Abelian Chern-Simons theory & Geometric Quantization

abelian CS theory toes the line between interesting & trivial

M 3-mfld $\quad \mathfrak{g} = \text{Lie}\, G \quad$ connection 1-form $A \in \Omega^1(M, \mathfrak{g})$

$$S_G(A) = \int_{M^3} tr\left(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A\right)$$

Abelian CS → G cmpt abelian. take $G = U(1)$:

$\text{Lie}(U(1)) \cong \mathbb{R}$, so $A \in \Omega^1(M, \mathbb{R})$

$$S_{U(1)}(A) = \int_{M^3} A \wedge dA$$

Quadratic in A!

$$Z(M^3) = \int_{\Omega^1(M,\mathbb{R})/\mathcal{G}} e^{ik S_{U(1)}(A)} \, DA$$

level $k \in \mathbb{Z}$

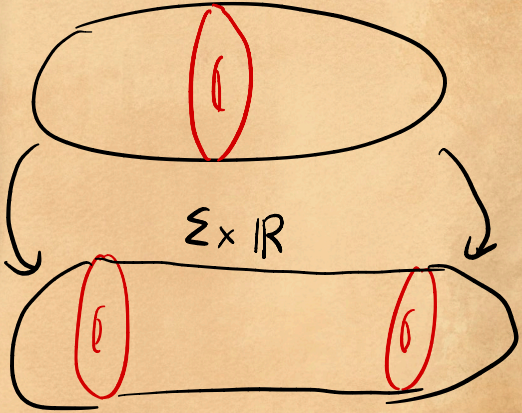
grp of gauge transforms

has exact stationary phase approximation

Abelian Chern simons is Free field theory

Classical solutions

$\Sigma^2 \subset M^3$



Classical solutions

Critical points of $S_{U(1)}(A)$

$\Rightarrow$ curvature $F_A = dA = 0$ ← $\underline{\text{Linear}}$ in A

classical states attached to Σ

$= $ solutions to $F_A = 0$ on $\Sigma \times \mathbb{R} / \mathcal{G}$

$= \{ A \in \Omega^1(\Sigma, \mathbb{R}) \mid F_A = 0 \} / \mathcal{G}$

Space of classical states

= moduli space of <u>flat</u> $U(1)$ connections on trivial $U(1)$ bundle over Σ

$:= \mathrm{Bun}_{U(1)}(\Sigma)$

Space of Quantum states

= <u>Quantization</u> of $\mathrm{Bun}_{U(1)}(\Sigma)$

Geometric Quantization

Geometric Quantization

Symplectic

$$\begin{array}{ccc} (M, \omega) & f \in C^\infty & \{f_1, f_2\} \\ \text{Quantize} \Big\downarrow & \Big\downarrow & \| \\ \text{Hilbert Space } \mathcal{H} & \text{operator } O_f & [O_{f_1}, O_{f_2}] \end{array}$$

Dream: canonical map $(M, \omega) \rightsquigarrow \mathcal{H}$

Geometric Quantization: $L \to M$ line bundle
$\mathcal{H} = \Gamma(M, L)$

Beautiful idea, only half works

Shopping List

Shopping list

for manifold M^{2n}, we need

- a Symplectic form ω
- a Line bundle (L, ∇) w/ curvature ω

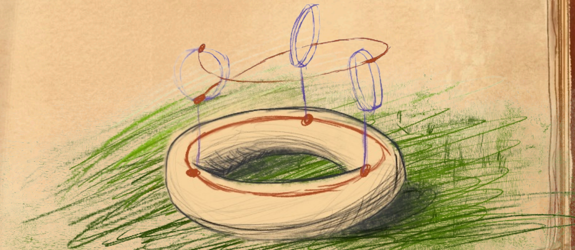
  $\Rightarrow$ cohomology $[\omega] = [c_1(L)] \in H^2(M, \mathbb{Z})$

- A "polarization": cuts down $\Gamma(M^{2n}, L)$ from $2n$ variables to n variables

  ↪ Kahler polarization: Holomorphic structures on M & L

$$\mathcal{H} = H^0(M, L)$$

$\mathrm{Bun}_{U(1)} \Sigma$ as a manifold

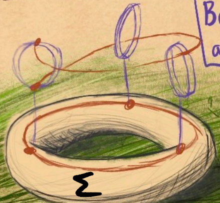


$\mathrm{Bun}_{U(1)}(\Sigma)$ as a manifold

flat connections up to gauge transforms determined by monodromy

$$
\begin{aligned}
\mathrm{Bun}_{U(1)}(\Sigma) &\simeq \mathrm{Hom}(\pi_1(\Sigma), U(1)) \\
&\simeq \mathrm{Hom}(H_1(\Sigma,\mathbb{Z}), U(1)) \quad \text{abelianize!} \\
&\simeq H^1(\Sigma, U(1)) \quad H_1(\Sigma,\mathbb{Z}) \text{ torsion free} \\
&\simeq H^1(\Sigma,\mathbb{R}) / H^1(\Sigma,\mathbb{Z}) := V/\Lambda \\
&\simeq \mathbb{R}^{2g}/\mathbb{Z}^{2g} \simeq U(1)^{2g} \quad \Sigma \text{ genus } g
\end{aligned}
$$

$\mathrm{Bun}_{U(1)}(\Sigma)$ is a torus!

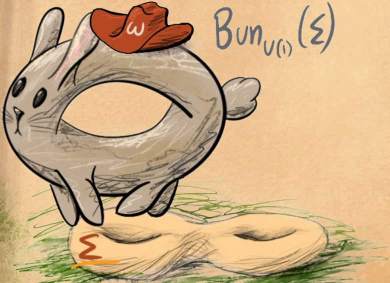


$\mathrm{Bun}_{U(1)}(\Sigma)$ symplectic structure

Geometric Quantization Shopping list

- (M,ω) symplectic
- (L,∇) w/ curvature ω
- holo structure on M & L

$\Rightarrow \mathcal{H} = H^0(M,L)$



$\mathrm{Bun}_{U(1)}(\Sigma)$

$\mathrm{Bun}_{U(1)}(\Sigma)$ symplectic structure

natural symplectic form on $H^1(\Sigma, \mathbb{R})$

for classes $[a], [b] \in H^1(\Sigma, \mathbb{R})$, define

$$\omega([a],[b]) = [a] \cup [b] = \int_\Sigma a \wedge b$$

Translation-invariant form on V

$\hookrightarrow$ induces symplectic form ω on V/Λ

$\omega \in H^2(V/\Lambda, \mathbb{Z}) \iff \forall \lambda_1, \lambda_2 \in \Lambda, \ \omega(\lambda_1, \lambda_2) \in \mathbb{Z}$

Geometric Quantization Shopping list

- ✓ (M, ω) symplectic
- (L, ∇) w/ curvature ω
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$\Rightarrow \mathcal{H} = H^0(M, L)$

Line bundle $\mathcal{L} \to \mathrm{Bun}_{U(1)}$

line bundle $\mathcal{L} \to Bun_{U(1)}$

Plan: 1. construct $\mathcal{L} \to (V, \omega)$

2. pass to $\mathcal{L} \to V/\Lambda = Bun_{U(1)}$

Step 1: inspiration from quantizing V

choose coords q_i, p_i s.t $\omega = \sum dq_i \wedge dp_i$

$\{q_i, p_j\} = \delta_{ij}$, $\{q_i, q_j\} = \{p_i, p_j\} = 0$

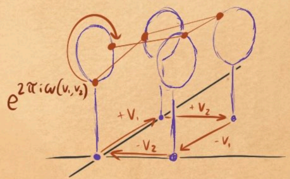
heisenburg Lie algebra $\underline{\text{heis}}(V)$

To Quantize, construct heisenberg Lie group

$$\text{Heis}(V) := V \rtimes U(1) \qquad v_{1,2} \in V, \; z_{1,2} \in U(1)$$

$$(v_1, z_1) \cdot (v_2, z_2) = (v_1 + v_2, e^{2\pi i \omega(v_1, v_2)} z_1 z_2)$$

Heisenburg group $U(1) \rtimes V$



geometrically, $\begin{array}{c} \mathrm{Heis}(V) \\ \downarrow \\ V \end{array}$ Principle $U(1)$ bundle

$\Rightarrow$ Line bundle $\mathcal{L}$

$\mathrm{Heis}(V) \circlearrowright L^2(\mathcal{L})$ reducible representation

Fact: Group structure $\Rightarrow$ natural connection $\nabla = d + \lambda$

connection 1-form $\lambda = \sum p_i dq_i$

curvature $d\lambda = \omega$!

(or, writing $V = T^*W$, λ is tautological 1-form & ω canonical symplectic form)

2.

$$\begin{array}{ccc} \mathrm{Heis}(V) & \dashrightarrow & ??? \\ \downarrow & & \downarrow \\ V & \longrightarrow & V/\Lambda \end{array}$$

build $\mathcal{L}$ as $U(1)$ central extention of V/Λ

define Discrete heisenburg Group $v_0 \in V/\Lambda$

$$\mathrm{Heis}(\Lambda, v_0) = \{(\lambda, e^{2\pi i \omega(v_0, \lambda)}) \mid \lambda \in \Lambda\}$$

$\mathrm{Heis}(V)/\mathrm{Heis}(\Lambda, v_0) \longrightarrow V/\Lambda$ gives $U(1)$ bundle

"θ line bundle"

Define associated line bundle $\mathcal{L}_\theta$:

$$V \times \mathbb{C} \Big/ (v,z) \sim \big(v+\lambda,\ \underbrace{e^{2\pi i\,\omega(v,\lambda)}\, e^{2\pi i\,\omega(v_0,\lambda)}}_{\text{factor of automorphy}} z\big) \quad \forall \lambda \in \Lambda$$

$e^{2\pi i\,\omega(v,\lambda)}$ contributes curvature ω

$e^{2\pi i\,\omega(v_0,\lambda)}$ contributes curvature 0

flat line bundle on V/Λ paramertized by point $v_0 \in V/\Lambda$

$$\begin{array}{c} \mathcal{L} \\ \downarrow \\ \mathrm{Bun}_{U(1)}\Sigma \end{array}$$

w/ curvature ω

$\mathrm{Bun}_{U(1)}\Sigma$ as a Kähler mfld

Geometric Quantization Shopping list

- (M,ω) symplectic
- (L,∇) w/ curvature ω
- holo structure on M & L

$\Rightarrow \mathcal{H} = H^0(M,L)$

$\mathrm{Bun}_{U(1)}$ as a Kahler mfld

To finish quantizing, need to choose polarization

Choose complex structure j on Σ

Lets us use hodge decomposition:

$$H^1(\Sigma,\mathbb{R}) \simeq H^{1,0}(\Sigma,\mathbb{C}) \simeq \mathbb{C}^g$$ complex vector space

$$\Rightarrow \mathrm{Bun}_{U(1)}(\Sigma) \simeq H^{1,0}(\Sigma,\mathbb{C})/H^1(\Sigma,\mathbb{Z})$$ complex torus

exponential sequence $\simeq H^1(\Sigma,\mathcal{O}^*)$ Picard Group of deg 0 holo line bundles

$$\boxed{\mathrm{Bun}^{flat}_{U(1)}(\Sigma) \simeq \mathrm{Bun}^{hol}_{U(1)}(\Sigma)}$$

topology algebraic geometry

Paramertizing complex tori

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Paramertizing complex tori

let $V \simeq \mathbb{C}^g$, $\Lambda = \mathrm{Span}_{\mathbb{Z}}(a_i, b_i)_{i=1,\dots,g} \simeq \mathbb{Z}^{2g} \subset \mathbb{C}$

as a matrix: $\begin{array}{cc} & 2g \\ g & \left[\begin{array}{ccc|ccc} | & & | & | & & | \\ a_1 & \cdots & a_g & b_1 & \cdots & b_g \\ | & & | & | & & | \end{array}\right]\end{array}$ a_i, b_i satisfy $\omega(a_i, b_j) = \delta_{ij}$, $\omega(a_i, a_j) = \omega(b_i, b_j) = 0$

row reduce! $\Lambda \sim \left[\begin{array}{ccc|c} 1 & & 0 & \\ & \ddots & & \tau \\ 0 & & 1 & \end{array}\right]$ τ "period matrix"

$\Lambda \simeq \mathbb{Z}^g \oplus \tau(\mathbb{Z}^g)$

τ determines complex structure of torus

Geometric Quantization Shopping list

- ✓ (M, ω) symplectic
- ✓ (L, ∇) w/ curvature ω
- ✓ holo structure on M & L

$\Rightarrow \mathcal{H} = H^0(M, L)$

Theta functions

Theta functions

using $\Lambda = \mathbb{Z}^g \oplus \tau(\mathbb{Z}^g)$, for $\vec{n} \in \mathbb{Z}^g$,

- $S(z+\vec{n}) = S(z)$
- $S(z+\tau\vec{n}) = \underbrace{e^{-2\pi i\left(\frac{1}{2}\langle n,\tau n\rangle + \langle n,z\rangle\right)}}_{\varepsilon_{\vec{n}}(z)} S(z)$

$H^0(\mathrm{Bun}_{U(1)}, \mathcal{L}_\theta)$ is the <u>holomorphic</u> $S(z)$ satisfying above

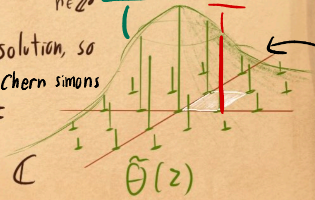
solutions are θ-functions!

$$\Theta(z,\tau) = \sum_{n\in\mathbb{Z}^g} e^{-2\pi i\left(\frac{1}{2}\langle n,\tau n\rangle + \langle n,z\rangle\right)}$$

fourier transform: $\tilde{\theta} = \sum_{n\in\mathbb{Z}^g} e^{-\pi i\langle n,\tau n\rangle}\delta_n$

this is <u>unique</u> solution, so Hilbert space of "level 1" Chern simons

$\mathcal{H}(\Sigma, j, \text{level } 1) = H^0(\mathrm{Bun}_{U(1)}, \mathcal{L}_\theta) \simeq \mathbb{C}$



Torus $T_\tau = V/\mathbb{Z}^g \oplus \tau(\mathbb{Z}^g)$

Sections of θ line bundle $\mathcal{L}_\theta \downarrow T_\tau$ are holomorphic functions $S(z)$ s.t

the forier transform of θ function is a comb of dirac deltas, with amplitude controlled by a gaussian.

This looks an awful lot like the forier transform of the fundamental solution to the heat eqn!

Level $k \in \mathbb{Z}^{\geq 0}$

Consider $H^0(\mathrm{Bun}_{U(1)}, \mathcal{L}_\Theta^{\otimes k})$

sections of $\mathcal{L}_\Theta^k$ transform as

- $s(z+n) = s(z)$
- $s(z+\Omega n) = \varepsilon_{\vec{n}}(z)^k \, s(z)$

solutions indexed by $m \in \mathbb{Z}_k^g = (\mathbb{Z}/k\mathbb{Z})^g$

$$\Theta_m(z,\tau) = \sum_{\substack{n\in\mathbb{Z}^g \\ n = m \bmod k}} e^{\frac{\pi i}{k}\langle n, \tau n\rangle + 2\pi i \langle z, n\rangle}$$

$\Rightarrow \dim H^0(\mathrm{Bun}_{U(1)}, \mathcal{L}_\Theta^k) = k^g$

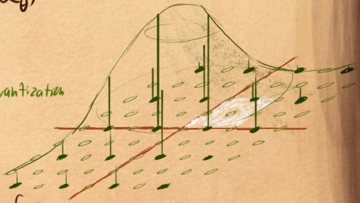
$(\mathrm{Bun}_{U(1)}, k\omega)$

$\rightsquigarrow$ geometric quantization

$\mathcal{H}(\Sigma, j, \text{level} = k) =$

$H^0(\mathrm{Bun}_{U(1)}(\Sigma), \mathcal{L}_\Theta^k) =$

space of level k Θ-fns



Invariance of polarization

Invariance of Polarization

We made up a complex structure j on Σ to Quantize $Bun_{U(1)}(\Sigma) \rightsquigarrow \mathcal{H}(\Sigma, j)$

Q: how does $\mathcal{H}(\Sigma, j)$ depend on j?

only j-dependence comes thru complex structure on $Bun_{U(1)}(\Sigma)$, measured by period matrix τ

Riemann bilinear relations: $\begin{matrix} \tau = \tau^T \\ \operatorname{im} \tau > 0 \end{matrix}$ ✡

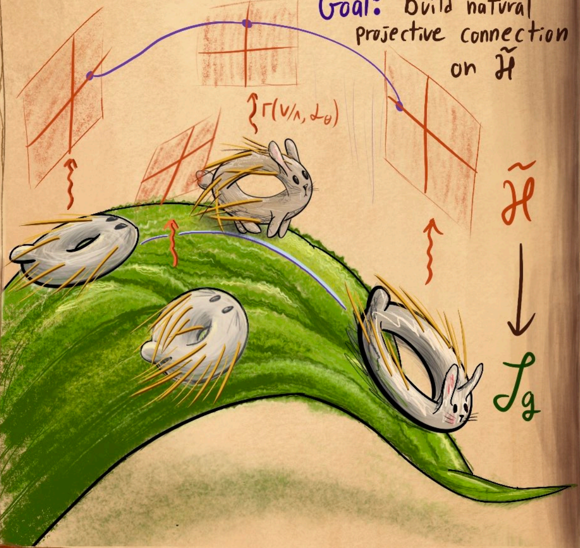
space of τ satisfying ✡ is called the <u>Segal upper half-space</u> $\mathcal{L}_g$

Facts: $\mathcal{L}_g = Sp(2g, \mathbb{R}) / U(g)$

$\begin{Bmatrix} \text{complex structures} \\ \text{on genus } g \ \Sigma \end{Bmatrix} \hookrightarrow \begin{Bmatrix} \text{space of polarized} \\ \text{complex tori} \end{Bmatrix} = Sp(2g, \mathbb{Z}) \backslash \mathcal{L}_g$

Build vector bundle $\tilde{\mathcal{H}}$ over $\mathcal{J}_g$ whose fiber over τ is $\mathcal{H}_\tau^k = H^0(V/\mathbb{Z}^g \oplus \tau\mathbb{Z}^g, \mathcal{L}_\theta^k)$

Goal: build natural projective connection on $\tilde{\mathcal{H}}$



Projective connection & Heat equation

Projective connection & Heat equation

First, Kahler Quantize (V, ω):

$Heis(V)$ gives line bundle $\mathcal{L} \to V$ w/ curvature ω

Say a complex structure $J: V \to V$ is compatible if $J^2 = -1$, $\omega(Jv, v) > 0$

Fact: for J compatible $H^0(V, \mathcal{L})$ is irrep of $Heis(V)$

Thm (Stone-von neumann): $Heis(V)$ has unique irrep up to scale

Fact: compatible J also paramertized by $\mathcal{J}_g$
$\Rightarrow$ must be a projetivly flat connection on
$\mathcal{H}_\tau(V) := H^0_{J_\tau}(V, \mathcal{L})$

Thm: The connection on $\overset{\tilde{\mathcal{H}}(V)}{\downarrow} \atop \mathcal{J}_g$ is $\delta^{\tilde{\mathcal{H}}(V)} = \delta - \Theta$
where the connection 1-form $\Theta \in \Omega^1(\mathcal{J}_g, End(\tilde{\mathcal{H}}))$
is $\Theta = \underbrace{d\tau_{ij}}_{\Omega^1(\mathcal{J}_g)} \underbrace{\partial_{z_i} \partial_{z_j}}_{\in\, End\, H^0_{J_\tau}(V, \mathcal{L})}$

τ_{ij} are coords on $\mathcal{J}_g$
$\partial_{z_i}\partial_{z_j} \Theta(z, \tau)$ is another Θ-fn,
so $\partial_{z_i}\partial_{z_j} \in End\, H^0_{J_\tau}(V, \mathcal{L})$

Projective connection & Heat equation 2

so, parralel transport of sections achieved by solving heat equation:

if $\tau(t)$ path in $\mathcal{J}_g$, then a section S is parralel

if $\frac{\partial}{\partial t} S(z,\tau(t)) = \sum \dot{\tau}_{ij}(t)\, \partial_{z_i}\partial_{z_j} S(z,\tau(t))$

this carries over immediatly to quantizations of V/Λ, w/ sections $\underset{\mathbb{Z}^g}{\underset{\cap}{\Theta_m^k}}(z,\tau) = \sum\limits_{\substack{n=m \\ \text{mod } k}} e^{2\pi i(\langle z,n\rangle + \frac{1}{2}\langle n,\tau n\rangle)}$

<u>Thm</u>: $\frac{\partial}{\partial \tau_{ij}} \Theta_m^k(z,\tau) = \frac{i}{4k\pi} \frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j} \Theta_m^k(z,\tau)$

Q: why heat equation ??

A: Deformations of $\mathcal{J}$ come from flows of <u>Quadratic</u>, positive Hamiltonians & these act by 2nd order, elliptic operators after Quantization

Θ-functions as Heat Kernel

$$\frac{\partial \theta(z,\tau)}{\partial \tau_{ij}} = \frac{\partial^2 \theta(z,\tau)}{\partial z_i \, \partial z_j}$$

take $\tau_{ij}(t) = t\,\delta_{ij}$:

$$\Theta(z, tI) = \sum_{n \in \mathbb{Z}^g} e^{-\pi i \langle \vec{n}, \vec{n} \rangle t} e^{2\pi i \langle \vec{n}, z \rangle}$$

Then $\frac{\partial}{\partial t} \Theta(z, \tau(t)) = \sum_i \frac{\partial^2}{\partial z_i^2} \Theta(z, \tau(t))$ heat equation!

in fact, $\Theta(z, \tau(t))$ is fundamental solution of $\partial_t \Theta = \Delta \Theta$

meaning $\Theta(z, \tau(t)) \xrightarrow{t \to 0} \delta_0$

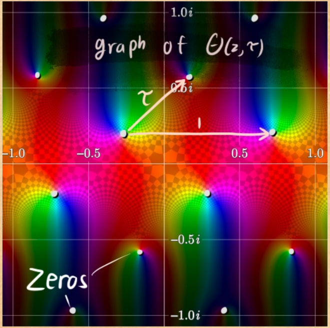
"Proof": $\Theta(z,0) = \sum_{\lambda \in \mathbb{Z}^g} e^{2\pi i \langle \lambda, z \rangle} = \sum_{\lambda \in \mathbb{Z}^g} \delta_\lambda$

Poisson summation formula

$= \delta_0$ on $V^{\mathbb{R}} / \mathbb{Z}^g$ real part of V

Example: $g = 1$

$$\underset{\{\tau \in \mathbb{C} \mid \operatorname{im} \tau > 0\}}{\Theta(z, \tau)} = \sum_{n \in \mathbb{Z}} e^{2\pi i \left(\frac{1}{2} n^2 \tau - n z\right)}$$



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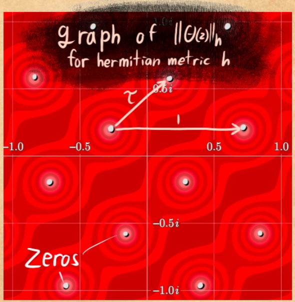

https://samuelj.li/complex-function-plotter/#abs(sum(exp(2*pi*i*(n%5E2*(a%2Bi*b)%2F2%2Bn*z))%2Cn%2C-10%2C10))*exp(-pi%2Fb*abs(imag(z))%5E2)

hermitian metric $\quad h(z) = e^{-\frac{\pi}{\mathrm{im}\,\tau}|\mathrm{im}\, z|^2}$

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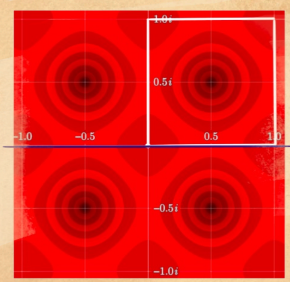
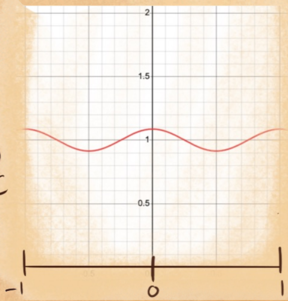
graph of $\|\Theta(z,\tau)\|_h$



$\tau = 1$

evaluate along line

graph of $\|\Theta(x,\tau)\|_{h(z)}$ on $x\in\mathbb{R}\subset\mathbb{C}$



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$= \delta_0$ on $V^{\mathbb{R}}/\mathbb{Z}^g$ real part of V

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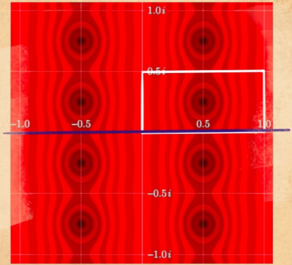
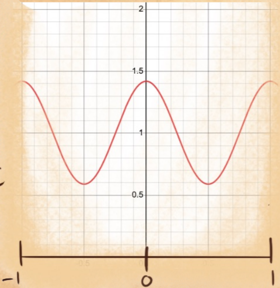
graph of $\|\theta(z,\tau)\|_h$



$\tau = 0.5$

evaluate along line

graph of $\|\theta(x,\tau)\|_{h(z)}$ on $x\in\mathbb{R}\subset\mathbb{C}$



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take $\tau_{ij}(t) = t\,\delta_{ij}$:

$$\Theta(z,\delta I) = \sum_{n\in\mathbb{Z}^g} e^{-\pi i\langle \vec{n},\vec{n}\rangle t}\, e^{2\pi i\langle \vec{n}, z\rangle}$$

Then $\frac{\partial}{\partial t}\,\Theta(z,\tau(t)) = \sum_i \frac{\partial^2}{\partial z_i^2}\,\theta(z,\tau(t))$ heat equation!

in fact, $\Theta(z,\tau(t))$ is fundamental solution of $\partial_t\theta = \Delta\theta$

meaning $\Theta(z,\tau(t)) \xrightarrow{t\to 0} \delta_0$

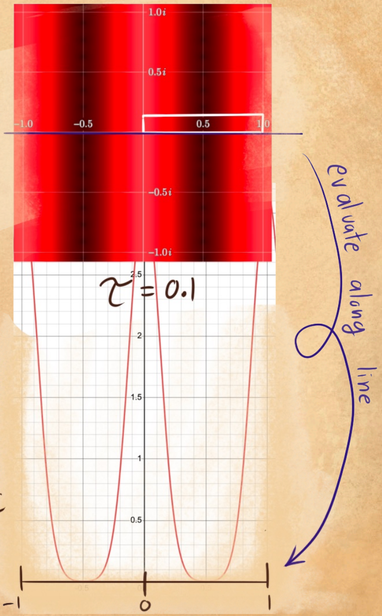
"Proof": $\Theta(z,0) = \sum_{\lambda\in\mathbb{Z}^g} e^{2\pi i\langle\lambda, z\rangle} \underset{\text{Poisson summation formula}}{=} \sum_{\lambda\in\mathbb{Z}^g} \delta_\lambda$

$= \delta_0$ on $V^{\mathbb{R}}/\mathbb{Z}^g$ real part of V

Example: $g=1$

$$\underset{\{\tau\in\mathbb{C}\,|\,\mathrm{im}\,\tau>0\}}{\Theta(z,\tau)} = \sum_{n\in\mathbb{Z}} e^{2\pi i\left(\frac{1}{2}n^2\tau - nz\right)}$$

graph of $\|\theta(z,\tau)\|_h$



graph of $\|\Theta(x,\tau)\|_{h(z)}$ on $x\in\mathbb{R}\subset\mathbb{C}$

Θ-functions as Heat Kernel

$$\frac{\partial \theta(z,\tau)}{\partial \tau_{ij}} = \frac{\partial^2 \theta(z,\tau)}{\partial z_i \, \partial z_j}$$

take $\tau_{ij}(t) = t\,\delta_{ij}$:

$$\Theta(z, \delta I) = \sum_{n\in\mathbb{Z}^g} e^{-\pi i \langle \vec{n}, \vec{n}\rangle t} e^{2\pi i \langle \vec{n}, z\rangle}$$

Then $\frac{\partial}{\partial t}\Theta(z,\tau(t)) = \sum_i \frac{\partial^2}{\partial z_i^2}\Theta(z,\tau(t))$ heat equation!

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Example: $g=1$

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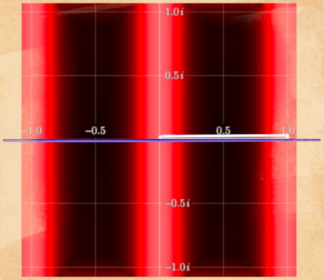
graph of $\|\theta(z,\tau)\|_h$



$\tau = 0.02$

evaluate along line

graph of $\|\Theta(x,\tau)\|_{h(z)}$ on $x\in\mathbb{R}\subset\mathbb{C}$

Converges to δ as $\tau\to 0$