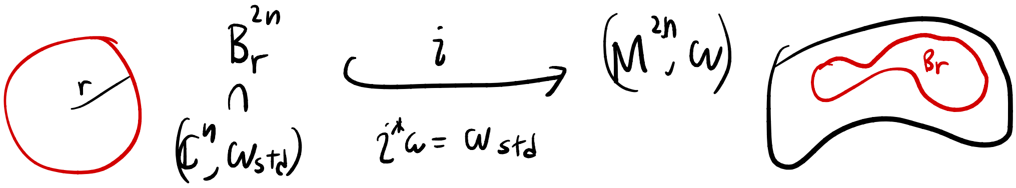
The Biran Decomposition

Following Biran 2001, "Lagrangian barriers and symplectic embeddings"

The M.O. of symplectic geometry is to pretend to be complex geometry

today, we will use this philosophy to understand symplectic embeddings into Kahler manifolds.
Question: can we obstruct symplectic embeddings of balls?

$$\begin{array}{ccc} B_r^{2n} & \xrightarrow{\ i\ } & (M^{2n}, \omega) \\ \cap & & \\ (\mathbb{C}^n, \omega_{std}) & i^*\omega = \omega_{std} & \end{array}$$

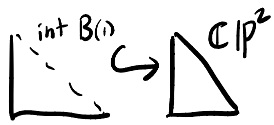


Volume bound: if $\mathrm{vol}(M,\omega) < \mathrm{vol}(B_r^{2n})$, there are no symplectic balls of radius r in M

Some times the volume bound is sharp:

Example: $(\mathbb{C}P^n, \omega_{FS})$ ω_{FS} is fubini studi form, normalized s.t $\int_{\mathbb{C}P^1} \omega_{FS} = \pi$, where $\mathbb{C}P^1 \subset \mathbb{C}P^n$ is a line.

Then $(\mathbb{C}P^n \setminus \mathbb{C}P^{n-1}, \omega_{FS}) \simeq (\mathrm{int}(B_1^{2n}), \omega_{std})$
$\{[z_0, \dots, z_n] \mid z_n \neq 0\}$



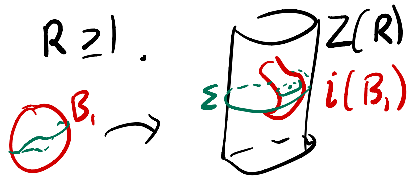
that is, a symplectic ball of radius 1 fully fills $(\mathbb{C}P^n, \omega_{FS})$
→ embedding which saturates volume bound

sometimes volume is not sharp:

Example: $Z^{2n}(R) = B_R^2 \times \mathbb{C}^{n-1}$ with product symplectic form

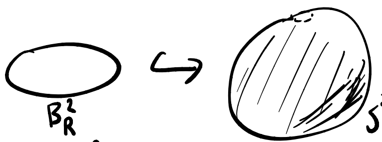
note $\mathrm{vol}(Z^{2n}(R)) = \infty$, so there is no volume obstuction. Yet, embeddings are obstucted!

Thm (Gromov non-squeezing):
$\exists$ symplectic embedding $B_1^{2n} \hookrightarrow Z^{2n}(R)$ iff $R \geq 1$.



Proof sketch: Suppose $\exists$ embedding

1. compactify $Z(R)$ along Disc: $Z(R) \hookrightarrow \overline{Z(R)} = S_R^2 \times \mathbb{C}^{n-1}$



2. find J-holo curve Σ in class $[S_R^2] \times pt \subset H_2(\overline{Z(R)})$ thru center of $i(B_1)$ (GW theory)
   Σ has area πR^2
3. pull back Σ to holomorphic curve in B_1^{2n}. Minimal surface theory $\Rightarrow$ area $\Sigma \geq \pi \Rightarrow$ $\boxed{R \geq 1}$ ∎

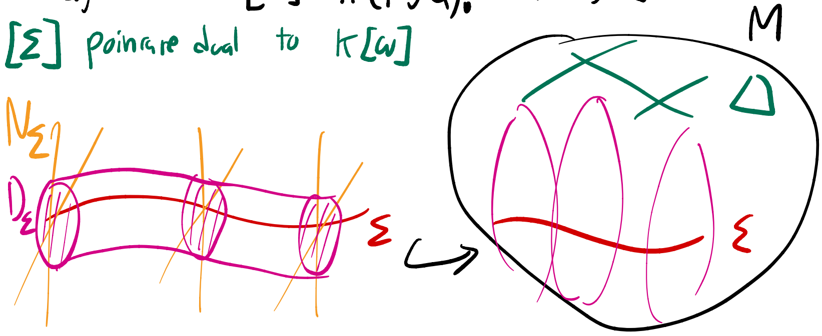
we can sometimes obstruct symplectic balls by cutting out lagrangian submanifolds
these are called lagrangian barriers

Thm (Biran): consider $\mathbb{C}P^n \setminus \mathbb{R}P^n = \{[z_0, \dots, z_n] \mid z_i \notin \mathbb{R}\}$ ← lagrangian submanifold
$\exists$ symplectic embedding $B_r^{2n} \hookrightarrow \mathbb{C}P^n \setminus \mathbb{R}P^n$ iff $r < 1/\sqrt{2}$
$\Longleftrightarrow$ Symplectic balls B_r w/ $\frac{1}{\sqrt{2}} < r < 1$ must intersect $\mathbb{R}P^n$

This arises from a very good symplectic understanding of Kahler manifolds:

Thm: (Biran Decomposition)

Let (M^{2n},ω) be a compact Kahler manifold, with $[\omega]\in H^2(M,\mathbb{Q})$. Then, $\exists$:
- $\Sigma^{2(n-1)} \subset M^{2n}$ a complex hypersurface w/ $[\Sigma]$ poincare dual to $k[\omega]$
- an isotropic CW complex $\Delta \subset M$
- a symplectomorphism $M\setminus\Delta \simeq D_\Sigma$ unit disc bundle of normal bundle N_Σ w/ symplectic area $1/k$

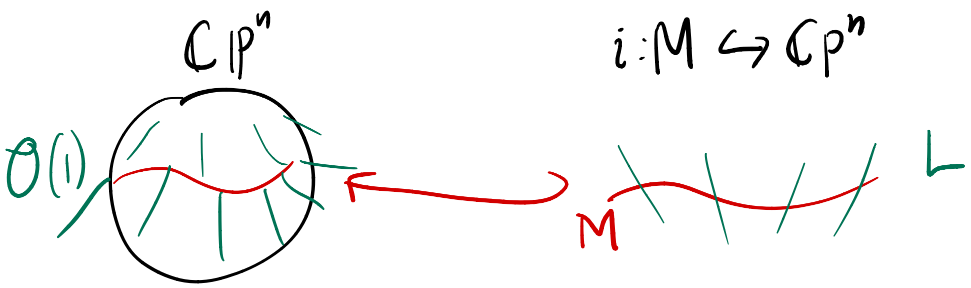


Part I: Lefshetz's dream

Biran's decomposition is inspired by the philosophy of complex geometry, in particular lefshetz's program for studying the topology of projective manifolds. To give us ground to build on we describe lefshetz's classical paradigm:

We are interested in projective manifolds, complex submanifolds of projective space

$\mathbb{C}P^n$ has many structures, which M inherits



Kahler structure	ω_{FS}	$\omega = i^*\omega_{FS}$
line bundle	$\mathcal{O}(1)$, $c_1(\mathcal{O}(1))=[\omega_{FS}]$	$L = i^*\mathcal{O}(1)$, $c_1(L)=[\omega]$
hermitian metric on line bundle	curvature ω_{FS}	h, w/ curvature $F_h=\omega$

There are several equivalent perspectives on projective manifolds:

- M projective $\Leftrightarrow$ M algebraic variety Chow's theorem
- M projective $\Leftrightarrow$ M has ample line bundle (curvature is positive definite (1,1) form)
  Kodaira embedding theorem

This motivates our geometric setup:

(M,ω) Kahler w/ line bundle $\begin{matrix}(L,h)\\ \downarrow \\ M\end{matrix}$ s.t h has curvature ω "Prequantum line bundle"

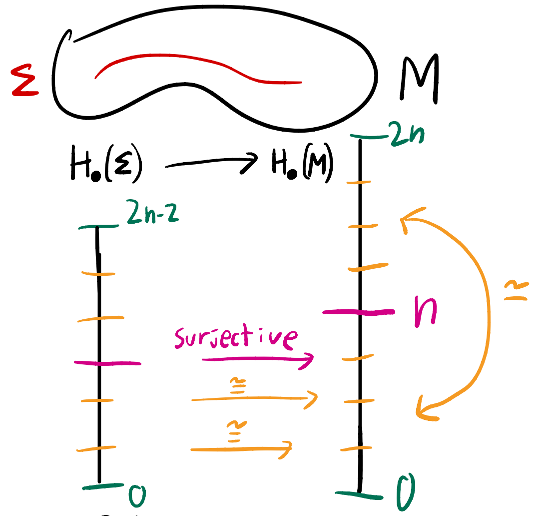
Goal: understand the topology of M via sections of L

let $s \in H^0(M,L)$ be a holo. section "Quantum state"

consider zero set $\Sigma = s^{-1}(0)$ (choose s s.t Σ is smooth)

Thm (Lefshetz hyperplane theorem)

$H_k(\Sigma,\mathbb{Z}) \to H_k(M,\mathbb{Z})$ is
- an isomorphism for $k<n-1$
- surjective for $k=n-1$



Thm: (hard Lefshetz theorem)

$H_k(M,\mathbb{Z}) \simeq H_{2n-k}(M,\mathbb{Z})$

nearily all the topology of M is contained in Σ!

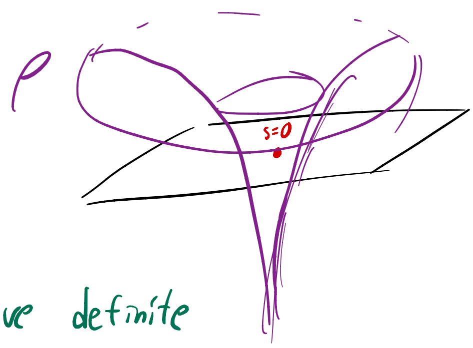
Lefshetz's program: study M using Σ, and induct on dimension!

Biran's decomposition is a symplectic enhancement of lefshetz hyperplane thm

Proof (lefshetz hyperplane theorem) w/ morse theory!

Bott 1959, "On a theorem of Lefschetz"

for $s \in H^0(M,L)$, define $\rho = \log \|s\|_h^2$



Fact: $\frac{i}{2\pi}\partial\bar\partial\rho = F_h = \omega$ $\quad$ ρ is plurisub harmonic

Prop: all critical pts of ρ have index $\geq n$

$\partial\bar\partial\rho = \sum \frac{\partial^2\rho}{\partial z_i \partial \bar z_j} dz_i \wedge d\bar z_j = \underline{-2\pi i\, \omega}$ negative definite

$\Rightarrow$ Hess ρ is negative on n-dimensional subspace span $\langle \partial z_i \rangle$

$\Rightarrow$ Hess $\rho|_x$ has at least n negative e.vals @ all crit pts x

use morse theory of ρ to build M:

$\rho^{-1}(-\infty) = s^{-1}(0) = \Sigma$

$\Rightarrow$ M built from Σ by attaching d-cells w/ $d \geq n$

$\Rightarrow$ applying homology LES, get Lefshetz hyperplane thm $\blacksquare$

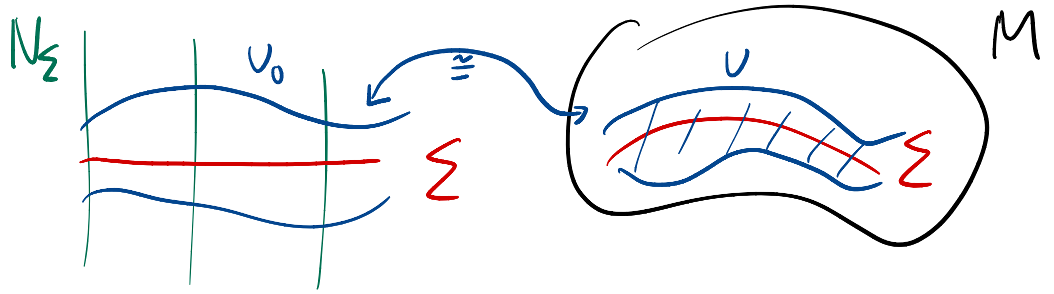
note: $\dim W_p^u = 2n - \text{index } p$, but W_p^u is isotropic, so $\dim(W_p^u) \leq n$

this gives an altrenate proof that $\text{index}(p) \geq n$

Next we turn to the symplectic geometry of the zero set Σ:

- $\Sigma \subset M$ is <u>symplectic</u> submanifold (as $\Sigma \subset M$ complex)

Symplectic neighborhood theorem: let N_Σ be the normal bundle of $\Sigma \hookrightarrow M$.
N_Σ has standard symplectic form ω_0 s.t $\Sigma \xhookrightarrow{0 \text{ section}} N_\Sigma$ is symplectic
then, $\exists$ nbhds U_0 of Σ in N_Σ, U of Σ in M, & symplectomorphism $\varphi: U_0 \to U$

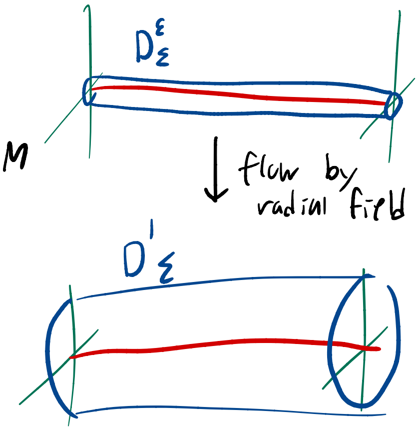


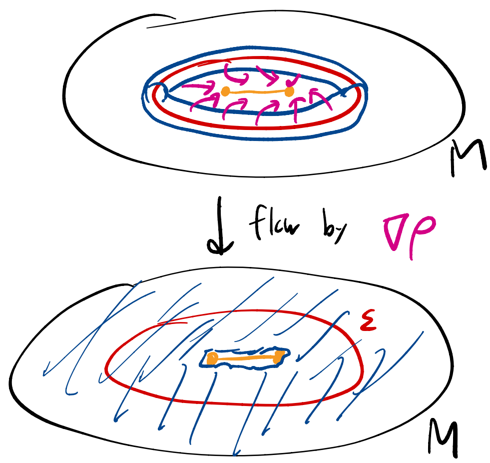
we have a good symplectic model for a tubular nbhd of Σ in M.
we can only make this neighborhood radius 1 in N_Σ. more specifically,
let D^1_Σ be the unit disc bundle in N_Σ w/ radius 1

theorem: D^1_Σ is symplectomorphic to $M \setminus \Delta$

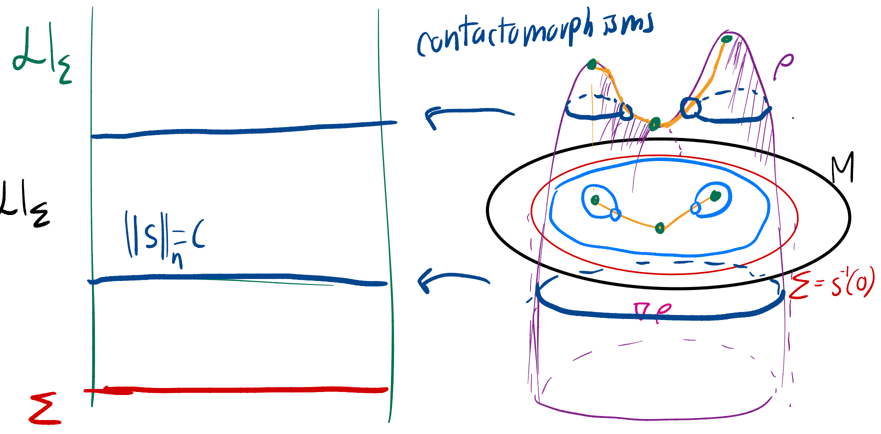
intuition:
start w/ small Disc bundle D^ε_Σ
& it's symplectomorphic image in M

then flow w/ $\nabla\rho$!
expand D^ε_Σ until it fills M
Since $\nabla\rho$ liouville,
$\phi^{+t}(D^\varepsilon_{\Sigma,\omega_0}) \text{"="} (D^1_\Sigma, \omega_0)$





Proof sketch
$N_\Sigma \cong \mathcal{L}|_\Sigma$ by adjunction formula
$\mathcal{L}|_\Sigma$ has coords (x, f) $f \in \mathcal{L}|_x$
$|f|_h$ is a plurisubharmonic function on $\mathcal{L}|_\Sigma$
want to equate $|f|_h$ on D^1_Σ w/ ρ on M
so, equate contact manifolds
$|f|_h^{-1}(c) \cong \rho^{-1}(c)$



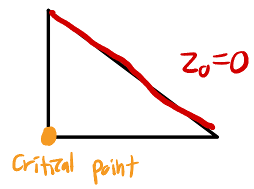
Examples:

$(M,\omega) = (\mathbb{CP}^2, \omega_{FS})$

$L = \mathcal{O}(1)$ in projective coordinates $[z_0, \dots, z_n]$ hermitian metric is $h([z_0:\dots:z_n]) = \frac{1}{\sum |z_i|^2}$

$H^0(\mathbb{CP}^2, \mathcal{O}(1))$ consists of linear functions

choose $s = z_0 \in H^0(\mathbb{CP}^2, \mathcal{O}(1))$: $|s|_h^2 = \frac{|z_0|^2}{\sum |z_i|^2}$



$\Sigma = s^{-1}(0) = \{z_0 = 0\}$

critical points of s are only $[1:0:\dots:0] \Rightarrow \Delta = [1:0:\dots:0]$

Biran Decomposition: $\boxed{\mathbb{CP}^2 \simeq D^1_{\mathbb{P}}(\mathcal{O}(1)) \sqcup pt}$

$(M,\omega) = (\mathbb{CP}^2, 2\omega_{FS})$

$L = \mathcal{O}(2)$ $h = \sum |z_i|^2$ hermitian metric doesn't change w/ tensor powers of L

$H^0(\mathbb{CP}^2, \mathcal{O}(2))$ are homogenous quadratic functions

choose $s = z_0^2 + z_1^2 + z_2^2$:

$\Sigma \simeq \mathbb{CP}^1 \subset \mathbb{CP}^2$ is the quadric

crit $s = \mathbb{RP}^2$ (in a morse bott way), so $\Delta = \mathbb{RP}^2$

$\boxed{(\mathbb{CP}^2, 2\omega_{FS}) \simeq D^1_{\mathbb{P}}(\mathcal{O}(2)) \sqcup \mathbb{RP}^2}$

Part III: Symplectic embeddings

we have decomposition $M = D^1(N_\varepsilon) \sqcup \Delta$

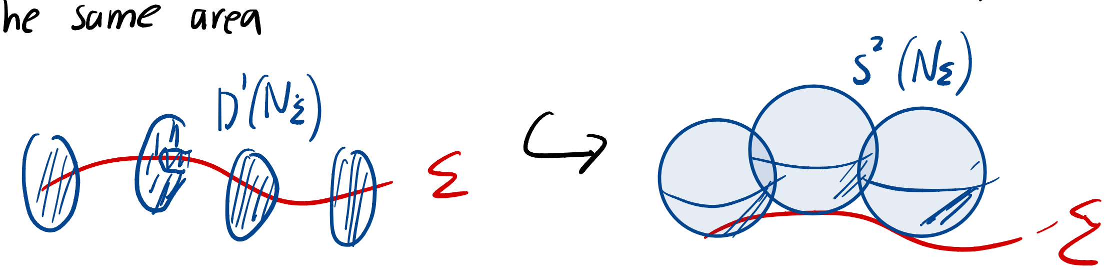
if there are no symplectic embeddings $B^r \hookrightarrow D^1(N_\varepsilon)$, then every symplectic ball $B^r \subset M$ must intersect Δ: Δ is a barrier

Thm: $\exists$ symplectic ball $i: B^r \hookrightarrow D^1(N_\varepsilon) \iff r \leq 1$

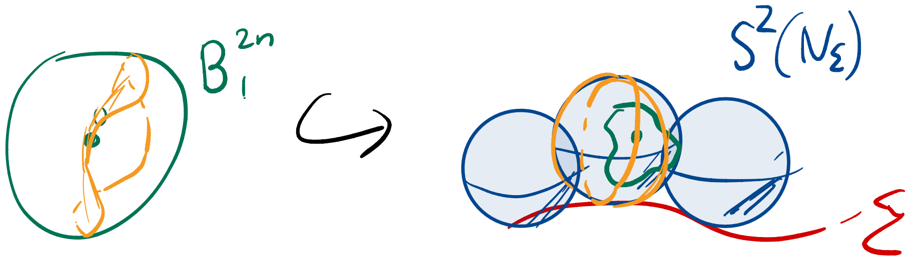
Proof $\Rightarrow$: we wish to obstruct symplectic balls of radius >1 rigidity

follow the argument of Gromov non-squeezing:

1. compactify $D^1(N_\varepsilon)$ from disc bundle to $\mathbb{P}^1$-bundle $S^2(N_\varepsilon)$ w/ the fibers having the same area



2. find a J-holomorphic curve through center of ball $i(0)$, in homology class of the fiber



To do this, we compute that the Gromov-Witten invariant counting these curves is 1. To verify GW Theory works here takes some work & technical assumptions on M. To get the count 1, we count holo. curves in the original Kahler structure

3. pull back J-curve to B_1, & bound 2D area of B_1 using area of J-curve

(this part is identical to the non-squeezing theorem)

Proof $\Leftarrow$: wish to show $\exists$ embedding $\text{int } B_1^{2n} \hookrightarrow D^1(N_\varepsilon)$

we will realize Lefschetz's dream in a symplectic world, & construct B_1^{2n} inductively

lemma: if $\mathcal{E}_{\vec{a}} \subset \mathbb{C}^n$ is a symplectic ellipsoid of radii $a_1, \dots, a_n$, then the disc bundle $D^r(\mathcal{E}_{\vec{a}} \times \mathbb{C})$ is symplectomorphic to $\mathcal{E}_{\vec{a},r} \subset \mathbb{C}^{n+1}$

Warning: the symplectic disc bundle does not carry the product symplectic structure

symplectic form is $(1-r\rho^2)\pi^*\omega_{std} + r\rho\, d\rho \wedge d\theta$

this is noted, for example, in lemma 2.1 of

Opshtein 2006, "Maximal symplectic packings of P2"

in particular, if $B_1^{2(n-1)} \hookrightarrow \Sigma$ is a symplectic ball

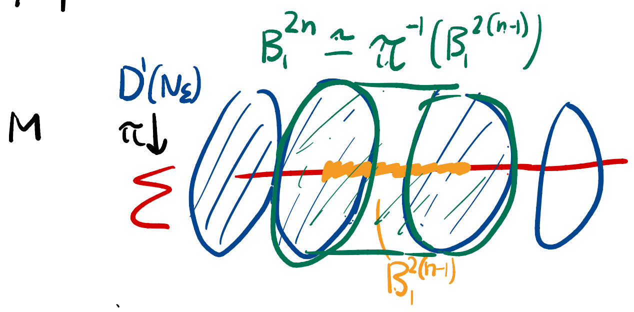
then $\pi^{-1}(B_1^{2(n-1)}) \simeq \mathcal{E}_{1,\dots,1} = B_1^{2n} \hookrightarrow M$

so, symplectic ball $B_1^{2(n-1)} \hookrightarrow \Sigma \Rightarrow$ symplectic ball $B_1^{2n} \hookrightarrow M$

when $n=0$, have symplectic embedding $B_1^0 = \{pt\} \hookrightarrow M^0 = \{pt\}$

induct on dimension!

$\Rightarrow$ $\boxed{\text{every integral Kahler manifold contains a symplectic ball of radius 1}}$



Remarks:

corollary: every rational kahler manifold is fully filled by an ellipsoid

This idea was used to prove packing stability for all rational symplectic manifolds:

1. use donaldson submfld to construct Σ
2. full packings of $M \Leftrightarrow$ full packing of Σ (induct on dimension)
3. reduce to 4D problem, & use ECH capacities

Buse, Hind 2013: "Ellipsoid embeddings and symplectic packing stability"

Part IV: extentions & applications

extention to non-kahler manifolds

we can get a biran type decomposition for arbitrary rational manifolds.

First we need a canidate for Σ: this is provided by donaldson's appraoch to symplectic submanifolds. instead of setting $\Sigma = s^{-1}(0)$ for s holomorphic, we attempt to find a section s.t $\bar{\partial}_J s = 0$ for some almost complex structure J. alas, $\bar{\partial}_J s = 0$ generally has no solutions.

We suffice w/ a family of almost holomorphic sections $s_k \in \Gamma(L^k)$, satisfying:

- $\|\bar{\partial}_J s\|_{L^\infty} < C/\sqrt{k}$
- $|\bar{\partial}_J s| < |\partial_J s|$ on $s^{-1}(0)$

for k sufficently large, $\Sigma = s^{-1}(0)$ is symplectic, w/ $[\Sigma] = k\,PD([\omega])$

Donaldson proved Σ always exist. Biran proved they have an analagous Biran decomposition for a 4-manifold.

Biran 1999, "A stability property of symplectic packing"

Using this, Biran proved packing stability for rational 4-manifolds:
$\exists N$ s.t $\forall n > N$, $\exists$ full filling by N equal radius balls.

full filling by distinct balls