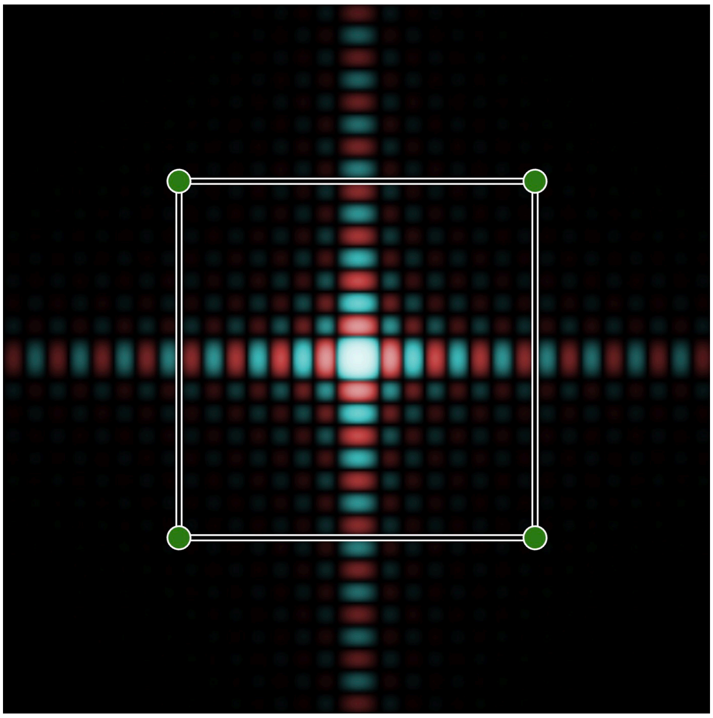
Fourier transforms of Polytopes

fourier transform of indicator fn.

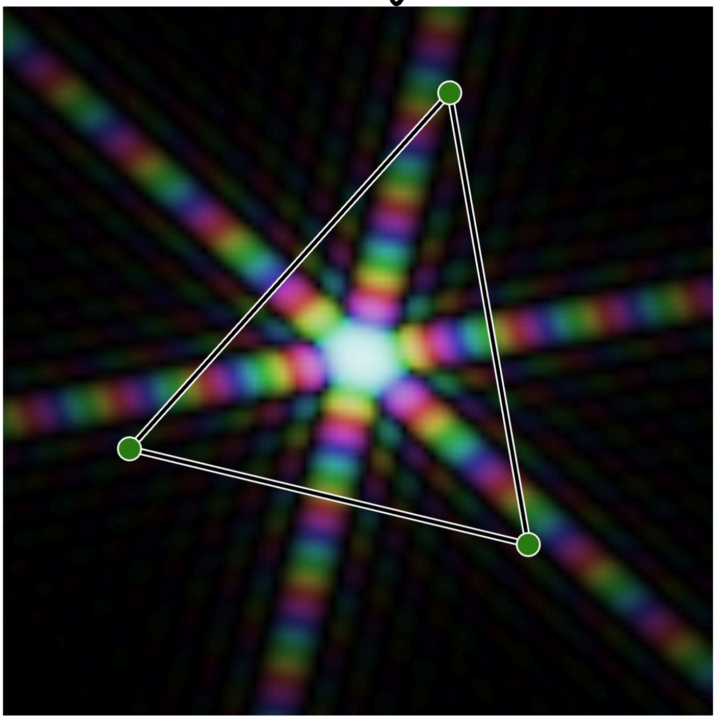
Question: for $P \subset \mathbb{R}^n$ a convex polytope, what is $\hat{1}_P(\xi) = \int_P e^{i\langle x,\xi\rangle}\,dx$

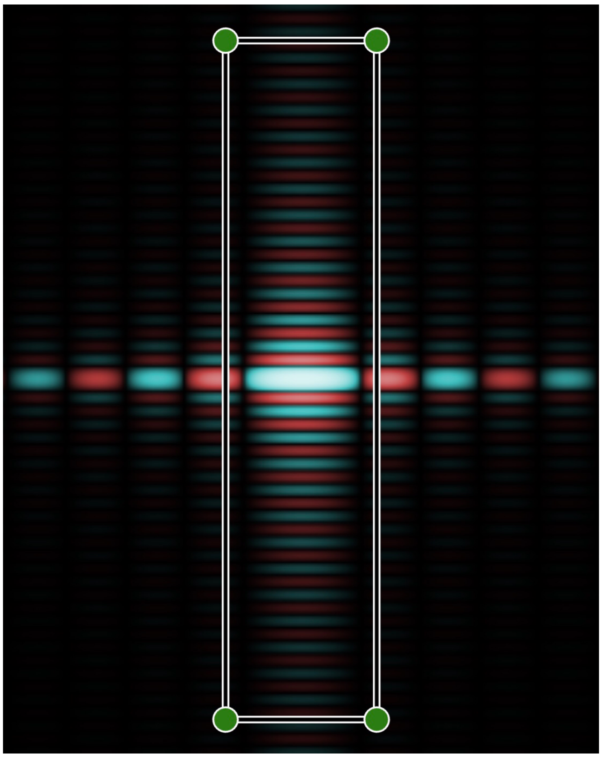
Let's look at some 2D examples, each image shows a polygon & it's F.T.

a square centered at 0



a triangle

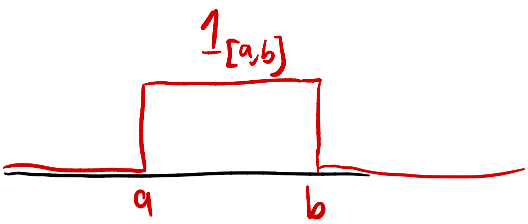




Some empherical observations about $\hat{1}_P$:

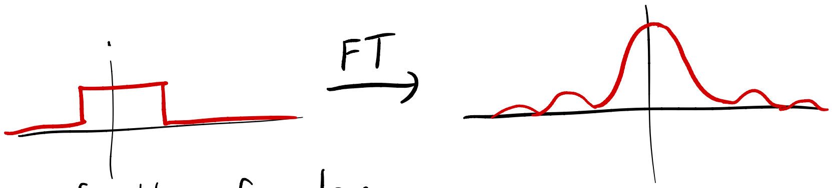
- concentrates along line through 0, I will call these "Beams"
- one beam per edge, Perpindicular to the edge
- beam thickness $\propto$ 1/edge length
- beam brightness $\propto$ edge length

Let's compute a 1-dimensional example



$$\hat{1}_{[a,b]}(\xi) = \int_a^b e^{i\xi x}\,dx = \frac{1}{i\xi} e^{i\xi x}\Big|_a^b = \frac{e^{i\xi b}}{i\xi} - \frac{e^{i\xi a}}{i\xi}$$

e.g: $\hat{1}_{[-1,1]}(\xi) = \frac{\sin(x)}{x}$



some Key properties of this formula:

(A) $\hat{1}_{[a,b]}(\xi)$ is a sum of contributions from the end points a, b

(B) 0 is a <u>removable</u> singularity of $\hat{1}_{[a,b]}$. indeed

$$\lim_{\xi\to 0} 1_{[a,b]}(\xi) = \lim_{\xi\to 0} \frac{e^{i\xi b} - e^{i\xi a}}{i\xi} \underset{\text{L'hopital's}}{=} \lim_{\xi\to 0} \frac{i(b-a)(e^{i\xi b} - e^{i\xi a})}{i} = b-a = \text{vol}([a,b])$$

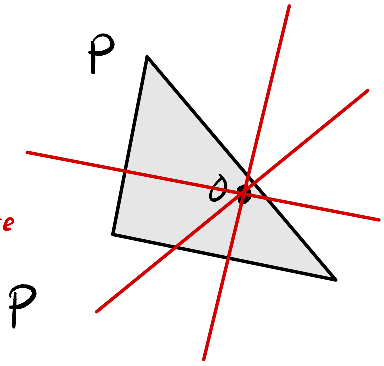
(C) $\hat{1}_{[a,b]}(\xi)$ decays as $1/\xi$ as $\xi \to \infty$

We will show how these properties extend to $\hat{1}_P$ for convex polytopes P

Part 1: Beams

Morally: for $|\xi| \gg 1$, $\hat{1}_P(\xi)$ is large when ξ is perpindicular to a facet of P.

facet = codim 1 face



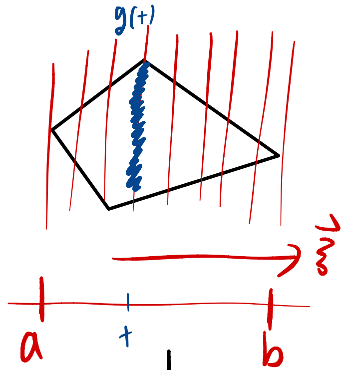
Thm: $\hat{1}_P(\lambda\xi) = \begin{cases} O(1/\lambda) & \xi \perp \text{ to a face of } P \\ O(1/\lambda^2) & \text{otherwise} \end{cases}$

This generalizes property (c) of $1_{[-a,b]}(\xi)$: in dimension ≥ 2, $\hat{1}_P(\lambda\xi)$ only has $1/\lambda$ decay in certain directions. otherwise, it decays faster.

Proof (heuristic):

$$\hat{1}_P(\lambda\xi) = \int_P e^{i\lambda\langle x,\xi\rangle}\,dx$$

fubini thm (assume $|\xi|=1$) $= \int_a^b g(t)\, e^{i\lambda t}\,dt$



$a = \min_{x\in P} \langle x,\xi\rangle$

$b = \max_{x\in P} \langle x,\xi\rangle$

$g(t) = \mathrm{vol}(P \cap \{\langle \xi,x\rangle = t\})$

area of slices $\perp$ to ξ

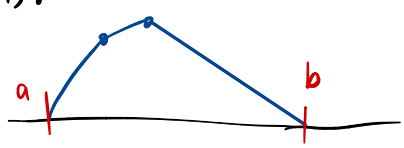
integration by parts

$u = g \qquad dv = e^{i\lambda t}dt$

$du = g'dt \qquad v = \frac{1}{i\lambda}e^{i\lambda t}$

$$= \frac{ge^{i\lambda t}}{i\lambda}\Big|_a^b - \frac{1}{i\lambda}\underbrace{\int_a^b g' e^{i\lambda t}}_{O(\lambda^{-1})}$$

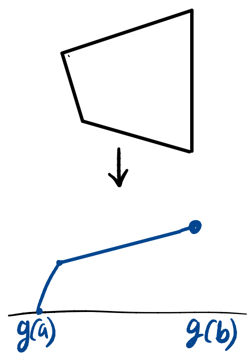
$g(t)$:



$g(a) = g(b) = 0 \Rightarrow \int_a^b g e^{i\lambda t} = O(\lambda^{-2})$ $\qquad$ $g(a)$ or $g(b) \neq 0 \Rightarrow \int_a^b g e^{i\lambda t} = O(\lambda^{-1})$

$\hat{1}_P(\lambda\xi) = O(\lambda^{-1}) \iff g(a)$ or $g(b) \neq 0$

$\iff \xi$ is $\perp$ to a face of P ∎



for ξ is perpindicular to a face F of P, then

$$|\hat{1}_P(\lambda\xi)| = \left|\frac{g e^{i\lambda t}}{i\lambda}\Big|_a^b + O(\lambda^{-2})\right| = \frac{\mathrm{vol}(F)}{\lambda} + O(\lambda^{-2})$$

amplitude of beam $\propto$ area of face

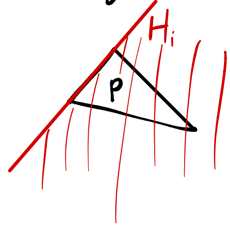
Part 2: Brion's formula

The two ways to describe a convex set / convex polytopes / vector space

- as the convex hull of a set of points — verticies — V
- as an intersection of half spaces — codim. 1 faces — V^*

These two perspectives give the fundamental duality of convex geometry

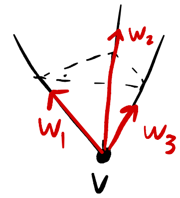
The indicator function $\mathbb{1}_P : V \to \mathbb{R}$ is best described using half-spaces



↳ suppose $P = \bigcap_i H_i$ (half-space). Then, $\mathbb{1}_P = \prod_i \mathbb{1}_{H_i}$

It's fourier transform $\hat{\mathbb{1}}_P : V^* \to \mathbb{R}$ is best described using verticies.

Thm (Brion 1988): Suppose a convex polytope $P \subset \mathbb{R}^n$ is simple, meaning each vertex v has n incident edges $\{\vec{w}_1^v, \ldots, \vec{w}_n^v\}$. construct matrix $M_v = [w_1^v \cdots w_n^v]$. Then the fourier transform $\hat{\mathbb{1}}_P$ satisfies

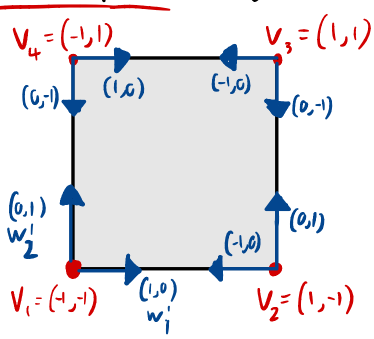


$$\boxed{\hat{\mathbb{1}}_P(\xi) = \frac{1}{i^n} \sum_{v \text{ vertex}} \frac{|\det(M_v)|\, e^{i\langle \xi, v\rangle}}{\prod_i \langle w_i^v, v\rangle}}$$

valid for ξ where the denominator is nonzero i.e ξ is not perpindicular to any edge of P.

The Brion formula is agnostic to the choice of edge vectors $\vec{w}_i^v$.
↳ Indeed, if $w_i \mapsto \lambda w_i$, then both $\det(M_v)$ & $\prod \langle w_i^v, v\rangle$ scale by λ.

Example: square in $\mathbb{R}^2$



let $\xi = (\xi_x, \xi_y)$. Here are contributions from each vertex

$$v_1: \quad \frac{\left|\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right| e^{-i\xi_x - i\xi_y}}{\xi_x \ \xi_y} = \frac{e^{-i\xi_x - i\xi_y}}{\xi_x \ \xi_y} \qquad v_3: \quad \frac{e^{i\xi_x + i\xi_y}}{(-\xi_x)(-\xi_y)}$$

$$v_2: \quad \cdots = \frac{e^{i\xi_x - i\xi_y}}{(-\xi_x)\ \xi_y} \qquad v_4: \quad \frac{e^{-i\xi_x + i\xi_y}}{\xi_x\ (-\xi_y)}$$

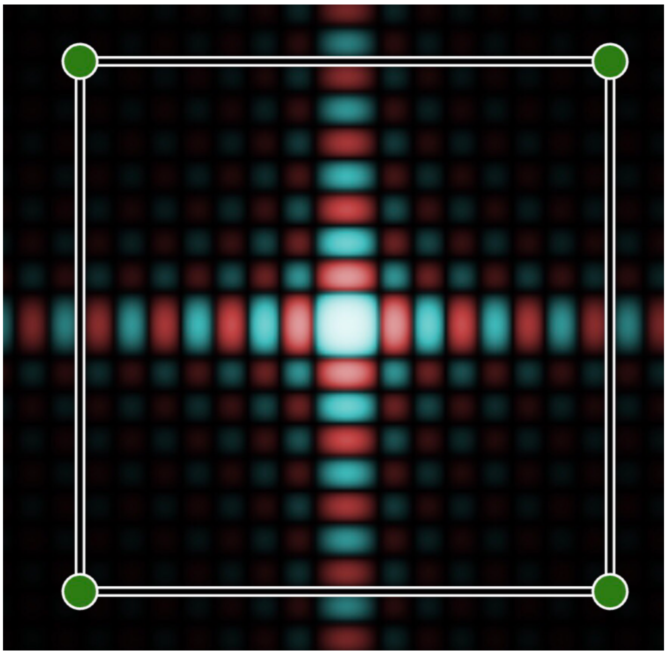
$$\hat{\mathbb{1}}_\square(\xi) = \frac{1}{i^2}\left(\frac{e^{-i\xi_x} e^{-i\xi_y}}{\xi_x \xi_y} - \frac{e^{i\xi_x} e^{-i\xi_y}}{\xi_x \xi_y} - \frac{e^{-i\xi_x} e^{i\xi_y}}{\xi_x \xi_y} + \frac{e^{i\xi_x} e^{i\xi_y}}{\xi_x \xi_y}\right) = -\frac{(e^{-i\xi_x} - e^{i\xi_x})(e^{-i\xi_y} - e^{i\xi_y})}{\xi_x \ \xi_y}$$

$$\boxed{\hat{\mathbb{1}}_\square(\xi) = \frac{\sin(\xi_x)}{\xi_x} \cdot \frac{\sin(\xi_y)}{\xi_y}}$$

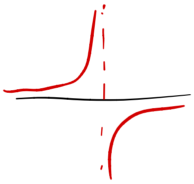
like we'd expect from 1D example!

See Picture

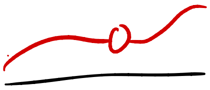
Key properties of Brion's formula:



(A) $\hat{1}_P(\zeta)$ is a sum of contributions from vertices of P. The contribution of v:
- only depends on the nbhd of v
- has essential discontinuity when ζ is $\perp$ to edges incident to v



(B) $B_P(\zeta) = \sum \frac{|\det M_v| e^{i\langle v,\zeta\rangle}}{\prod \langle w_i, \zeta\rangle}$ has removable discontinuities



$B_P(\zeta)$ is sum of rational fn times exponentials
$\Rightarrow B_P(\zeta)$ only has essential or removable discontinuities.
But, $B_P(\zeta) = \hat{1}_P(\zeta)$ outside union of hyperplanes, and $\hat{1}_P(\zeta)$ is bounded, so $B_P(\zeta)$ is bounded. Thus, all discontinuities are removable.

Remarks:

- using (B), we can use limits of Brion's formula to compute $\hat{1}_P(\zeta)$ beyond the domain of definition of Brion's formula. For example,

  $\mathrm{Vol}(P) = \hat{1}_P(0) = \lim_{\zeta \to 0} B_P(\zeta)$ w/ limit along a path avoiding hyperplanes $\perp$ to the edges of P.

- Though each vertex contribution is singular (property (A)), their sum is nonsingular (property (B)). Brion's formula suggests cancelation of terms like $\frac{1}{\zeta}$.

- Despite its simple form, Brion's formula wasn't discovered until the late 80's.

  It arose from the dictionary between convex polytopes and <u>Toric varieties</u>.

convex geometry	Toric geometry
$P \subset \mathbb{R}^n$ convex	X_P 2n-dimensional mfld w/ torus action $(S^1)^n \mathrel{\reflectbox{$\circlearrowleft$}} X_P$
vertex of P	fixed points of action
$\hat{1}_P(\zeta)$	topology of torus action
Brion formula	Localization formula: Topology of torus action entirely determined by fixed points

Despite its high falutent origin, the Brion formula has an elementry proof

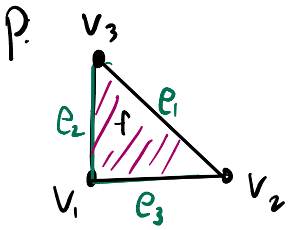
step 1: write 1_P as a sum of indicator functions of cones

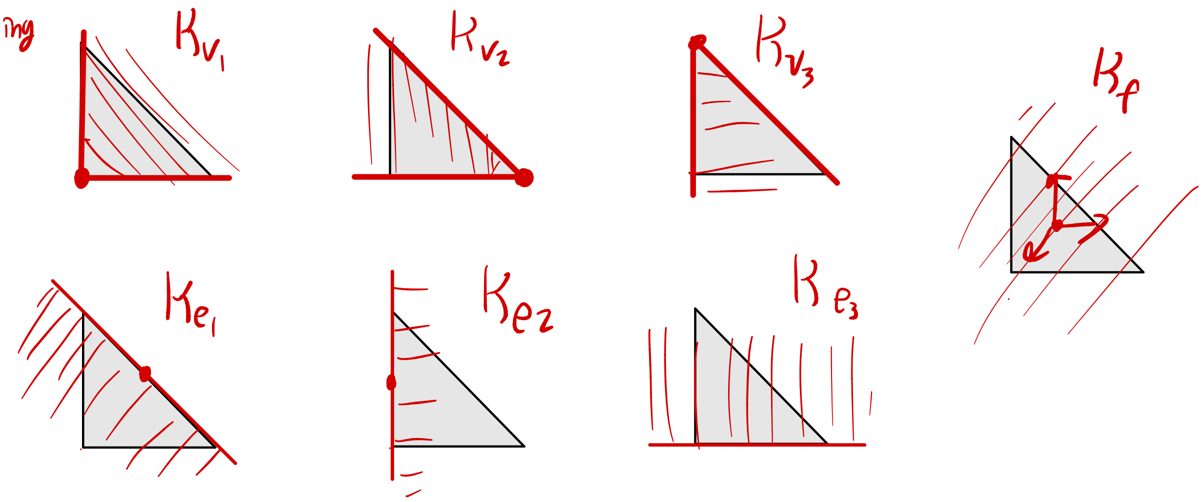
step 2: compute the fourier transform of a cone

Def: the <u>Tangent cone</u> of a face $F \subset P$ is defined as

standing at a point on F, what directions can I walk while still being inside P?

$$K_F = \{ q + \lambda(p-q) \mid q \in F,\ p \in P,\ \lambda \in \mathbb{R}_{\geq 0} \}$$



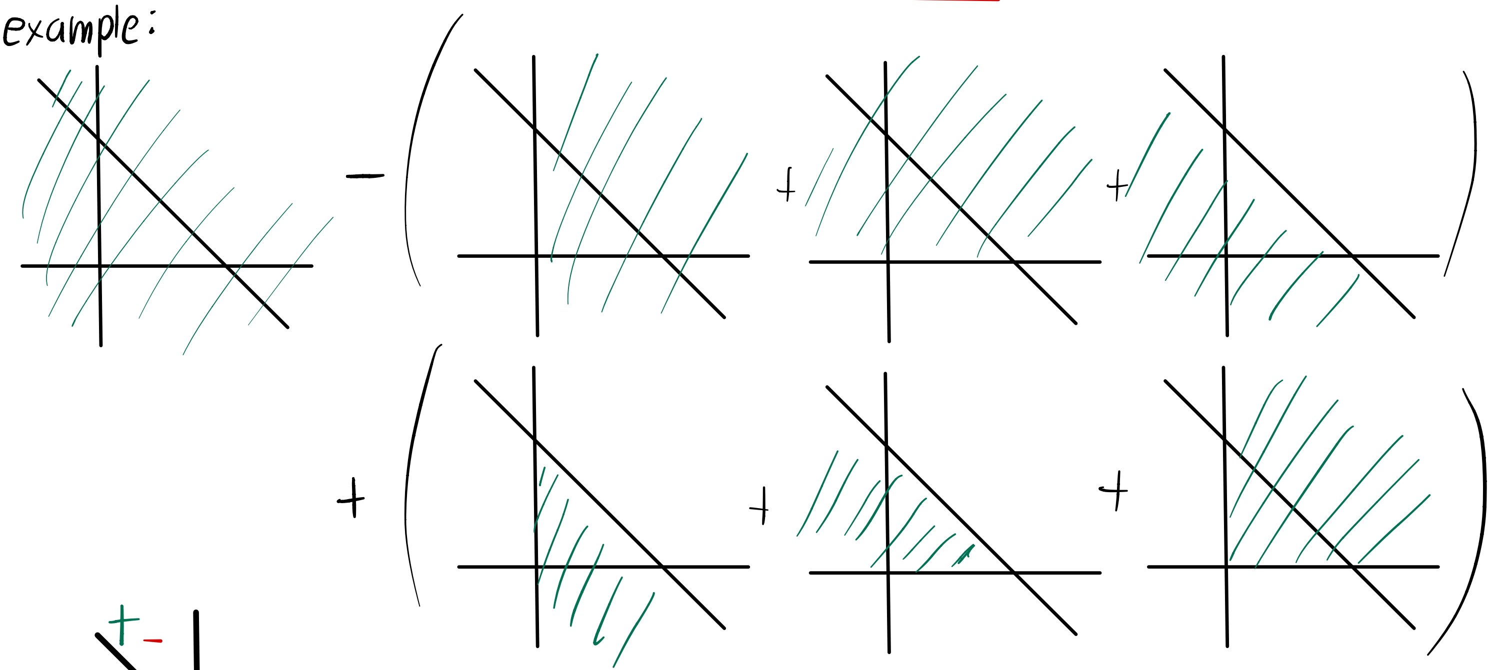


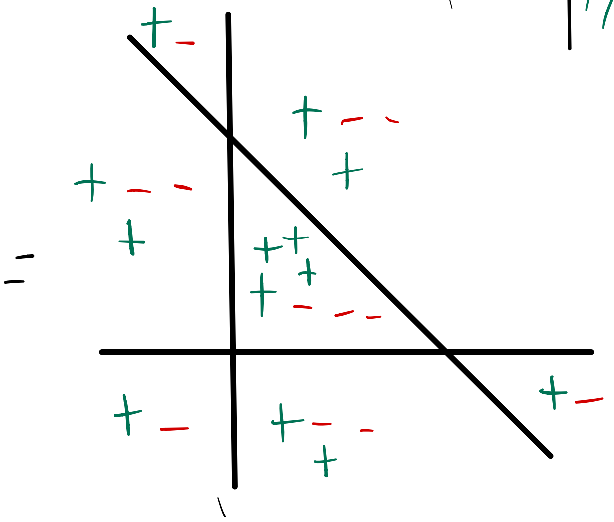
now we express $\mathbb{1}_P$ in terms of the tangent cones of P

Thm (Briachon-Gram identity): let P be a convex polytope, $F \subseteq P$ the faces

$$\boxed{\mathbb{1}_P = \sum_{F \subseteq P} (-1)^{\dim F}\, \mathbb{1}_{K_F}}$$

example:





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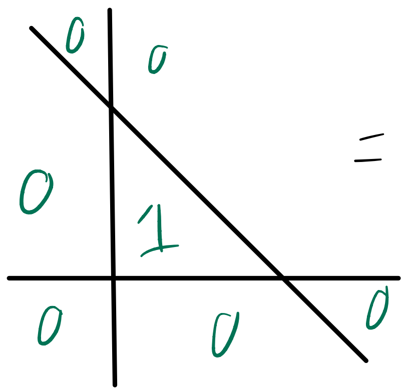


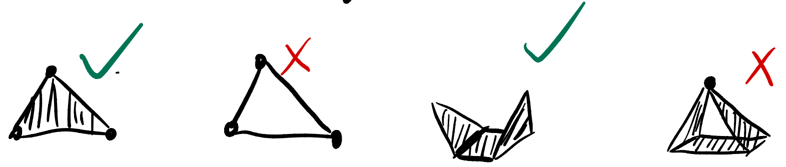
$= \mathbb{1}_P$

briachon-gram identities are equivelent to euler's formula

a <u>Polyhedral complex</u> $\mathcal{P}$ is something that looks like

Thm (Euler-Poincare) if $\mathcal{P}$ is contractible, then $\sum_{F \in \mathcal{P}} (-1)^{\dim F} = 1$ euler charecteristic of a point

alternitivly, the altrenating sum of the # of d-dimnl faces is 1.



"contractible" means "can be continuosly deformed to a point"
every convex polytope is contractible.

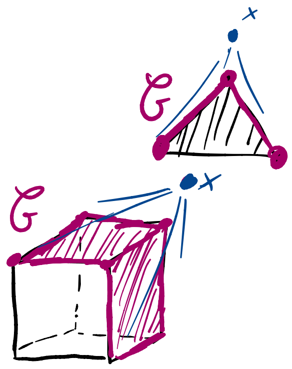
Proof (Brianchon-Gram): want to show $1_P \overset{?}{=} \sum_{F \in P} (-1)^{\dim F} 1_F$

case 1: $x \in P$. Then $1_P(x) = 1$.

$x \in P \Rightarrow x \in K_F$ for every face F. so, $\sum_{F \in P} (-1)^{\dim F} 1_{K_F}(x) = \sum_{F \in P} (-1)^{\dim F} = 1$ as desired.

case 2: $x \notin P$. Then $1_P(x) = 0$

let $\mathcal{G}$ be the collection of faces $F \subseteq P$ s.t $x \notin K_F$
$\mathcal{G}$ contains all faces of P which can be seen from x.



$\mathcal{G}$ is topologically a disc on the surface ∂P.
In particular, $\mathcal{G}$ is contractible.

$$\sum_{F \subseteq P} (-1)^{\dim F} \underbrace{1_F(x)}_{\substack{1 \text{ if } F \notin \mathcal{G} \\ 0 \text{ otherwise}}} = \sum_{F \in P} (-1)^{\dim F} - \sum_{F \subseteq \mathcal{G}} (-1)^{\dim F} = 1 - 1 = 0,$$ as desired.

Using Brianchon-Gram, $\hat{1}_P(\xi) = \sum_{F \in P} (-1)^{\dim F} \hat{1}_{K_F}(\xi)$ The F.T of tangent cones are:

Thm: 1: if $\dim F \geq 1$, then $\hat{1}_{K_F}(\xi) = 0$. So, $\hat{1}_P(\xi) = \sum_{v \text{ vertex of } P} \hat{1}_{K_v}(\xi)$

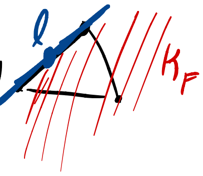
2: for v a simple vertex, $\hat{1}_{K_v}(\xi) = \dfrac{|\det M_v| \, e^{i\langle v, \xi \rangle}}{(i)^n \prod \langle w_i, \xi \rangle}$

together, these yeild Brion's formula.

<u>Remark</u>: cones K_F are unbounded, so 1_{K_F} is not L^1. we must somehow regularize their fourier transform. For example, $\hat{1}_{K_F} := \lim_{\varepsilon \to 0} \mathcal{F}\{ 1_{K_F} \underbrace{e^{-\varepsilon |x|^2}}_{\text{gaussian cutoff}} \}$

In the derivation below, we use the equivelent but heuristic regularization $e^{ix}\big|_{\pm\infty} = 0$.

Proof of Brion formula:



Part 1: if $\dim F \geq 1$, then K_F contains a line l. (any line tangent to F)

$\Rightarrow K_F = l \times K$, cartesian product of l and lower dimensional cone

choose orthonormal coordinates $(x_1, \dots, x_n)$ s.t $l = \{x_2 = \dots = x_n = 0\}$

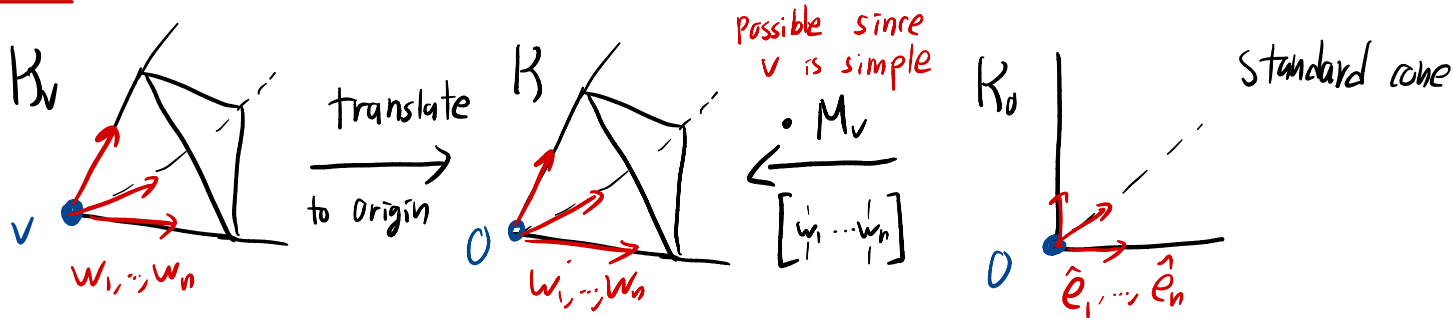
then $\mathbb{1}_{K_F}(x_1, \dots, x_n) = \mathbb{1}_l(x_1)\, \mathbb{1}_K(x_2, \dots, x_n) = \mathbb{1}_K(x_2, \dots, x_n)$

so $\hat{\mathbb{1}}_{K_F}(\xi) = \int_{K_F} e^{i\langle x, \xi \rangle} dx \overset{\text{fubini}}{=} \int_K e^{i(x_2\xi_2 + \dots)} \left(\int_{\mathbb{R}} e^{ix_1\xi_1} dx_1 \right) dx_2 \cdots dx_n$

but $\int_{\mathbb{R}} e^{ix\xi} dx = \frac{1}{i\xi} e^{ix\xi}\Big|_{-\infty}^{\infty} = 0$ in our regularization scheme.

Thus, $\hat{\mathbb{1}}_{K_F}(\xi) = 0$.

Part 2: change variables to put K_v into a standard form.



$\mathbb{1}_{K_v}(x)$ $\qquad$ $\mathbb{1}_K(x) = \mathbb{1}_{K_v}(x - v)$ $\qquad$ $\mathbb{1}_{K_0}(x) = \mathbb{1}_K(M_v x)$

$$\mathbb{1}_{K_v}(x) = \mathbb{1}_K(x + v) \Rightarrow \hat{\mathbb{1}}_{K_v}(\xi) = e^{i\langle v, \xi \rangle} \hat{\mathbb{1}}_K(\xi)$$

$$\mathbb{1}_K(x) = \mathbb{1}_{K_0}(M_v^{-1} x) \Rightarrow \hat{\mathbb{1}}_K(\xi) = \frac{1}{|M_v^{-1}|} \hat{\mathbb{1}}_{K_0}(M_v^T \xi) = |M_v| \hat{\mathbb{1}}_{K_0}(M_v^T \xi)$$

Therefore, $\hat{\mathbb{1}}_{K_v}(\xi) = |M_v| e^{i\langle v, \xi \rangle} \hat{\mathbb{1}}_{K_0}(M_v^T \xi)$

Computing $\hat{\mathbb{1}}_{K_0}$:

$$\hat{\mathbb{1}}_{K_0}(\xi) = \int_{\mathbb{R}^n_{\geq 0}} e^{i\langle x, \xi \rangle} dx \overset{\text{fubini}}{=} \left(\int_0^\infty e^{i\xi_1 x_1} dx_1 \right) \cdots \left(\int_0^\infty e^{i\xi_n x_n} dx_n \right)$$

$\int_0^\infty e^{ix\xi} dx = \frac{1}{i\xi} e^{ix\xi}\Big|_0^\infty = \frac{1}{i\xi}(1 - 0) = \frac{1}{i\xi}$, in our regularization.

so $\hat{\mathbb{1}}_{K_0}(\xi) = \left(\frac{1}{i\xi_1}\right) \cdot \dots \cdot \left(\frac{1}{i\xi_n}\right) = \frac{1}{i^n} \frac{1}{\xi_1 \cdot \dots \cdot \xi_n}$

$$\hat{\mathbb{1}}_{K_v}(\xi) = |M_v| e^{i\langle x, \xi \rangle} \left(\frac{1}{i^d} \frac{1}{(M_v^T \xi_1) \cdot \dots \cdot (M_v^T \xi_n)} \right) = \boxed{\frac{|M_v|\, e^{i\langle x, \xi \rangle}}{(i)^d \prod \langle w_i^v, \xi_i \rangle}}$$ as desired.

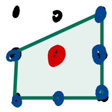
Part 3: Lattice point enumeration

The number of lattice points inside a region approximates its volume.

Picks formula: Let $P \subset \mathbb{R}^2$ be a lattice polygon, meaning each vertex lies in $\mathbb{Z}^2$. Then,

$$\operatorname{vol}(P) = |\operatorname{int} P \cap \mathbb{Z}^2| + \underbrace{\frac{|\partial P \cap \mathbb{Z}^2|}{2}} - 1$$

"Boundary points count half"

example: 

1 interior pt
6 boundary points

$\Rightarrow$ area $1 + \frac{6}{2} - 1 = 3$

Fourier transforms let us count lattice points. Let $\mathcal{L}(C) = |C \cap \mathbb{Z}^n| = \sum_{x \in \mathbb{Z}^n} 1_C(x)$

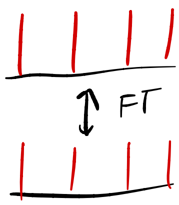
"lattice point enumerator"

A really good idea: poisson summation formula!

$$\boxed{\sum_{x \in \mathbb{Z}^n} 1_C(x) = \sum_{\xi \in \mathbb{Z}^n} \widehat{1_C}(\xi)}$$

in terms of distributions, Poisson summation says

$$\widehat{\sum_{x \in \mathbb{Z}^n} \delta_x} = \sum_{x \in \mathbb{Z}^n} \delta_x$$



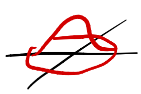
F.T takes a lattice to its dual lattice.

$\hat{1}_C(0) = \operatorname{vol}(C)$. so, $\mathcal{L}(C) = \operatorname{vol}(C) + \underbrace{\sum_{\xi \in \mathbb{Z}^n \setminus \{0\}} \hat{1}_C(\xi)}$

$\mathcal{L}(C)$ approximates $\operatorname{Vol}(C)$ with this error

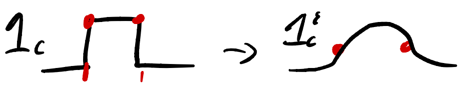
Van-der-corput: If $C \subset \mathbb{R}^2$ is a circle, $|\mathcal{L}(rC) - V(rC)| = O(r^{2/3})$ gauss circle problem

Like all good ideas, This doesn't quite work. Poisson summation only applies to schwartz functions, and 1_C is not smooth enough.

Mollify 1_C with a radial bump function $\varphi =$  $\varphi_\varepsilon(x) = \varepsilon^n \varphi\left(\frac{x}{\varepsilon}\right)$

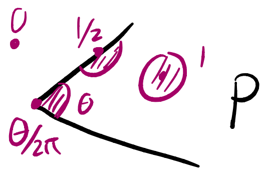
$1_C^\varepsilon = 1_C * \varphi_\varepsilon$ is a schwartz fn $\quad \int_{\mathbb{R}^n} \varphi_\varepsilon = 1$

Apply Poisson summation to 1_C^ε, then take limit as $\varepsilon \to 0$.



$$\tilde{1}_C := \lim_{\varepsilon \to 0} 1^\varepsilon_C(x) = \lim_{\varepsilon \to 0} \frac{\operatorname{vol} B_x(\varepsilon) \cap C}{\operatorname{vol} B_x(\varepsilon)}$$

regularized charecteristic fn

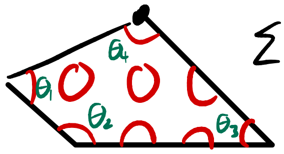


example: if P is a 2D polygon,

$$\tilde{1}_P(x) = \begin{cases} 0 & x \notin P \\ 1 & x \in \operatorname{int} P \\ 1/2 & x \text{ is boundary pt, but not vertex} \\ \theta/2\pi & x \text{ is vertex} \end{cases}$$

The Discrete volume is $\operatorname{Vol}_{\mathbb{Z}}(P) = \sum_{x \in \mathbb{Z}^n} \tilde{1}_P(x)$

for a lattice polygon, the discrete volume is the LHS of Pick's formula:



$$\sum \tilde{1}_P(x) = (\#\text{ interior pts}) + \tfrac{1}{2}(\#\text{ bdry pts}) \underbrace{- \tfrac{1}{2}(\#\text{ vertices})}_{\text{avoid double counting}} + \tfrac{1}{2\pi}\sum\theta_i$$

the interior algle formula says $\sum\theta_i = (n-2)\pi$ w/ $n = \#$ vertices

so $$\sum \hat{1}_P(x) = (\#\text{ interior pts}) + \tfrac{1}{2}(\#\text{ bdry pts}) - 1.$$

Pick's formula, v. 2: for P lattice polygon, $Vol(P) = Vol_{\mathbb{Z}}(P)$

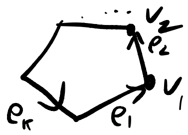
The discrete volume is accessible through poisson summation

$$Vol_{\mathbb{Z}}(P) = \sum_{x\in\mathbb{Z}^2}\tilde{1}_P(x) = \lim_{\varepsilon\to 0}\sum_{x\in\mathbb{Z}^2} 1^{\varepsilon}_P(x) = \lim_{\varepsilon\to 0}\sum_{\xi\in\mathbb{Z}^2}\hat{1}^{\varepsilon}_P(\xi) = \sum_{\xi\in\mathbb{Z}^2}\hat{1}_P(\xi)$$

$$\Rightarrow vol_{\mathbb{Z}}(P) = vol(P) + \sum_{\substack{\xi\in\mathbb{Z}^2\\ \xi\neq 0}}\hat{1}_P(\xi)$$

Lemma: for P lattice, $\xi\in\mathbb{Z}^2$ not $\perp$ to any facets of P. Then $\hat{1}_P(\xi)=0$

Use Brion's formula: write edges $e_1,\dots,e_k$



vertex v_i has tangent cone w/ edges $\{-e_i, e_{i+1}\}$

$\det[e_i,e_{i+1}] = \omega(e_i,e_{i+1})$ for $\omega = dx\wedge dy$ standard area 2-form

$$\hat{1}_P(\xi) = \sum_i \frac{e^{2\pi i\langle v_i,\xi\rangle}}{(2\pi i)^2}\frac{|\omega(-e_i,e_{i+1})|}{\langle -e_i,\xi\rangle\langle e_{i+1},\xi\rangle}$$

- $v,\xi\in\mathbb{Z}^2 \Rightarrow \langle v,\xi\rangle\in\mathbb{Z}$. so, $e^{2\pi i\langle v,\xi\rangle} = 1$ ★ simplifying! No phase information!
- since e_i ordered, for P convex $\omega(e_i,e_{i+1}) = |\omega(e_i,e_{i+1})|$

$$= \sum_i \frac{1}{(2\pi i)^2}\frac{\omega(e_i,e_{i+1})}{\langle e_i,\xi\rangle\langle e_{i+1},\xi\rangle} = 0$$

This sum telescopes, meaning $$\frac{\omega(e_i,e_{i+1})}{\langle e_i,\xi\rangle\langle e_{i+1},\xi\rangle} + \frac{\omega(e_{i+1},e_{i+2})}{\langle e_{i+1},\xi\rangle\langle e_{i+2},\xi\rangle} = \frac{\omega(e_i,e_{i+2})}{\langle e_i,\xi\rangle\langle e_{i+2},\xi\rangle}$$

let $f_i(\xi) = \omega(e_i,e_{i+1})\langle e_{i+2},\xi\rangle + \omega(e_{i+1},e_{i+2})\langle e_i,\xi\rangle + \omega(e_{i+2},e_i)\langle e_{i+1},\xi\rangle$. WTS: $f_i(\xi)=0$

To see this, identify $e_i = (e_i^x, e_i^y, 0)$. Then, $e_i\times e_j = (0,0,\omega(e_i,e_j))$

$$(e_i\times e_j)\times e_k = \omega(e_i,e_j)\underbrace{(-e_k^y, e_k^x, 0)}_{90^\circ\text{ rotation}}$$

Defining $J = \begin{bmatrix}0&1\\-1&0\end{bmatrix}$, $(-e_i^y, e_i^x, 0)\cdot J\xi = \langle e_i,\xi\rangle$. Thus,

$$\omega(e_i,e_{i+1})\langle e_{i+2},\xi\rangle = \langle (e_i\times e_{i+1})\times e_{i+2}, J\xi\rangle$$

$$f_i(\xi) = \langle (e_i\times e_{i+1})\times e_{i+2}, J\xi\rangle + \text{cyclic} = \langle \overbrace{(e_i\times e_{i+1})\times e_{i+2} + \text{cyclic}}^{0}, J\xi\rangle = 0 \;\square$$

by Jacobi identity.

This shows there is no contribution to $Vol_{\mathbb{Z}}(P) - Vol(P)$ from lattice pts where the Brion formula applies. Unfortunatly, we do get contributions when ξ is $\perp$ to an edge of P. I think they cancel:

If $\xi \neq 0$, $\xi \in \mathbb{Z}^n$, & $\xi \perp$ to an edge of P, then $\hat{1}_P(\xi) = -\hat{1}_P(-\xi)$ (This seems to hold, looking @ my graph). Then, the sum over integral points would vanish, so, $\sum_{\xi \in \mathbb{Z}^2 - \{0\}} \hat{1}_P(\xi) = 0$

$\Rightarrow Vol(P) = Vol_{\mathbb{Z}}(P)$ (picks formula)

to extract this from Brion's formula, we need to take a limit. This is tricky to do explicitly, & makes Brion's formula somewhat tempermental.