Asymtotic holomorphic geometry

a partia of symplectic flexability

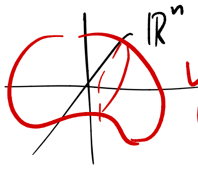
Theorem (Donaldson '96): Every closed symplectic manifold (M,ω) has a $\boxed{\text{closed symplectic submanifold}}$ $i: V \hookrightarrow M$ (i.e $i^*\omega$ is symplectic on V)

This result is best understood in contrast with $M = (\mathbb{R}^{2n}, \omega_{std})$

Prop: there are no closed symplectic submanifolds of $(\mathbb{R}^{2n}, \omega_{std})$

Proof: Let $i: V \hookrightarrow \mathbb{R}^{2n}$ be a closed manifold of dimension $2m$. compute the symplectic volume:

$$\int_V i^*\omega^m = \underbrace{\langle [V], [\omega]^m \rangle}_{\text{homological pairing in } H^{2m}(\mathbb{R}^{2n})} = 0 \quad \text{as } [V] \in H^{2m}(\mathbb{R}^{2n}) = 0$$



So, $i^*\omega^m$ must equal 0 at some point of V.

Here, $i^*\omega$ is degenerate. Hence, $i^*\omega$ cannot be symplectic

This proof shows that symplectic submanifolds are topologically nontrivial objects.

Donaldson's construction of symplectic submanifolds was ground breaking because it introduced a new techique for constructing topological structures which behave nicely wrt. the symplectic structure. Moreover, it is "elementary": No use of J-curve theory or seiberg-witten theory.

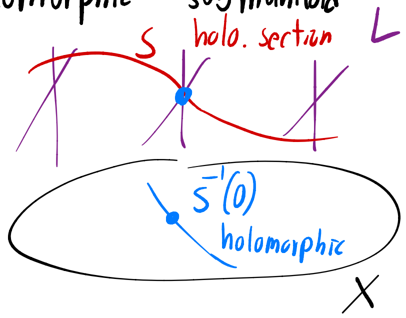
Like all good symplectic things, Donaldson borrowed a construction from complex geometry.

Theorem: Every Kahler manifold (X, ω) w/ $[\omega] \in H^2(M, \mathbb{Z})$ has a holomorphic submanifold

Proof: consider a holomorphic line bundle $\begin{array}{c} L \\ \downarrow \\ X \end{array}$, & $s \in H^0(X, L)$.

then the zero locus $s^{-1}(0)$ is holomorphic.

If I can choose s s.t 0 is a regular value then $s^{-1}(0)$ is a holo submanifold.



Thm (Bertini): a generic section $s \in H^0(X,L)$ has $s^{-1}(0)$ smooth, if $\dim H^0(X,L) \geq 2$

So, we seek a line bundle with ≥ 2 independent holo. sections. Here's a construction:

1. choose L w/ $c_1(L) = [\omega]$. "Prequantum line bundle" exists b.c $[\omega] \in H^2(M, \mathbb{Z})$
   (M,ω) is "integral"

   L carries a connection with curvature $\omega \Rightarrow L$ is "positive / ample"

2. Consider L^k for $k \gg 0$

. Riemann-Roch thm: $\chi(L^k) = \sum (-1)^i H^i(X, L^k) = \frac{k^n}{n!} \int_M c_1(L)^n + \dots$

$$= \frac{k^n}{n!} \mathrm{vol}(X) + O(k^{n-1})$$

for large powers of the line bundle, euler charecteristic is large.

Kodaira vanishing thm: for L positive, $k \gg 0$, $H^i(X, L^k) = 0$ for $i > 0$

"all cohomology concentrates in degree 0"

$$\Rightarrow H^0(X, L^k) = \frac{k^n}{n!} \mathrm{vol}(X) + O(k^{n-1}) \text{ for } k \gg 0$$

"Large powers of prequantum line bundle have many sections"

All together, for some k, $\exists$ $s \in H^0(X, L^k)$ such that $s^{-1}(0)$ is a smooth holomorphic submanifold.

$\dim s^{-1}(0) = 2n-2$, $[s^{-1}(0)] = Pd\, c_1(L^k) = Pd\,[k\omega]$

Thm (Donaldson), more specifically: if (M^{2n}, ω) is integral symplectic manifold, $\exists$ codimension 2 symplectic submanifold $V^{2n-2} \subset M^{2n}$, s.t $[V^{2n-2}] = Pd\,[k\omega]$ for $k \in \mathbb{Z}$, $k > k_0$. we call V a symplectic divisor / symplectic hyperplane section

Corollary: Every symplectic manifold has a symplectic submanifold

if $[\omega] \in H^2(M, \mathbb{Q})$, then $\exists\, q$ s.t $[q\omega] \in H^2(M, \mathbb{Z})$. by above $\exists$ symplectic submanifold V^{2n-2} s.t $[V^{2n-2}] = [kq\omega]$.

Rational symplectic forms are dense among all symplectic forms, and the condition that $i^*\omega$ is symplectic is open in space of forms ω. To find a form in (M, ω), Deform ω to $\omega' \in H^2(M, \mathbb{Q})$, & choose V symplectic wrt ω'. then, V is also symplectic wrt ω.

Donaldson's construction

Recall the hierarchy of subspaces $i: W^{2m} \hookrightarrow (V^{2n}, \omega, J)$

$$\det g|_W \geq (i^*\omega)^m \geq 0$$

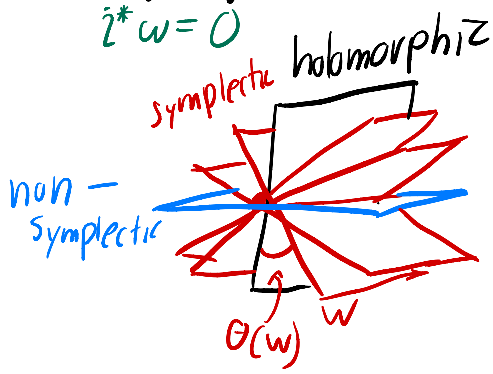
riemannian volume — symplectic volume

J-holomorphic $\subset$ symplectic $\cdots$ Lagrangian

$\det g|_W = (i^*\omega)^m/m!$ $\quad$ $\frac{i^*\omega^m}{m!} > 0$ $\quad$ $(i^*\omega)^m = 0$ $\quad$ $i^*\omega = 0$

$\Updownarrow$

$JW \subset W$



$\Rightarrow$ angle $\cos^{-1}\left(\frac{1}{m!}\frac{(i^*\omega)^m}{\det g|_W}\right) := \theta(W)$ measures distance from holomorphy

W holomorphic $\Leftrightarrow \theta(W) = 0$

W symplectic $\Leftrightarrow \theta(W) < \pi/2$ $\quad$ open condition

so a perturbation of a holomorphic subspace is a symplectic subspace

theme of the day: Quantitative measures of topological properties

$\theta(W)$ is a Quantitative measure of W being symplectic

Prop: if $|\bar\partial_J s| < |\partial_J s|$, then $s^{-1}(0)$ is a symplectic submanifold

analyze $s^{-1}(0)$ infinitesimally: $T s^{-1}(0) = \ker ds \subset TM$

if $\bar\partial_J s = 0$, $\ker ds = \ker \partial_J s$ is holomorphic

calculation: $\theta(\ker ds) \leq 2\frac{|\bar\partial_J s|}{|\partial_J s|}$ $\quad$ $\bar\partial s$ small $\Rightarrow$ $\ker ds$ nearly holomorphic

fact of linear algebra: if $a: \mathbb{R}^{2n} \cong \mathbb{C}^n \to \mathbb{C}$ is an $\mathbb{R}$-linear map, $a = a^{1,0} + a^{0,1}$, ($ds = \partial_J s + \bar\partial_J s$; $a^{1,0}$ $\mathbb{C}$-linear, $a^{0,1}$ $\mathbb{C}$-antilinear)

$|a^{0,1}| < |a^{1,0}| \Rightarrow \ker a \subset (\mathbb{R}^{2n}, \omega)$ symplectic

Donaldson constructs symplectic submanifolds by finding "asymptotically holo. sections"

Theorem: for $k \gg 0$, there are sections s_k of L^k w/

$$\boxed{|\bar\partial s_k| < \frac{C}{\sqrt{k}}|\partial s_k|}$$ on zero set of s_k

Cor: $s_k^{-1}(0)$ is symplectic, w/ $[s_k^{-1}(0)] = PD[k\omega]$ (in fact, they are nearly J-holo.)

Steps:

1. construct local sections $\sigma_p \in \Gamma(L^k)$ supported near p s.t $|\bar\partial \sigma_k| \le \frac{C}{\sqrt{k}}$
2. local transversality: for functions w/ $|\bar\partial f| \le x$ on a ball, there is a level set which has $|\bar\partial f| < |\partial f|$
3. local to global: patch together σ_{p_i} into $s_k = \sum w_i \sigma_{p_i}$ such that $|ds_k| \ge \varepsilon$ on $s_k^{-1}(0)$ (ε uniform in k)

Step 1: local model for sections of L^k

consider $(M,\omega) = (\mathbb{C}^n, \sum dz_i \wedge d\bar z_i = \omega_0)$

$\mathbb{C}^n$ has prequantum line bundle $L = \mathbb{C}\times\mathbb{C}^n \downarrow \mathbb{C}^n$, & holo. sections of L are holo. fns.

need to define hermitian metric $\langle\,,\rangle$ on L with curvature ω

in the natural trivialization of L, hermitian metric $\langle s_1, s_2\rangle = h\, s_1 \bar s_2$ $\quad h: \mathbb{C}^n \to \mathbb{R}$

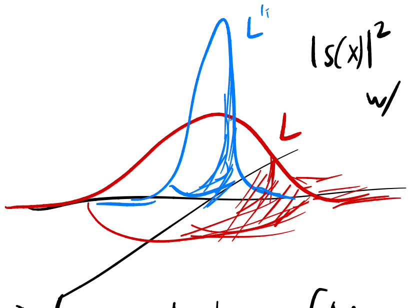
fact: curvature of chern connection associated to h is $-\partial\bar\partial \log h$

choose $h = e^{-|z|^2/2}$ $\qquad -\partial\bar\partial \log h = \frac{1}{4}\partial\bar\partial |z|^2 = \omega_0$

likewise, $h = e^{-k|z|^2/2}$ is prequantum metric on L^k

so the holomorphic section $s(x) = 1$ has $|s(x)|^2 = h = e^{-k|z|^2/2}$

gaussian section!!



$|s(x)|^2$ is gaussian bump, w/ standard deviation $1/\sqrt{k}$

Moral: passing from $L \mapsto L^k$ scales up $\omega \mapsto k\omega$
local sections are scaled down by $x \mapsto x/\sqrt{k}$

Now let (M, ω, J) be any symplectic manifold, L^k prequantum bundle

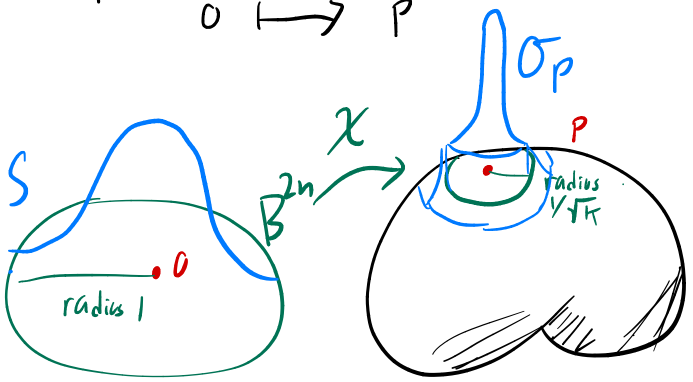
choose Darboux chart at point p:

$$\chi_p : B^{2n} \to V \qquad 0 \mapsto p$$

where $\chi_p^* \omega = k\,\omega_0$ scales up symplectic form

$\chi_p^* J|_0 = J_0$ agrees w/ complex structure at origin

$|D^n \chi_p| < C/\sqrt{k}$ uniform bound on all derivatives. B maps to small ball ⇒ derivatives are small



Then $|\chi_p^* J - J_0| < C/\sqrt{k}$

⇒ $s(x)=1$ satisfies $\left|\bar\partial_{\chi_p^* J} s\right| \le \frac{C}{\sqrt{k}}$

define σ_p as $s \circ \chi$ on $\chi(B)$, with smooth cutoff outside $\chi(B)$

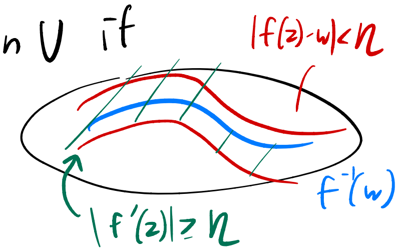
Then $\boxed{\left|\bar\partial_J \sigma_p\right| < \frac{C}{\sqrt{k}}}$

moral: transport a gaussian section on a darboux chart to (M,ω). as $k \to \infty$, the resulting section is more and more peaked, and closer & closer to being holomorphic

Step 2: Quantatitive transversality

Def: $f: U \to \mathbb{C}$ is η-transverse to w on U if

$$|f(z) - w| \le \eta \Rightarrow |df| > \eta$$



Remark: ordinary transversality is 0-transversality

if $\eta_1 > \eta_2$, then η_1-transverse ⇒ η_2-transverse $\qquad$ η measures "how transverse f is"

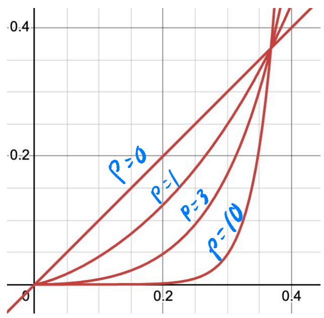
Thm: (Quantative Sards thm for almost holo fns): (thm 20 of Donaldson)

if $f: B^{2n} \subset \mathbb{C}^n \to \mathbb{C}$, $|f| \le 1$, $|\bar\partial f|_{C^1} \le \underset{\text{fixed}}{\rho}$, for any $\eta \in (0, \frac{1}{2})$,

$\exists$ w in $\mathbb{C}$ with $|w| \le \eta$ s.t. f is $\dfrac{\eta}{(-\ln(\eta))^p}$ transverse to w

integer depending on dimension

we can always find w near 0 which is transverse to f. But, the closer we want w to 0, the worse transversality we can get



graph of $\dfrac{\eta}{(-\ln(\eta))^p}$

goal: construct sections s_k w/ $|\bar{\partial} s_k| < \frac{C}{\sqrt{k}}$ with s_k ε-transverse to zero, with ε indepent of k. then, $s_k^{-1}(0)$ is symplectic for $k \gg 0$

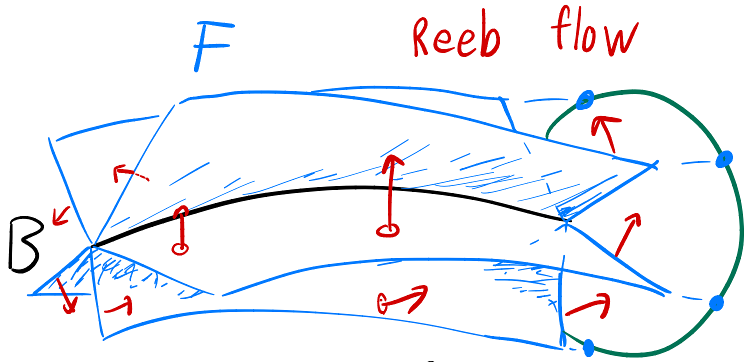
Step 3: local-to-global

ToDo

Open Book Decompositions

Def: an open book decomposition on a manifold M^n is (B^{n-2}, θ) s.t

- a binding $B^{n-2} \subset M^n$ of codimension 2
- a fibration $\theta : M \setminus B \to S^1$, with fiber $F = \theta^{-1}(c)$, $\partial F = B$



a contact form α is carried by (B, θ) if

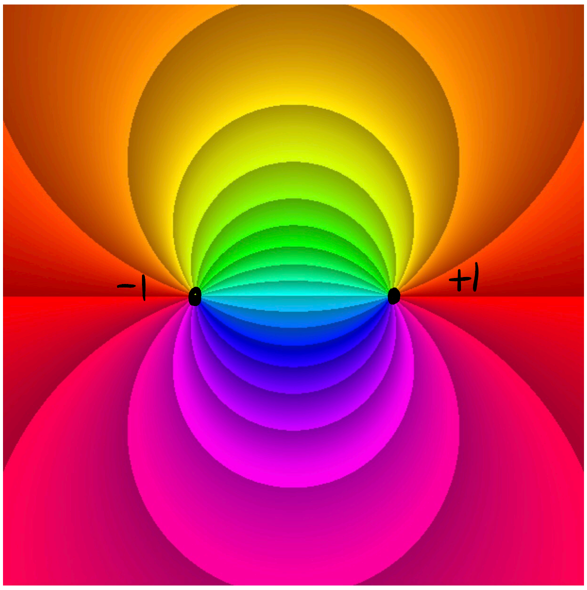
$$M \xrightarrow{\theta} S^1$$

- B is a contact submanifold (i.e. $B \pitchfork \ker\alpha$)
- $d\alpha$ is symplectic on F $\Rightarrow$ Reeb flow transverse to F
- the orientations on ∂F & B agree

Thm: (Giroux) Every contact structure is carried by an open book.

Like always, we turn to the Kahler case

Example $(M^2 = \mathbb{C})$



consider $f(z) : \dfrac{z-1}{z+1}$

$f : \mathbb{C} \to \mathbb{P}^1$

$B = \pm 1$ (Poles & zeros)

$\theta = \dfrac{f(z)}{|f(z)|} : \mathbb{C} - B \to S^1$

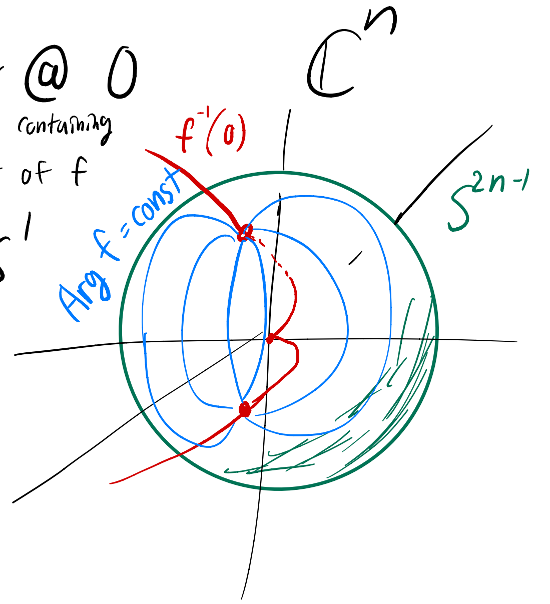
Example: Milnor fibration

$f: \mathbb{C}^n \to \mathbb{C}$ holomorphic w/ isolated singularity @ 0

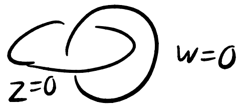
restrict f to $S^{2n-1}(\varepsilon) \subset \mathbb{C}^n$ — ε small, only containing 1 critical point of f

set $B = f^{-1}(0) \subset S^{2n-1}$; $\theta = \frac{f(z)}{|f(z)|} : S^{2n-1} - B \to S^1$

(B, θ) is open book decomp. of S^{2n-1}



e.g. $f(z,w) = zw$ $\quad f^{-1}(0) \cap S^3 =$

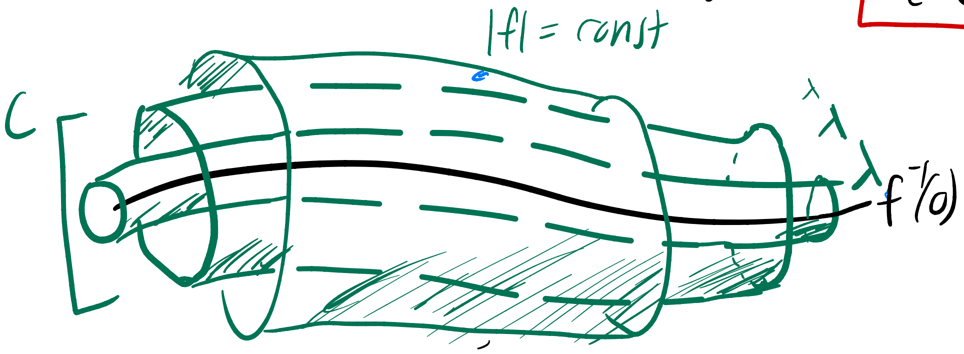


e.g. $f(z,w) = z^2 + w^3$ $\quad f^{-1}(0) \cap S^3 =$

Thm: for $f: \mathbb{C}^2 \to \mathbb{C}$ holomorphic w/ 0 isolated, the open book decomposition $\left(f^{-1}(0), \frac{f}{|f|}\right)$ on S^3 carries the standard contact structure

let $\lambda = \frac{1}{2}(x_1 dy_1 - y_1 dx_1) + \frac{1}{2}(x_2 dy_2 - y_2 dx_2)$ be the Liouville form on $\mathbb{C}^2 = (z_1, z_2)$

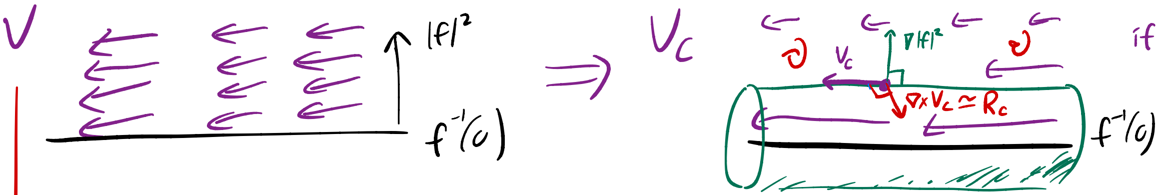
$\lambda|_{S^3}$ defines a contact form. Define $\boxed{\lambda_c = e^{-c|f|^2} \lambda}$



Gaussian decay of λ away from $f^{-1}(0)$, with width c

Claim: for $c \gg 0$, R_{λ_c} is everywhere transverse to the pages, so $d\lambda_c$ symplectic on pages

intuition: if $\lambda = \langle v, \cdot \rangle$, R_λ is parralel to $\nabla \times V \Rightarrow R_{\lambda_c}$ parralel to $\nabla \times (e^{-c|f|^2} V)$



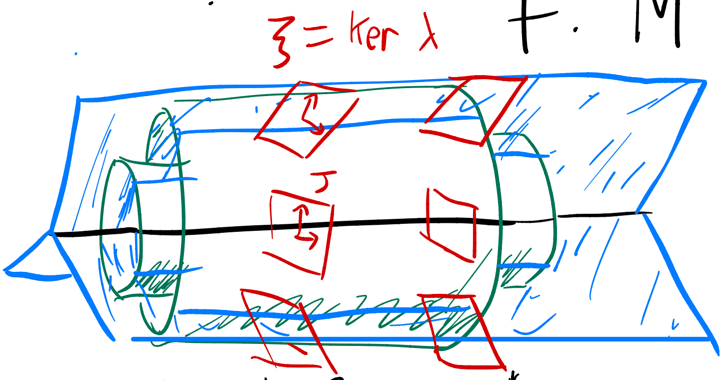
if V always points in same direction along $f^{-1}(0)$, $R_c = \nabla \times V_c$ rotates around $f^{-1}(0)$ for c large enough

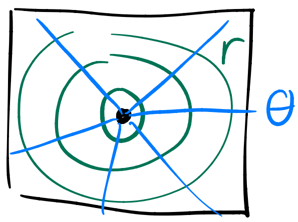
to check this: denote $d\theta = \theta^*(dx)$ from S^1. the Reeb field R_{λ_c} is positivly transverse iff $d\theta(R_{\lambda_c}) > 0 \iff d\theta \wedge d\lambda_c > 0$

$$d\theta \wedge d\left(e^{-c|f|^2}\lambda\right) = e^{-c|f|^2}\Big(\underbrace{d\theta \wedge d\lambda}_{|d\theta \wedge d\alpha| < M} - \underbrace{c}_{\gg 0}\, \underbrace{d\theta \wedge d|f|^2 \wedge \lambda}_{< 0}\Big)$$

if $d\theta \wedge d|f|^2 \wedge \lambda < 0$, then for c large enough, $d\theta \wedge d\lambda_c > 0$

$$f: M^3 \longrightarrow \mathbb{C}$$





$$d\theta \wedge d|f|^2 = -2f^*\omega \qquad \omega_{\mathbb{C}} = r\,dr\,d\theta = \tfrac{1}{2}\, d|f|^2 \wedge d\theta$$

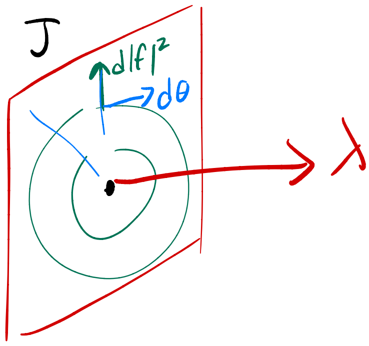
but f is <u>holomorphic</u>: Along contact structure ξ there is almost complex structure J, & $J \circ df|_\xi = df \circ j|_\xi$

so $f^*\omega_{\mathbb{C}}$ provides orientation on ξ agreeing w/ that of J

λ provides a positive co-orientation of ξ agreeing w/ that of J

$\Rightarrow \lambda \wedge f^*\omega_{\mathbb{C}}$ gives orientation of M!

$\Rightarrow d\theta \wedge d|f|^2 \wedge \lambda < 0$ $\blacksquare$



Now we translate this example to an arbitrary contact manifold, following Donaldson's template.

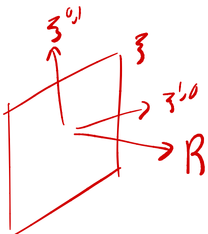
for a contact manifold (M^{2n+1}, ξ) w/ contact form λ, choose an almost complex structure J on ξ. A function $f: M \to \mathbb{C}$ is "holomorphic" if it is holo. on ξ

Thm: if $f: (M,\lambda) \to \mathbb{C}$ is holomorphic, then

- $f^{-1}(0) = B^{2n-1}$ is a contact manifold
- $\theta = f/|f| : M \setminus B \to S^1$ Defines an open book decomposition (B,θ) of M
- for $c \gg 0$, $\lambda_c = e^{-c|f|^2}\lambda$ satisfies $d\lambda_c$ symplectic on $\theta^{-1}(c)$. i.e, (B,θ) carries ξ

For a nonintegrable J, there are no holo. functions f. But, we can get asymptotically close. This w

Let $\begin{array}{c} L = M \times \mathbb{C} \\ \downarrow \\ M \end{array}$ be a trivial bundle w/ curvature $d\lambda$. this has a connection given by λ: $\nabla s = ds + i\lambda \cdot s$



∇s decomposes into 3 parts:

$$\nabla s = \nabla s\big|_{\xi}^{1,0} + \nabla s\big|_{\xi}^{0,1} + \nabla s(R)\,\lambda$$
$$\partial_{\xi,J} s + \bar\partial_{\xi,J} s + \nabla^{\perp} s$$

Thm: if s a smooth section of L w/ $|\bar\partial_{\xi,J} s| < |\partial_{\xi,J} s|$ along $B = s^{-1}(0)$, then B is a contact submanifold. Furthermore $\Theta: s/|s| : M - B \to S^1$ defines an open book decomposition.

(Giroux): (B, Θ) carries ξ

we will achieve this by constructing sections of $L^{\otimes k}$, $k \gg 0$ $\quad \nabla^k s = ds + ik\alpha s$

the following is proved using techniques analagous to donaldson:

Thm: (Ibort, Martinez-torres, Presas) for $k \gg 0$, there exists sections $s_k \in \Gamma(L^{\otimes k})$ s.t

- $|\bar\partial^k_{\xi,J} s| < \frac{C}{\sqrt{k}}$ (asymtotic holomorphicity)
- $|d_{\xi} s| > \eta$ along $s^{-1}(0)$ (zero set cut out transversly)

Cor: Every Contact manifold (M, ξ) is carried by the open book $\left(s_k^{-1}(0), \frac{s_k}{|s_k|}\right)$ for $k \gg 0$

Other applications of asymptotic holomorphic Methods

Anything you can build with holomorphic sections in Kahler geometry, you can approx. build using asymptotically holomorphic sections on an integral symplectic manifold

fix (X^{2n}, ω) Kahler, (M^{2n}, ω) symplectic, prequantum line bundles $L \to X$, $L \to M$

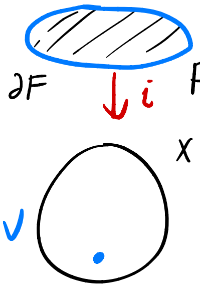
Weinstein complements

Thm (Kahler geometry): for generic $s \in H^0(M, L^k)$, $(X - \underset{s^{-1}(0)}{V}, \omega)$ is weinstein

Thm (symplectic), Giroux '18: for $k \gg 0$, there exists a symplectic divisor $V \subset M$ w/ $[V] = PD[k\omega]$ & $(M - V, \omega)$ weinstein https://arxiv.org/abs/1803.05929

more precisly, there is a weinstein domain $(F, d\lambda)$ w/ ∂F the unit normal bundle of V, & map $i: F \to X$

- i : int $F \to X - V$ is a symplectomorphism
- $i : \partial F \to V$ collapses the fibers of the normal bundle



Proof sketch (Kahler case): need a vector field v on $X - V$

- $\mathcal{L}_v \omega = \omega$ (Liouville)
- morse function ρ s.t $v(\rho) < 0$

suffices to find $v = \nabla \rho$ liouville

choose a hermitian metric $\langle \rangle$ on L w/ curvature ω

fact: for any holo section s, let $\rho(x) = \log |s(x)|^2$, defined on $X - s^{-1}(0) = X - V$. then $\partial\bar{\partial} \rho = \omega$

$\partial\bar{\partial}\rho = dd^c \rho$, where $d^c\rho(v) := d\rho(Jv)$

Prop: $\nabla \rho$ is a liouville vector field.

Proof: $\mathcal{L}_{\nabla\rho} \omega = d i_{\nabla\rho} \omega = d\, \omega(\nabla\rho, v) = d\, \omega(JX_\rho, v) = d\, \omega(X_\rho, Jv)$

$= d(d\rho(Jv)) = dd^c\rho = \omega$

moral: norm of section gives a morse function whose gradient is a liouville v.f

On a symplectic (M, ω, J), we can fenagle an asymptotic holomorphic section with these same properties.

Lefshetz pencils

Morse theory sees topology through a function $f: X \to \mathbb{R}$

When X is holomorphic, we can do "complex morse theory" via $f: X \to \mathbb{C}P^1$ holomorphic

choose two holomorphic sections $s_0, s_\infty \in H^0(X, L)$, & take $f = s_0/s_\infty$

f exists outside of $s_0^{-1}(0) \cap s_\infty^{-1}(0)$

the fibers $f^{-1}(\lambda)$ are solutions to $s_0/s_\infty = \lambda \Rightarrow s_0 - \lambda s_\infty = 0$

OR: $\mathbb{P}^1$ family of holo sections $s_\lambda = s_0 - \lambda s_\infty$, w/ fibers $f^{-1}(\lambda) = s_\lambda^{-1}(0)$

The family of divisors $s_\lambda^{-1}(0)$ are called a Lefshetz Pencil

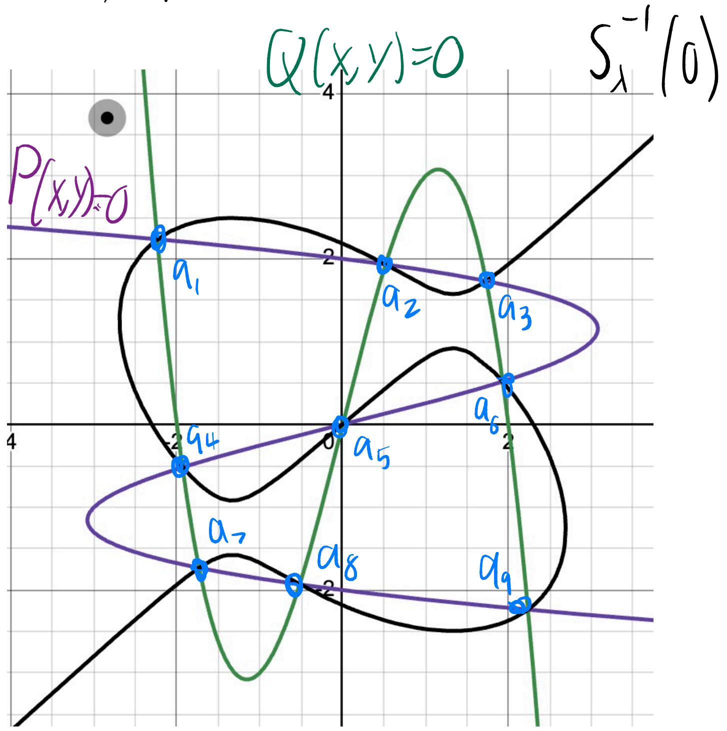
Example: $X = \mathbb{C}P^2$, $L = \mathcal{O}(3)$, $H^0(\mathbb{P}^2, L)$ = homogenous cubics on $\mathbb{C}^3$

Choose cubics P, Q. Their zero sets $P^{-1}(0)$, $Q^{-1}(0)$ intersect at 9 pts $\{a_1, \dots, a_9\}$

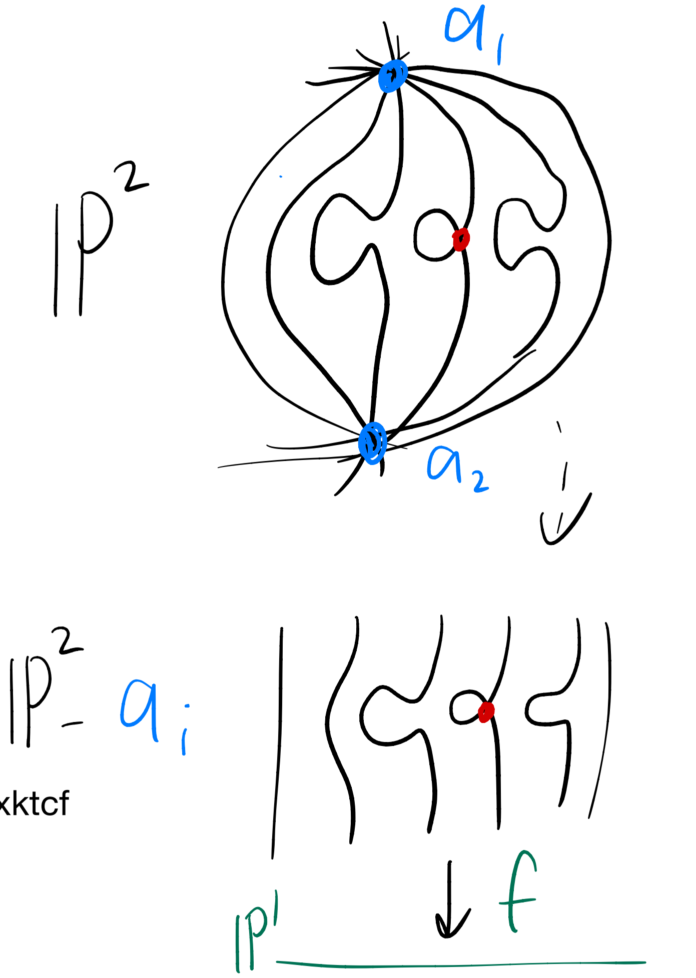
every cubic curve $P - \lambda Q = 0$ passes through $a_1, \dots, a_9$

P/Q defines $f: \mathbb{P}^2 - \{a_1, \dots, a_9\} \to \mathbb{P}^1$

every point in $\mathbb{P}^2 - \{a_1, \dots, a_9\}$ belongs to $s_\lambda^{-1}(0)$ for exactly 1 $\lambda \in \mathbb{P}^1$

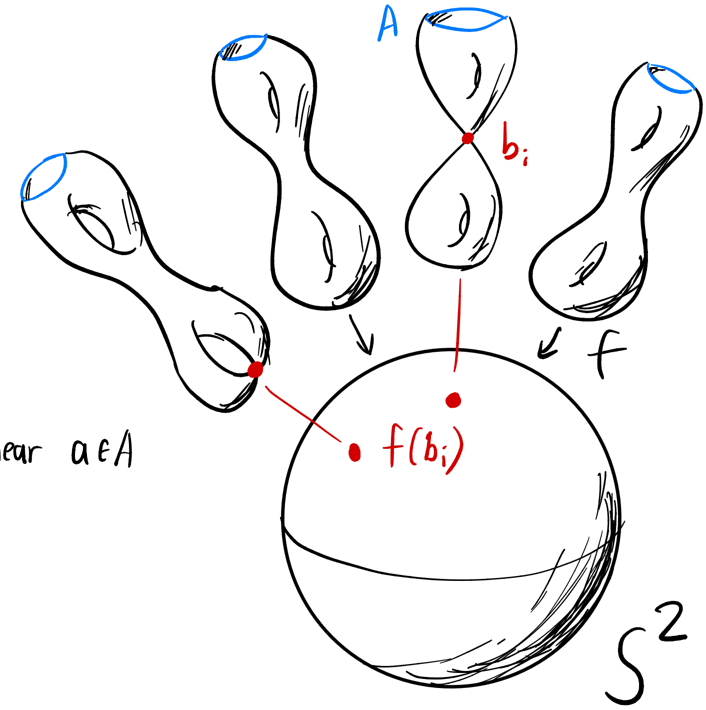


https://www.desmos.com/calculator/j3pudxktcf



Definition: A topological Lefshetz fibration on a symplectic manifold (M,ω) is:

- A codimension 4 set $A \subset M$
- A set of points $\{b_i\} \subset M \setminus A$ "critical points"
- A map $f: M \setminus A \to S^2$ which is
  - a submersion outside of $\{b_i\}$
  - $f(b_i) \neq f(b_j)$ for $i \neq j$



- Local complex coordinates $(z_1, \dots, z_n)$ near $a \in A$
  - $A = \{z_1 = z_2 = 0\}$
  - $f(z_1, z_2, \dots, z_n) = z_1/z_2 \in \mathbb{C}P^1$
- local complex coordinates near b_i where
  - $f(z_1, \dots, z_n) = f(b_i) + z_1^2 + \dots + z_n^2$ morse-type critical point

Thm (Lefshetz): Every Kahler (X,ω) has a topological lefshetz fibration

pick $s_0, s_\infty \in H^0(X, L^k)$. $f(x) = [s_0(x), s_\infty(x)]$

Define $A = s_0^{-1}(0) \cap s_\infty^{-1}(0)$, $f: X \setminus A \to \mathbb{P}^1$

note $[f^{-1}(\lambda)] = PD\, c_1(L^k) = PD\, [k\omega]$

for $k \gg 0$, can choose s_0, s_∞ generically enough to ensure f is nondegenerate.

Thm (Donaldson '96): Every integral symplectic mfld (M,ω) has a topological lefshetz fibration with symplectic fibers, & $[f^{-1}(\lambda)] = PD\,[k\omega]$ $k \gg 0$

choose two asymptotically holomorphic sections s_0, s_∞ of L^k, & mimic the above construction. we need enough freedom that $s_\lambda^{-1}(0)$ is cut out sufficiently transversely to be a symplectic submanifold. this has to work for every λ – a very souped up sard's thm.

What if we used 3 sections, instead of 2?

Thm (Auroux '00): every symplectic 4 manifold is topologically a branched cover of $\mathbb{C}P^2$, branched over a symplectic divisor.

the map $f: (M,\omega) \mapsto \mathbb{C}P^2$ is furnished by choosing three sections s_0, s_1, s_2, & defining $f(x) = [s_0(x) : s_1(x) : s_2(x)]$ https://link.springer.com/article/10.1007/s002220050019

Projective embeddings

mimic the construction for lefshetz pencils with many sections of L^k.

Kahler setting:

Choose basis $s_1, \dots, s_d$ of $H^0(X, L^k)$, $d = \dim H^0(X, L^k)$

Define $\Psi : X \to \mathbb{P}(H^0(X, L^k))$ Ψ is holomorphic

$x \longmapsto [s_1(x) : \dots : s_d(x)]$

Thm (Kodaira embedding): for $k \gg 0$, $X \hookrightarrow \mathbb{P}(H^0(X, L^k))$ is an embedding

Thm (Borthwick, uribe '98): https://arxiv.org/abs/math/9812041

There is an embedding $i_k : M \hookrightarrow (\mathbb{P}^k, \omega_{FS})$ for $k \gg 0$ s.t $i^* \omega_{FS}$ is symplectic. that is, every M is a symplectic submanifold of projective space.

fubini-Studi form

Moreover, for $k \gg 0$, and a compatible triple (ω, g, J) on M, i can be made asymptotically Kahler:

symplectic: $|i_k^* \omega_{FS} - \omega| = O(\frac{1}{k})$

isometric: $|i_k^* g_{FS} - g| = O(\frac{1}{k})$

holomorphic: $\|\partial i_k\| = O(k)$, $\|\bar\partial i_k\| = O(1)$

An elliptic approach to asymptotic holomorphicity

unlike the other results, Brothwick & uribe do not use an extension of donaldsons techniques. Instead, they construct asymptotically holomorphic sections as solutions of an elliptic PDE, the spin-c dirac equation

Kahler case: consider the dolbeault complex on X valued in L^k

$$\Omega^{0,0} \otimes L^k \xrightarrow{\bar\partial} \cdots \to \Omega^{0,n} \otimes L^k \quad \text{roll up} \quad \underbrace{\bigoplus_i \Omega^{0,2i} \otimes L^k}_{\mathcal{E}^+} \underset{\bar\partial + \bar\partial^*}{\overset{\bar\partial + \bar\partial^*}{\rightleftarrows}} \underbrace{\bigoplus_i \Omega^{0,2i+1} \otimes L^k}_{\mathcal{E}^-}$$

solutions to $(\bar\partial + \bar\partial^*)\alpha = 0$ are harmonic forms,

so $\ker(\bar\partial + \bar\partial^*)|_{\mathcal{E}^+} = \bigoplus_i H^{0,2i}(X, L^k)$ $\operatorname{coker}(\bar\partial + \bar\partial^*) = \ker(\bar\partial + \bar\partial^*)|_{\mathcal{E}^-} = \bigoplus_n H^{0,2i+1}(X, L^k)$

$\Rightarrow \operatorname{index}(\bar\partial + \bar\partial^*) = \sum_i (-1)^i \dim H^{0,i}(X, L^k) = \chi(L^k)$

for $k \gg 0$, $\ker(\bar\partial + \bar\partial^*)$ concentrates in degree 0.

Almost complex case

(M, ω, J) J compatible almost complex structure

Define spin$^{\mathbb{C}}$ dirac operator $\not{D}$ twisted by line bundle L^k:

$$\underbrace{\bigoplus \Lambda^{0,2n} T^*M \otimes L^k}_{\mathcal{E}^+} \underset{\not{D}^-}{\overset{\not{D}^+}{\rightleftarrows}} \underbrace{\bigoplus \Lambda^{0,2n+1} T^*M \otimes L^k}_{\mathcal{E}^-}$$

$\mathcal{E}^\pm$ is canonical spin$^{\mathbb{C}}$ spinor bundle defined by J

$\not{D} = \not{D}^+ + \not{D}^-$ is the associated dirac operator.

$\not{D}$ agrees w/ $\bar{\partial}_J + \bar{\partial}_J^*$ up to lower order terms, so $\text{index}(\not{D}) = \text{index}(\bar{\partial} + \bar{\partial}^*)$

Thm (Brothwick-Uribe): $\text{index}\, \not{D} = \dim \ker \not{D}^+$ for $k \gg 0$

https://arxiv.org/abs/dg-ga/9608006
Almost complex structures and geometric quantization

if J is integrable, then for large k, $H^0(M, L^k) \cong \ker \not{D}^+$.

as J deforms to an almost complex structure, though we lose any holomorphic sections of L^k, the # of solutions to $\not{D}\psi = 0$ remains constant

"harmonic spinors"

Conjecture: if $\not{D}\psi = 0$, the degree 0 component ψ_0 of $\psi \in \Gamma(\bigoplus_n \Lambda^{0,2n} T^*M \otimes L^k)$ is asymptotically holomorphic

i.e $|\bar{\partial}_J \psi_0(x)| \le \frac{C}{\sqrt{k}} \|\psi\|_{L^2}$

(remark: I think (?) I can extract this from the asymptotic isometry into projective space)

Conjecture: you can choose $\psi \in \ker \not{D}^+$ s.t ψ_0 has Quantitative transversality

$|\partial \psi_0|_{\psi_0^{-1}(c)} = O(1)$ while $|\bar{\partial} \psi_0|_{\psi_0^{-1}(c)} = o(\frac{1}{\sqrt{k}})$

This is part of an old dream: a proof of the donaldson submanifold theorem through microlocal analysis. The pursuit of this led to elaboration of the theory of Spin$^{\mathbb{C}}$ Quantization:

- Development of Almost kahler Quantization, Brothwick and Uribe
  https://arxiv.org/abs/dg-ga/9608006
- Investigation into asymptotic expansions of the bergman kernel for the spin$^{\mathbb{C}}$-Dirac operator: Brothwick-Uribe '98 https://arxiv.org/abs/math/9812041

  Ma, Marinescu '07: Bergman Kernel's on symplectic manifolds

- Shiffman & Zeldich have a different operator whose kernel gives asymptotic holomorphic sections. They actually achieved a microlocal proof of Donaldson's submanifold theorm. But, their operator is noncanonical & hard to write down.

ASYMPTOTICS OF ALMOST HOLOMORPHIC SECTIONS OF AMPLE LINE BUNDLES ON SYMPLECTIC MANIFOLDS

https://arxiv.org/abs/math/0212180