\rangle$ along the line

$\Rightarrow \alpha(J, \dot J)\Big|_{(\hat u, \vec v)} = \langle v, \dot J\rangle$ in these coordinates

The monodromy of the lift of $\gamma : S^1 \to \mathbb{T}$ is $\int_\gamma \alpha$

L is normal congruence $\Leftrightarrow \int_\gamma \alpha = 0 \;\; \forall \gamma : S^1 \to \Sigma \Rightarrow L^* d\alpha = L^*\Omega = 0$

$\Rightarrow$ for contractible loops $\gamma = \partial D^2$, $\int_{\partial D^2} \alpha = \int_{D^2} d\alpha = 0$ for all Discs D^2

$\Rightarrow$ curvature $\Omega := d\alpha$ must be zero along L

$\Rightarrow L^*\Omega = 0$

Thm: using the round metric g on S^2, define dualizing map $g : TS^2 \to T^*S^2$

T^*S^2 has canonical symplectic form ω_{std}

$g^*\omega_{std} = \Omega$

Correlarry: Ω is a symplectic form on $\mathbb{T}$

Correlary: if L is a normal congruence, then L is lagrangian in $\mathbb{T}$.

Proof: $\omega_{std} = d\lambda$ for λ the tautological 1-form

for a vector $(v, \dot v) \in T_{(q,p)} T^*S^2$, $\lambda(v, \dot v)\big|_{(q,p)} := p(v)$

that is: 1. project vector from T^*S^2 onto S^2

2. pair resulting vector w/ the point on T^*S^2, thought of as a 1-form

now observe $g^*\lambda = \alpha$:

indeed, $(J_0, \dot J) \in T_{(\hat u, \vec v)} TS^2$, $g^*\lambda(J_0, \dot J)\Big|_{(\hat u, \vec v)} := \langle \vec v, \dot J\rangle = \alpha(J_0, \dot J)$

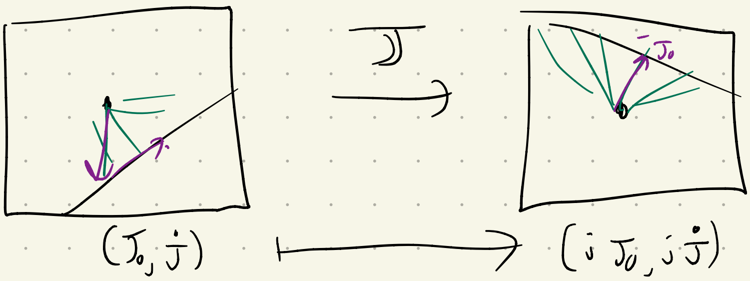
Pulling back by g replaces "pair w/ cotangent coordinate" in step 2 with "inner product w/ tangent coordinate"

$\Rightarrow \Omega = d\alpha = d g^*\lambda = g^* d\lambda = g^*\omega_{std}$ ∎

Complex structure on $\mathbb{T}$:

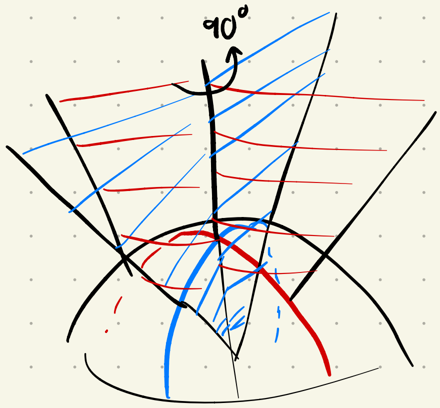
$\mathbb{J}_\ell$ rotates everything $90°$ around ℓ

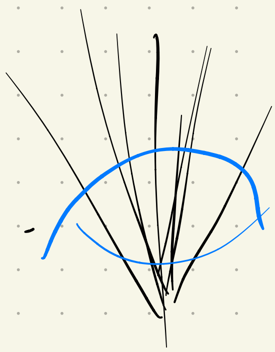
let j be the standard complex structure on TS^2



if $\mathbb{J}\, T_\ell N(\Sigma) = T_\ell N(\Sigma)$, ℓ is "complex point" of $N(\Sigma)$

thm: $N(p)$ complex $\iff$ p umbilic point





radially symmetric tangent vectors

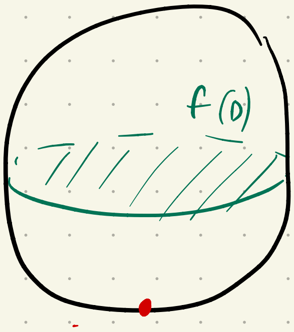
Kahler structure: $\mathbb{G}(\cdot,\cdot) = \Omega(\cdot, \mathbb{J}\,\cdot)$

$\mathbb{G}$ is nondegenerate w/ signature $(++--)$

Proof of the Carathéodory Conjecture

Suppose $\Sigma \subset \mathbb{T}$ lagrangian has only 1 complex point

i) find moduli space of J-curves



Boundary value problem:

$f: D \to \mathbb{T}$

(i) $\bar\partial f = 0 \iff$ all pts of $f(D)$ are complex

(ii) $f(\partial D) \subset \Sigma$

$f(D)$ is J-holomorphic, w/ boundary lying on the totally real part of Σ

moduli space of solutions $\mathcal{M}_\Sigma$

Thm: (Oh' 96) on Kahler mfld, can perturb Σ to lagrangian Σ' st $\mathcal{M}_{\Sigma'}$ is smooth (the set of lagrangians Σ which are regular, meaning coker $\bar\partial = 0$, is baire)

(needs to be adapted to neutral Kahler setting)

$$\dim \mathcal{M}_{\Sigma'} = \text{index } \bar\partial = \mu(D,\partial D)+2 = \sum_{\substack{\text{interior} \\ \text{cmplx pts } p}} \cancel{I(p)}^{\,0} + 2 = 2$$

after reparametrizations,

$$\dim \mathcal{M}_{\Sigma'}/\text{Aut}(D) = \text{index } \bar\partial - 3 = \boxed{-1}$$

$\Rightarrow \mathcal{M}_{\Sigma'}$ is empty!

$\boxed{\text{There are no J-holo discs w/ Boundary } \Sigma}$

2. construct a J-curve

use mean curvature flow: (converges to a minimal surface, J-curves are minimal)

nicer convergence properties in indefinite Kahler case than normal.

$\boxed{\text{There are J-curves w/ Bdry } \Sigma}$

Q:	A:	method:
hyperbolic PDE	Elliptic PDE	parabolic PDE
(gauss codazzi eqs)	($\bar\partial f = 0$)	(Mean curvature flow)