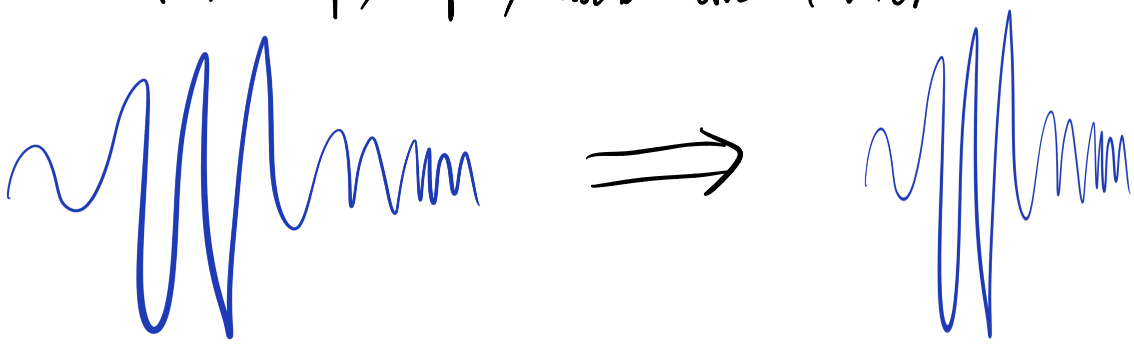
Harmonic analysis on Phase Space

~The Segal-Bargmann transform~

Part 1: Phase space representations

You know how you can watch youtube videos at 2 times speed?

First idea: simply play audio wave faster

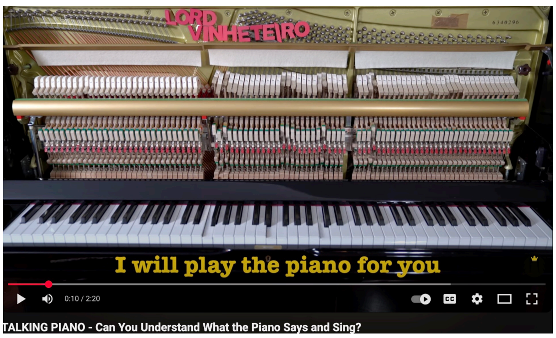


frequency is increased

But then, the pitch is increased. How do we avoid that?

This is easy for a musician: simply play the notes faster

so what if we convert audio signal into sheet music?



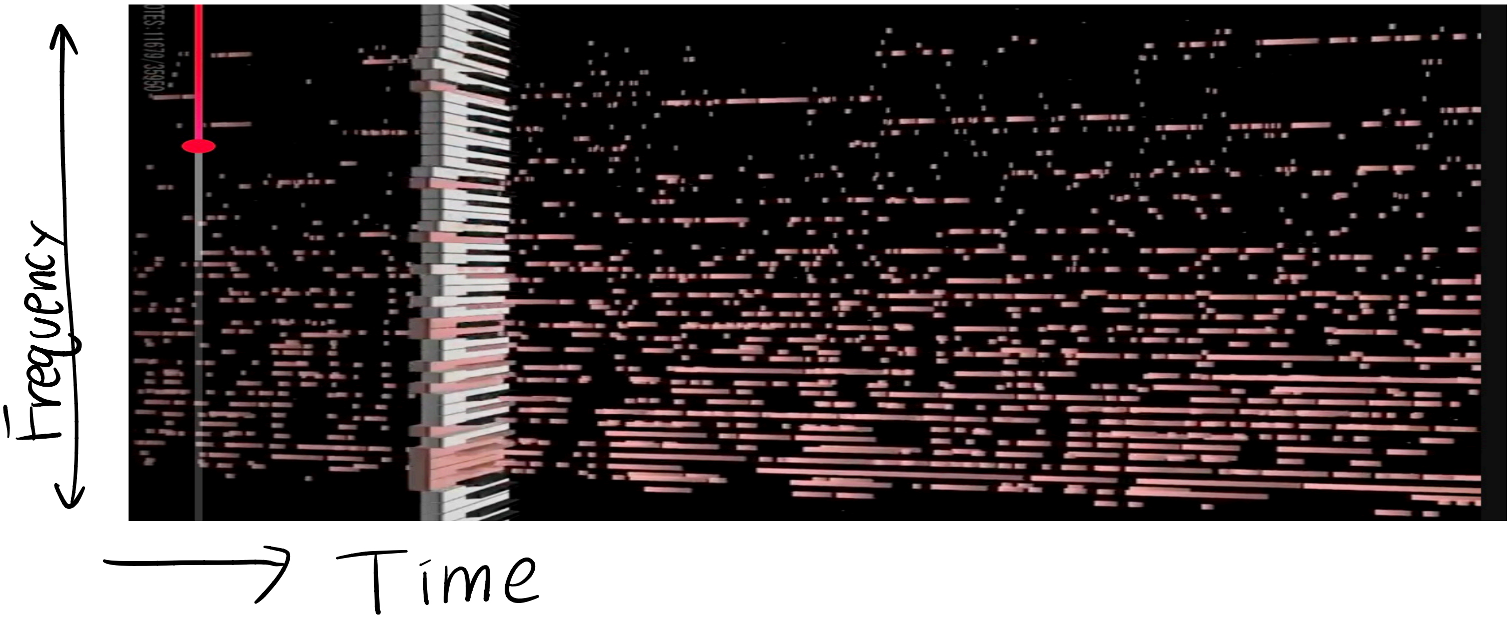
Talking piano (click for link)

1. split the audio into blocks
2. compute the Fourier transform of each block (a "chord of notes")
3. Play back the notes

To make the talking faster without increasing pitch, simply play notes faster.

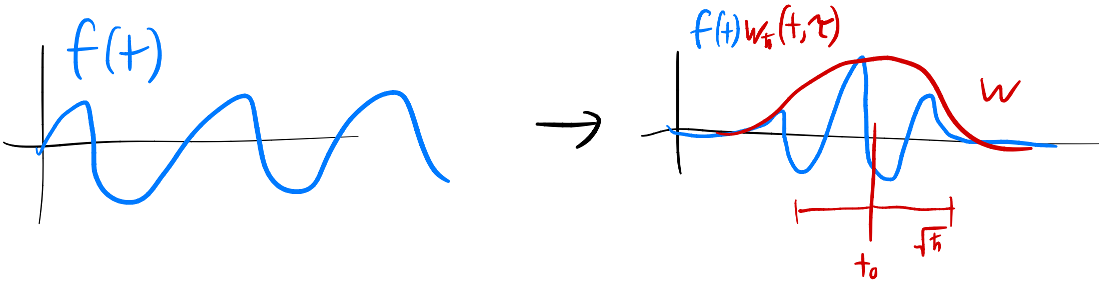
The resulting representation of the audio is 2-Dimensional: time & frequency.

billy joel piano man



Mathematically, This is achieved w/ Short-time fourier transform (STFT)

consider $f \in L^2(\mathbb{R})$. To take the fourier transform near time τ, we look at $f(t)$ through a window function $w_\hbar(t,\tau) = e^{-(t-\tau)^2/2\hbar}$ $\sqrt{\hbar}$ is window width



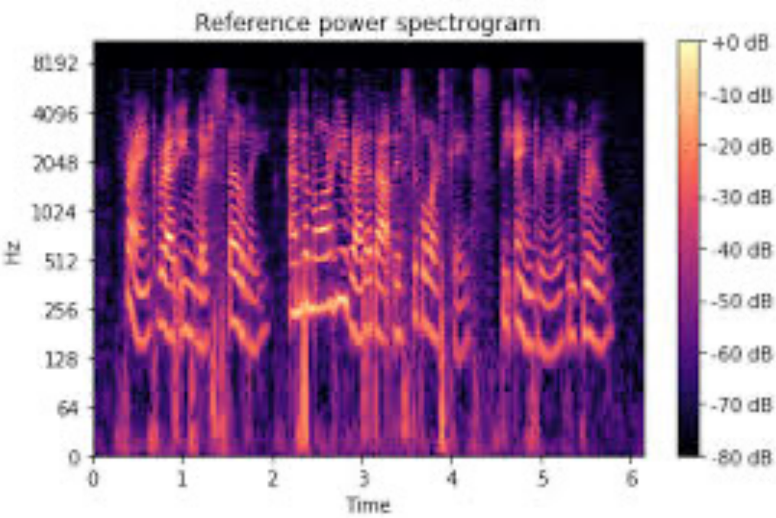
Then we take the fourier transform

Def: The STFT is $(\mathcal{V}_\hbar f)(t_0, \xi) = \int_{\mathbb{R}} f(t) \underbrace{e^{-(t-\tau)^2/2\hbar}}_{\text{short time}} \underbrace{e^{i\xi t/\hbar}}_{\text{fourier transform}} dt$ ξ scales w/ $\hbar$ $\mathcal{V}_\hbar f : \mathbb{R}^2 \to \mathbb{C}$

we call $\mathbb{R}^2 = \mathbb{R}_{time} \times \mathbb{R}_{frequency}$ the Phase space

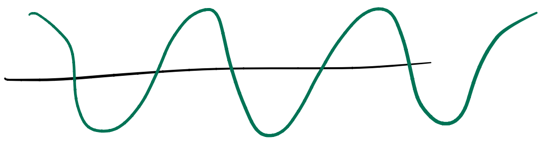
the density $|\mathcal{V}_\hbar f|^2 : \mathbb{R}^2 \to \mathbb{R}$ represents the ammount of a frequency @ a given time.

The graph of density is a spectrogram

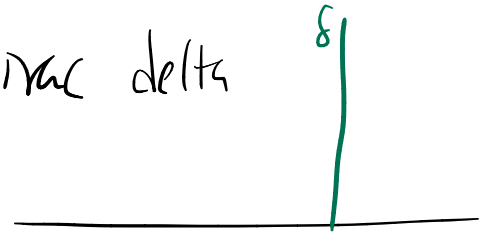


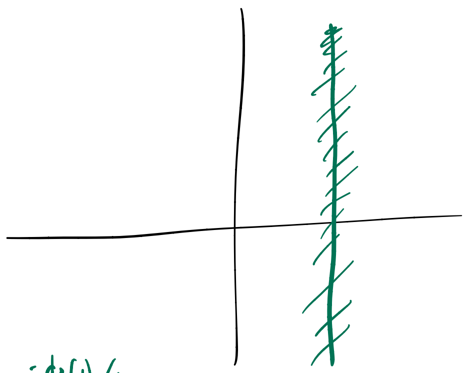
Examples:

$e^{ikx/\hbar}$

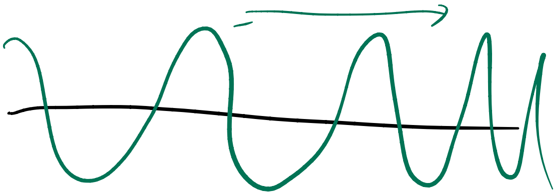


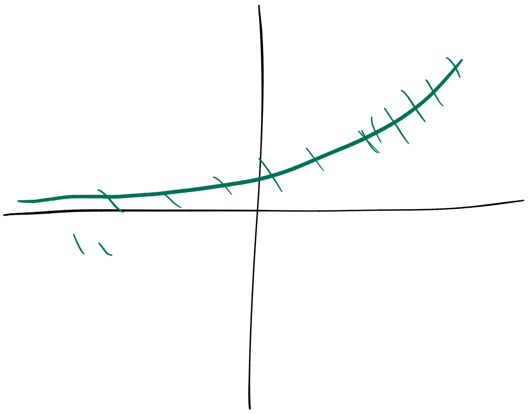
Dirac delta



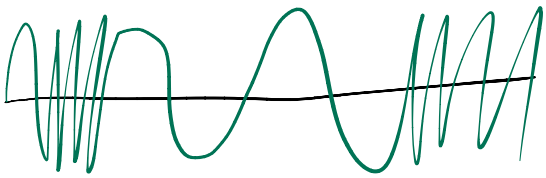


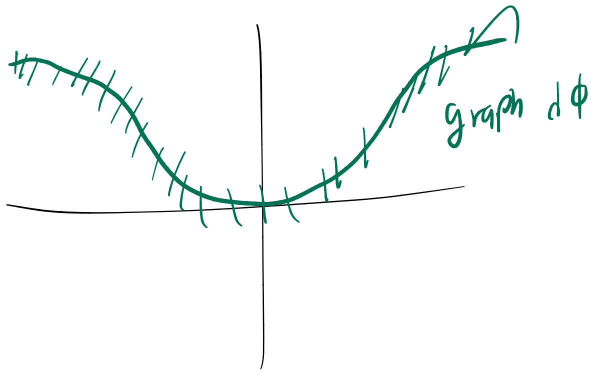
frequency increases w/ time



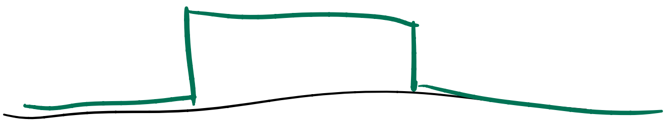


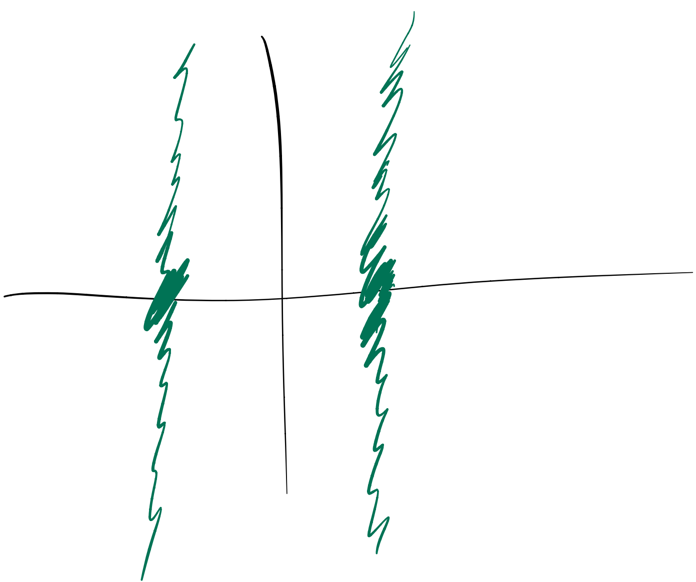
FM Radio $e^{i\phi(t)/\hbar}$



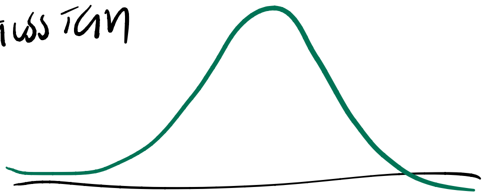


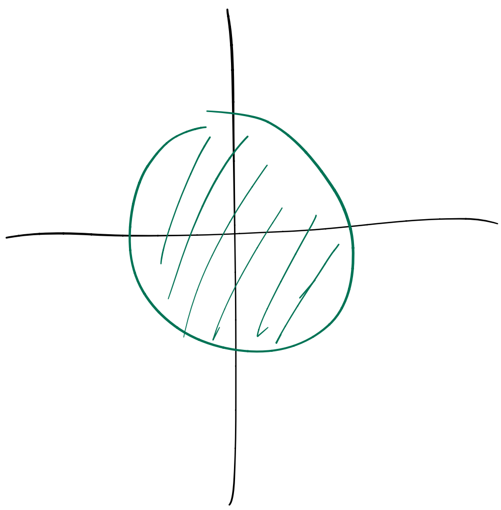
Square pulse





Gaussian





Part 2: the Segal-Bargmann transform

we can think of phase space $\mathbb{R}^2$ as $\mathbb{C}$, w/ coordinate $z = \tau + i\xi$

Goal: Modify the STFT to produce a holomorphic function

$V_\hbar f = \int f(t)\, K_\hbar^{STFT}((\tau,\xi),t)$, $K_\hbar^{STFT}((\tau,\xi),t) = e^{-(t-\tau)^2/2\hbar} e^{i\xi t/\hbar}$ is the kernel of STFT.

issue is, $K_\hbar^{STFT}((\tau,\xi),t)$ is not holomorphic in $\tau + i\xi$

Try kernel $e^{-(z-t)^2/2\hbar}$:

$$= e^{-((\tau-t)+i\xi)^2/2\hbar} = e^{-(\tau-t)^2/2\hbar - 2i\xi(\tau-t)/2\hbar + \xi^2/2\hbar}$$

$$= \underbrace{e^{-(\tau-t)^2/2\hbar} e^{i\xi t/\hbar}}_{K_\hbar^{STFT}((\tau,\xi),t)} \underbrace{e^{-i\xi\tau/\hbar + \xi^2/2\hbar}}_{\text{function depending only on } (\xi,\tau)}$$

Def: (preliminary Segal-Bargmann transform) $C_\hbar f(z) = \int f(t)\, e^{-\frac{(z-t)^2}{2\hbar}}\, dt$

- kernel is holomorphic $\Rightarrow$ $C_\hbar f(z)$ is holomorphic
- $C_\hbar f(z) = \underbrace{V_\hbar f(\tau,\xi)}_{L^2 \text{ in } \xi \,\&\, \tau} \cdot \underbrace{e^{i\xi\tau/\hbar + \xi^2/2\hbar}}_{\text{grows like } e^{\xi^2/2\hbar}}$

$\hookrightarrow C_\hbar f \cdot e^{-\xi^2/2\hbar} \in L^2(\mathbb{C})$

so $C_\hbar f \in \mathcal{H}L^2(\mathbb{C}, e^{-\frac{(\mathrm{im}\, z)^2}{2\hbar}})$ — Hilbert space of Holomorphic L^2 fns weighted by gaussian in ξ

Thm $C_\hbar$ defines a unitary map $C_\hbar : L^2(\mathbb{R}) \to \mathcal{H}L^2(\mathbb{C}, e^{-\frac{(\mathrm{im}\, z)^2}{2\hbar}})$

We will use a different perspective that puts ξ & τ on equal footing, at expense of a slightly more complicated transform.

for any ϕ holomorphic, there is an isometry $\mathcal{H}L^2(\mathbb{C}, \mu) \to \mathcal{H}L^2(\mathbb{C}, \frac{\mu}{|\phi|^2})$, $f \longmapsto \phi \cdot f$

using $\phi(z) = e^{z^2/4\hbar}$, get isometry $\mathcal{H}L^2(\mathbb{C}, e^{-\xi^2/2\hbar}) \to \mathcal{H}L^2(\mathbb{C}, e^{-|z|^2/\hbar})$ gaussian weight!

Def The Segal-Bargmann space is $\mathcal{H}_\hbar = HL^2(\mathbb{C}, e^{-|z|^2/\hbar}) = HL^2(\mathbb{C}, \mu_\hbar)$

There is an isometry $A_\hbar : L^2(\mathbb{R}) \longrightarrow \mathcal{H}_\hbar$ called the segal-Bargmann transform

$$A_\hbar f(z) = \int_{\mathbb{R}} e^{-(z^2 - 2\sqrt{2}\, zt + t^2)/2\hbar} f(t)\, dt$$

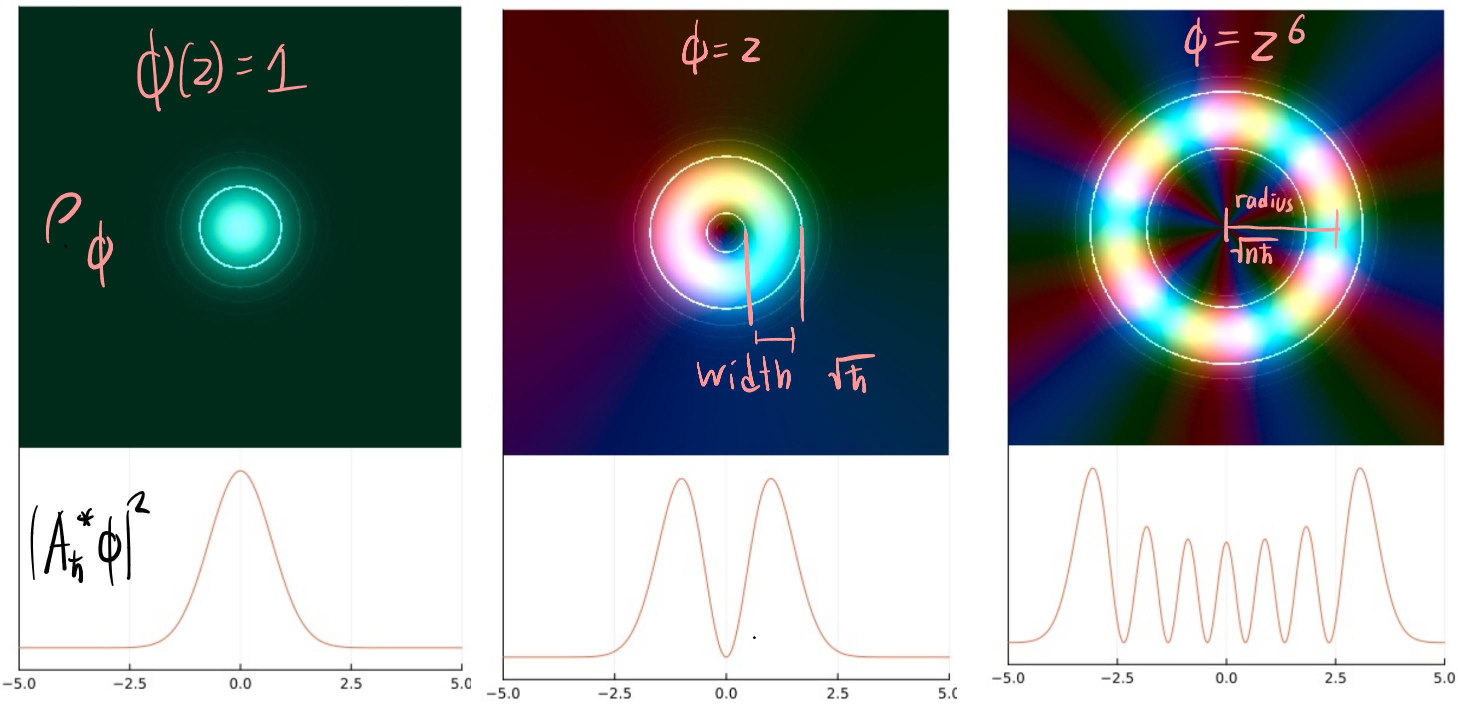
note: to match w/ above preliminary transform, need $z \mapsto z/\sqrt{2}$

Note that the construction of $HL^2(\mathbb{C}, \mu_\hbar)$ is agnostic between τ & ξ

There is an adjoint $A_\hbar^* : \mathcal{H}_\hbar \to L^2(\mathbb{R})$ s.t $A_\hbar^* A_\hbar = Id : L^2(\mathbb{R}) \circlearrowleft$

Let's get a feel for the segal-Bargmann transform

L^2-density of a holomorphic $\phi \in \mathcal{H}_\hbar$ is $\rho_\phi(z) = |\phi|^2 e^{-|z|^2/\hbar}$, $\rho_\phi : \mathbb{C} \to \mathbb{R}_{\geq 0}$



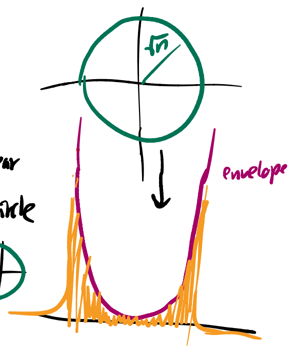
$\hookrightarrow$ $A_\hbar^* 1 = e^{-x^2/\hbar}$ is gaussian

- ρ_{z^n} concentrates on an annulus of radius $\sqrt{n\hbar}$, width $\sim \sqrt{\hbar}$

large z^n:

$A_\hbar^* z^n$ concentrates near vertical lines of circle

comes from $\mathrm{Gr}(d\phi) = \oplus$ intuition...



- $c_n z^n$, $c_n = \frac{1}{\sqrt{n!\,\hbar^n}}$ form orthonormal basis of $\mathcal{H}_\hbar$

- $A_\hbar^* c_n z^n$ is the n^{th} hermite function: the n^{th} e.function of the "Quantum harmonic oscillator" $-\hbar^2 \partial_x^2 + x^2 : L^2(\mathbb{R}) \circlearrowleft$

Operators on Segal-Bargmann space

$L^2(\mathbb{R})$ has two fundamental symmetries:

- translation $T_\tau : f(t) \mapsto f(t+\tau)$ generated by operator $i\partial_x$
- Phase modulation $M_\xi : f(t) \mapsto e^{i\xi t} f(t)$ generated by $\hat{x}$

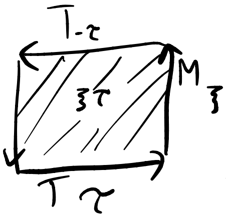
note translation & modulation are traded by the fourier transform

M_ω is "translation in frequency space"

commute up to phase

note that $[\hat{x}, i\partial_x] = i \cdot I \iff T_\tau M_\xi = e^{i\xi\tau} M_\xi T_\tau$

algebraically $L^2(\mathbb{R})$ is completely determined by these symmetries. This is encoded in



phase accumulated = area of path traversed in phase space.

Thm: (Stone-von neumann): suppose $\mathcal{H}$ is a Hilbert space w/ symmetries $\tilde{T}_\tau, \tilde{M}_\xi$ (unitary 1-parameter subgrps) s.t $\tilde{T}_\tau \tilde{M}_\xi \tilde{T}_\tau \tilde{M}_\xi = e^{i\tau\xi}$

& no subspace of $\mathcal{H}$ is preserved by P, Q. Then $\exists$ unitary map

$U : \mathcal{H} \to L^2(\mathbb{R})$ s.t

$$U T_\tau = \tilde{T}_\tau U$$
$$U M_\xi = \tilde{M}_\xi U$$

U intertwines the symmetries

This is very strong: $L^2(\mathbb{R})$ is unique Hilbert space w/ these symmetries

The segal-Bargmann space $\mathcal{H}$ also has translation symmetries:

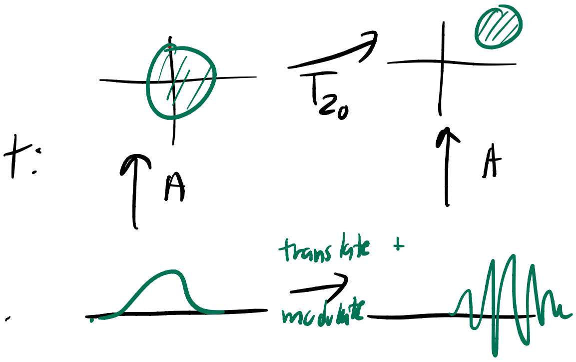
for $z_0 \in \mathbb{C}$, $T_{z_0} : f \longmapsto \underbrace{e^{|z_0|^2} e^{\bar{z}_0 z}}_{\text{modification to make translation unitary}} \overbrace{f(z-z_0)}^{\text{translation (what we want)}}$

(needed as μ_t is not translation invariant)

using T_{z_0}, every point in $\mathbb{C}$ is indistinguishable from perspective of $\mathcal{H}$.

Note $T_{z_0} 1 = \varphi_{z_0}$: coherent states translated by these operators.

T_{z_0} does what you expect:

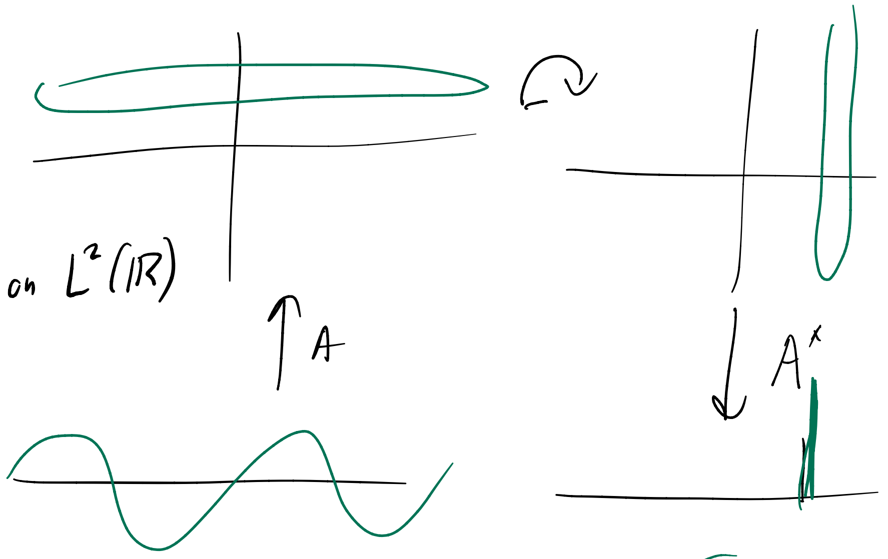


We can act on $L^2(\mathbb{R})$ by other symmetries of $\mathbb{C}$.

Rotatation by 90°:

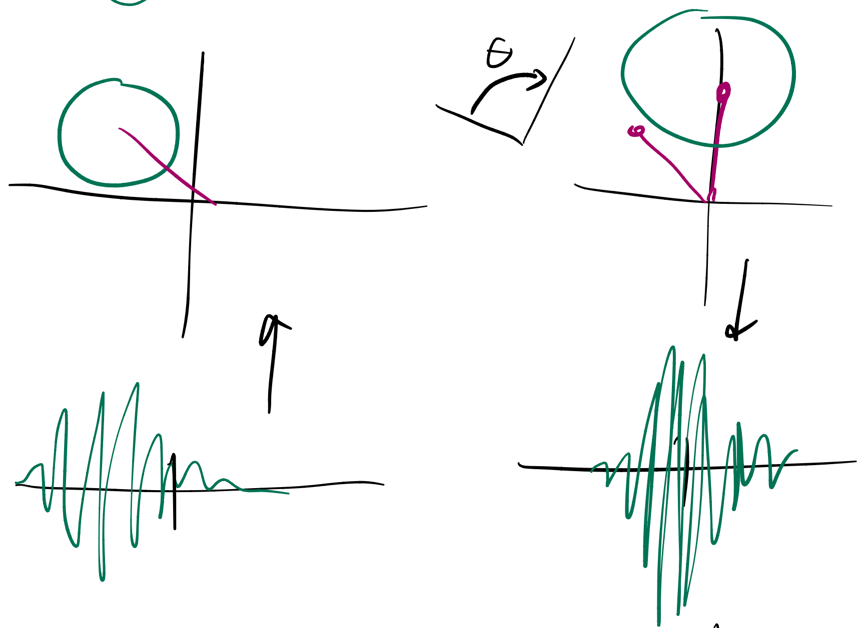
acts by Fourier transform on $L^2(\mathbb{R})$

this is why $\mathcal{F}^2 = -1$, not 1!



Rotation by θ°:

Defines a fractional fourier transform

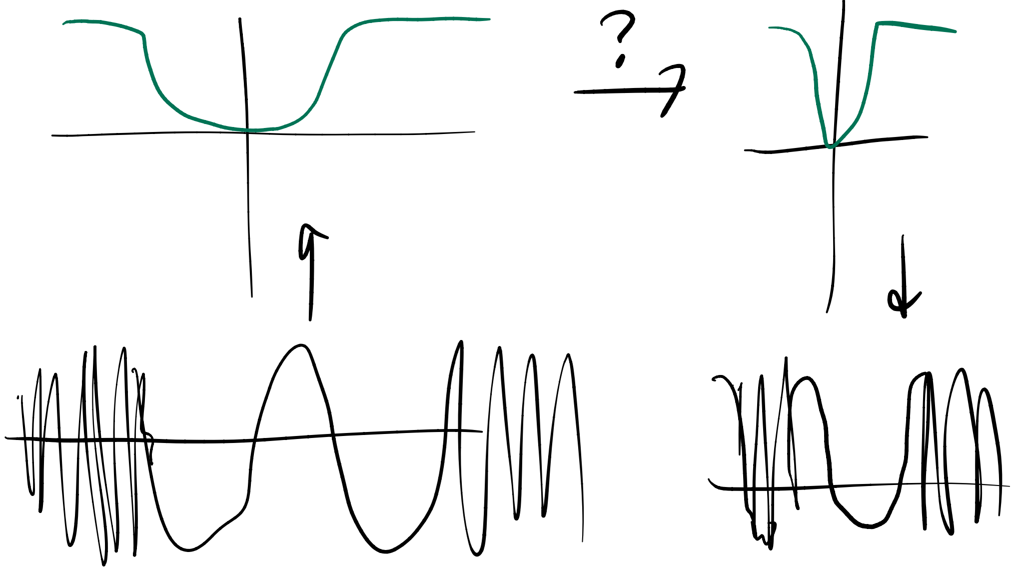


Squashing time

we can try the squash map

$$S : \phi(\tau + i\xi) \mapsto \phi\left(\frac{\tau}{2} + i\xi\right)$$

but this doesn't send holo functions to holo functions ∵ (Doesn't preserve angles). BUT we can just force the result to be holomorphic!



let $\pi : L^2(\mathbb{C},\mu) \to HL^2(\mathbb{C},\mu)$ be orthogonal projection.

i.e $\pi\big|_{HL^2(\mathbb{C},\mu)} = id$, & $\pi\big|_{HL^2(\mathbb{C},\mu)^\perp} = 0$. we can construct this explicitly using coherent states

$f(z) = \langle \psi_z, \phi \rangle = \int_{\mathbb{C}} \overline{\psi_z(w)}\, \phi(w)\, \mu$. so the integral transform w/ kernel $K(z,w) = \overline{\psi_z(w)}$ acts

as Identity on holomorphic functions, and kills anything $\perp$ to HL^2... so it is π!

π is an integral transform with kernel $K(z,w) = \overline{\psi_z(w)} = e^{\bar{w}z}$.

this is called the Repoducing kernel or Bergman kernel

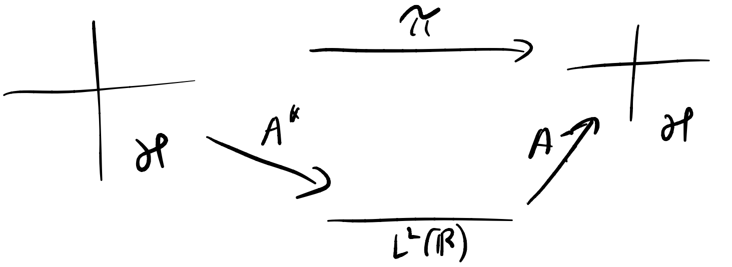
Define $S : \phi(\tau + i\xi) = \phi\left(\frac{\tau}{2} + i\xi\right)$, $S : HL^2(\mathbb{C},\mu) \to L^2(\mathbb{C},\mu)$.

Consider the Quantization $\pi \circ S : HL^2(\mathbb{C},\mu) \to HL^2(\mathbb{C},\mu)$.

The time-squashing map is $A^* \circ \pi \circ S \circ A : L^2(\mathbb{R}) \circlearrowleft$

Note: The inverse Segal-Bargmann transform is an integral transform $A^* : HL^2(\mathbb{C},\mu) \to L^2(\mathbb{R})$.

So, it extends to $A^* : L^2(\mathbb{C},\mu) \to L^2(\mathbb{R})$. This acts similarly to π.



$A A^* = \pi$

in particular, $A^* f = A^* \pi f$.

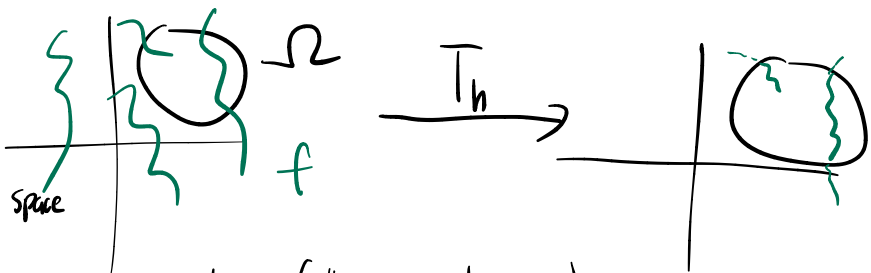
So, we can safley define time squashing as $A^* S A : L^2(\mathbb{R}) \circlearrowleft$

Multipling by functions: similarly, for $h : \mathbb{C} \to \mathbb{C}$, we can define multiplication operator $M_h : \phi \in HL^2(\mathbb{C},\mu) \to h \cdot \phi \in L^2(\mathbb{C},\mu)$. This does not preserve holomorphicity. So, we define the Toeplitz operator $T_h = \pi \circ M_h : \mathcal{H} \circlearrowleft$

The matrix elements are easy to compute: $\langle \psi, T_h \phi \rangle = \int_{\mathbb{C}} \bar{\psi}\, h \phi$

Example: $h = 1_\Omega$

we expect T_{1_Ω} to localize our function to a region in phase space



we expect T_{1_Ω} to approximatly be a projection (all evals 1 or 0)

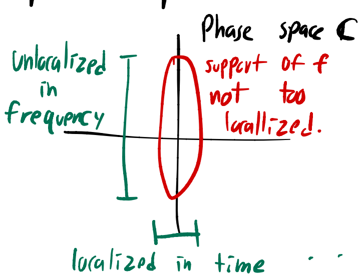
Reighligh Quotient $\frac{\langle \phi, T_{1_\Omega} \phi \rangle}{\langle \phi, \phi \rangle} = \frac{\int_\Omega |\phi|^2 \mu}{\|\phi\|^2_{L^2}}$ measures how much f is localized to Ω. the operator norm $\|T_{1_\Omega}\|_\infty = \frac{\sup \int_\Omega |\phi|^2 \mu}{\|\phi\|_{L^2}} \leq 1 - e^{-\text{Area}\,\Omega/\pi}$ as seen below

Uncertianty Principles in Segal-Bargmann Space

Last time: we constructed the Segal-Bargmann transform $L^2(\mathbb{R}) \to \mathcal{H}$, where $\mathcal{H} = HL^2(\mathbb{C}, e^{-|z|^2})$ is the space of holomorphic $f: \mathbb{C} \to \mathbb{C}$ w/ $\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} < \infty$

Notational note: last week, we used $f \in L^2(\mathbb{R})$ & $\phi \in \mathcal{H}$. This week, $f \in \mathcal{H}$ is holomorphic. We also specialize to $\hbar = 1$ and omit it from notation, since we won't be varying $\hbar$ today.

The Uncertianty principle schmatically states that any a function & its fourier transform cannot be simultanously localized. This manifests on phase space, saying that a phase space representation cannot be too localized.



For $f \in \mathcal{H}$ holomorphic, we measure it's distrabution in phase space via the L^2 density $\rho_f(z) = |f(z)|^2 e^{-|z|^2}$

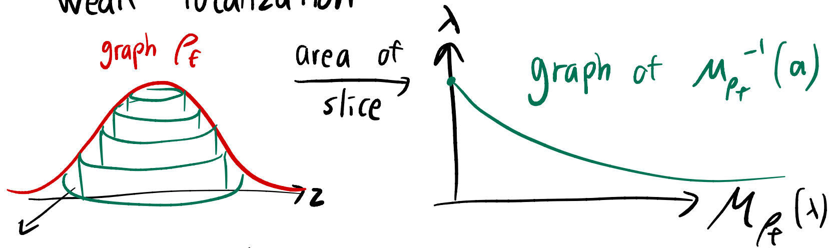
A manifestaton of the uncertianty principle bounds the "localization" of ρ_f from below.
there are many ways to quantify this...

One way to formalize "localization" is by measuring size of super level sets.

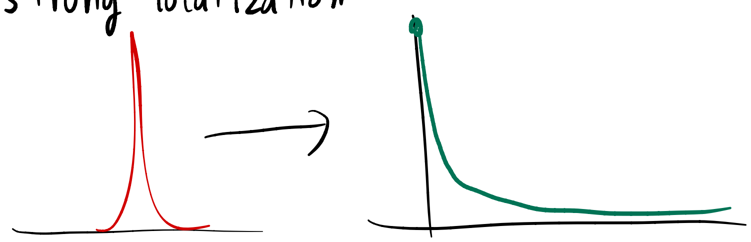
$$S^{\lambda}_{\rho_f} = \{ z \mid \rho_f(z) > \lambda \}$$
"region where ρ_f is large"

$$M_{\rho_f}(\lambda) = \mathrm{Area}(S^{\lambda}_{\rho_f})$$
"distrabution function"

Weak localization:



strong localization:

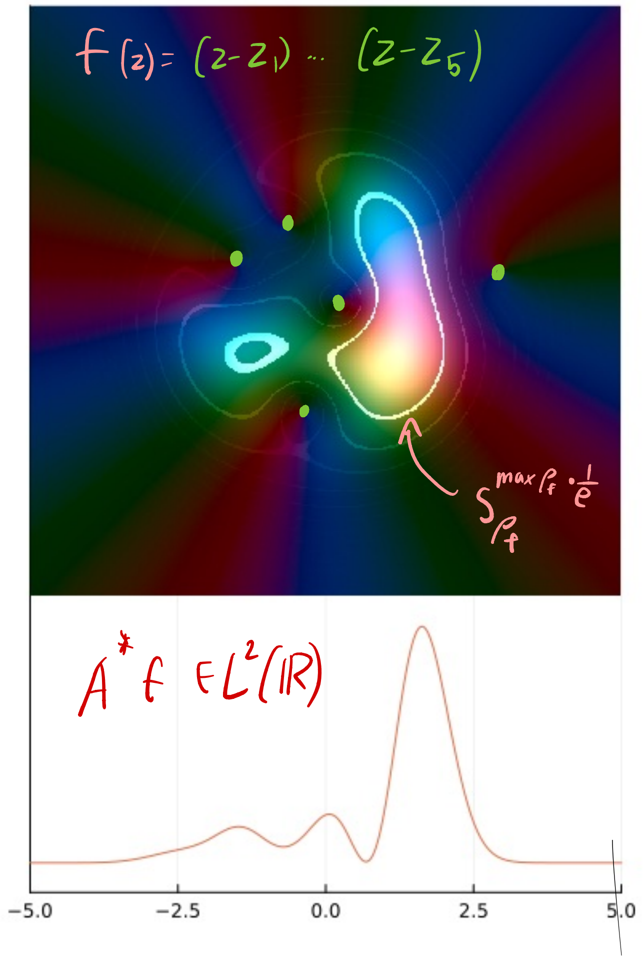


experementally, if f a polynomial, then

$$M_{\rho_f}(\tau \cdot \max \rho_f) \geq M_{\rho_1}(\tau)$$

All polynomials are localized (w.r.t. gaussian measure) But 1 is the most Localized!

Example: Polynomials

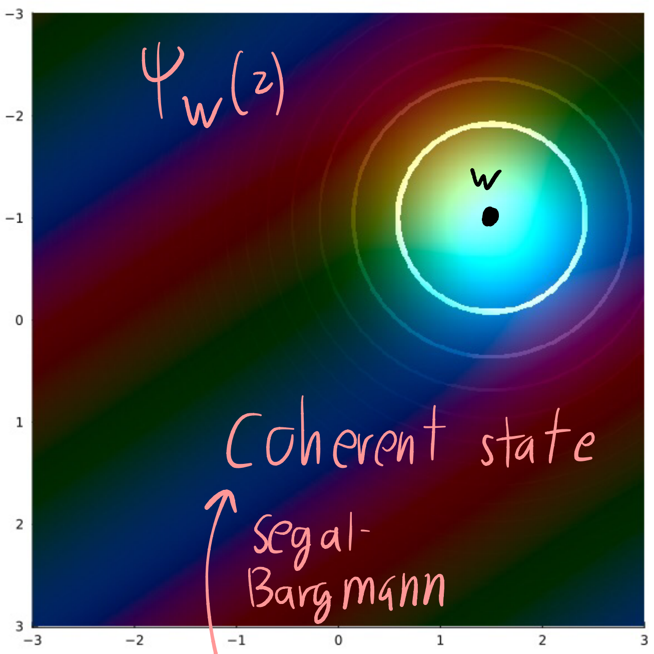


try it yourself:
https://openprocessing.org/sketch/2755210

Part 1: Coherent states ~

A coherent state is $\Psi_w(z) = e^{\bar{w}z}$

ρ_{Ψ_w} is a gaussian centered at w



Thm: The coherent states are repoducing

for $f \in \mathcal{H}$, $\langle \Psi_w, f \rangle_{\mathcal{H}} = \int \bar{\Psi}_w f e^{-|z|^2} = f(w)$

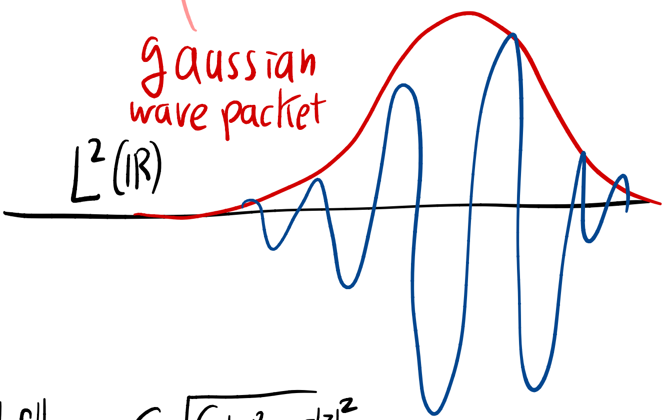
$\boxed{\text{example: } \Psi_0(z) = 1. \text{ Then, } \langle 1, f \rangle_{\mathcal{H}} = f(0)}$

recall $\{z^n/\sqrt{n!}\}$ is an o.n.b of $\mathcal{H}$

$\langle 1, f(z) \rangle = \langle 1, \sum a_n z^n \rangle = \sum a_n \langle 1, z^n \rangle = a_0 = f(0)$

We can construct Ψ_w abstractly through the evaluation map $ev_w : \mathcal{H} \to \mathbb{C}$ $\quad ev_w(f) = f(w)$

By the Reisz representation thm, there exists Ψ_w such that $ev_w = \langle \Psi_w, \cdot \rangle$ iff $ev_w : \mathcal{H} \to \mathbb{C}$ is continous



that is, there exists a constant C_w

such that $\forall f \in \mathcal{H}$, $\quad |ev_w f| = |f(w)| \leq C_w \|f\|_{\mathcal{H}} = C_w \sqrt{\int_{\mathbb{C}} |f|^2 e^{-|z|^2}}$

the existence of Ψ_w s.t $\langle \Psi_w, f \rangle_{\mathcal{H}} = f(w)$ $\iff$ Continuity of pointwise evaluation

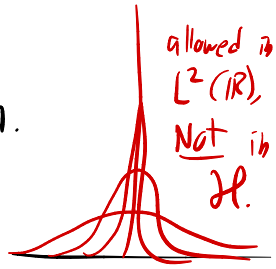
Knowing $\Psi_w(z) = e^{\bar{w}z}$ we can compute the minimal constant C_w

$$|f(w)|^2 = |\langle \Psi_w, f \rangle_{\mathcal{H}}|^2 \underset{\text{Cauchy-Schwartz}}{\leq} \langle \Psi_w, \Psi_w \rangle \langle f, f \rangle = \Psi_w(w) \|f\|_{\mathcal{H}}^2 = e^{|w|^2} \|f\|_{\mathcal{H}}^2$$

$\Rightarrow \boxed{|f(w)|^2 \leq e^{|w|^2} \|f\|_{\mathcal{H}}^2}$ with equality when $f \propto \Psi_w$.

the value of f at any point w is controlled by its $L^2(\mathbb{C}, e^{-|z|^2})$ norm.

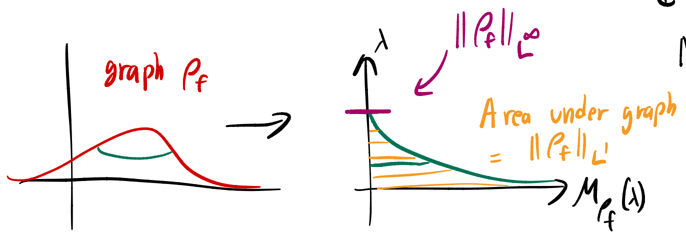
contrast w/ $L^2(\mathbb{R})$, which has functions $\|g_n\|_{L^2} = 1$, $g_n(0) \to \infty$.



This is an uncertainty principle: holomorphic functions can't be too peaked!

phrased in terms of L^2 density $\rho_f = |f|^2 e^{-|z|^2}$:

$|f(w)|^2 e^{-|w|^2} \leq \|f\|_{\mathcal{H}}^2 \Rightarrow \rho_f(w) \leq \int_{\mathbb{C}} \rho_f$. This holds for all w, so $\boxed{\|\rho_f\|_{L^\infty} \leq \|\rho_f\|_{L^1} \text{ w/ equality when } f \propto \Psi_w}$



Note: $\int_{\mathbb{C}} \rho_f = \int_{\mathbb{R}} \mu(\lambda)$ is the Lesbegue integral

Part 2: Bounding L^2 concentration

How concentrated can $\rho_f = |f|^2 e^{-|z|^2}$ (holomorphic) be on a set $\Omega \subset \mathbb{C}$?

Thm (Nicola-Tilli '22) See https://arxiv.org/abs/2212.14008

$$\boxed{\sup_{f \in \mathcal{H}} \frac{\int_\Omega \rho_f}{\int_{\mathbb{C}} \rho_f} \leq 1 - e^{-\mathrm{Area}(\Omega)/\pi}}$$

w/ equality when $f = \Psi_w$ is a coherent state, Ω is a ball centered at w

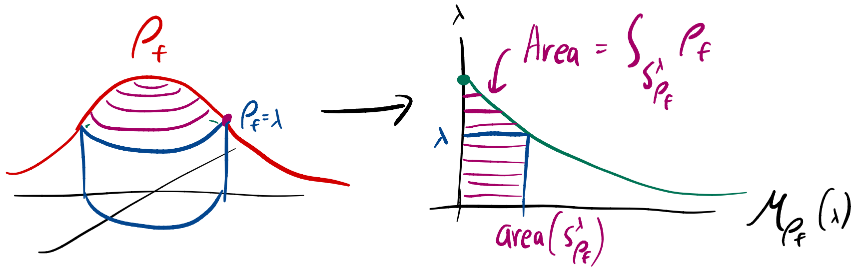
$\int_\Omega \rho_f / \int_{\mathbb{C}} \rho_f$ measures the portion of L^2-norm of f concentrated in Ω.
for small Ω, only a small portion of f may concentrate in Ω. An uncertainty principle!
The "smallest uncertainty" states are coherent states.

The upper bound is
$$\sup_{\mathrm{area}(\Omega) = a} \ \sup_{f \in \mathcal{H}} \frac{\int_\Omega \rho_f}{\int_{\mathbb{C}} \rho_f} = \sup_{f \in \mathcal{H}} \frac{\sup_{\mathrm{area}(\Omega)=a} \int_\Omega \rho_f}{\int_{\mathbb{C}} \rho_f}$$

to maximise $\int_\Omega \rho_f$ w/ fixed f, choose Ω a super level set (region where ρ_f is large)

so it suffices to bound $\int_{S^\lambda_{\rho_f}} \rho_f$

We will extract this bound from an estimate of growth rates of areas of super level sets of subharmonic functions



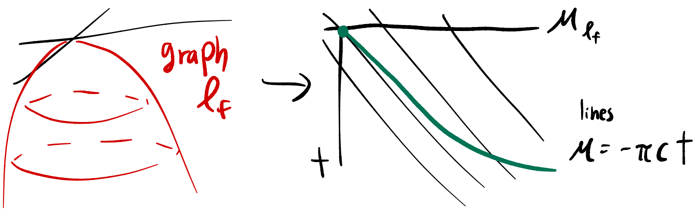
Define $\ell_f := \log \rho_f = \log |f|^2 - |z|^2 = \log f + \log \bar{f} - |z|^2 = 2 \,\mathrm{Re} \log f - |z|^2$. Observe $S^\lambda_{\rho_f} = S^{\log \lambda}_{\ell_f}$

$$\Delta \ell_f = \Delta(2\, \underbrace{\cancel{\mathrm{Re}\, \ell_f}}_{0}) - \Delta(|z|^2)$$
$$= -(\partial_x^2 + \partial_y^2)(x^2 + y^2) = -4$$

ℓ_f is holomorphic away from 0,
so its real part is harmonic

Lemma: Let $\ell : \mathbb{C} \to \mathbb{R}$ satisfy $\Delta \ell = -c$ ("uniformly subharmonic")
then $\mu_\ell(t) = \mathrm{Area}(S^t_\ell)$ satisfies $|\mu_\ell'(t)| \geq \frac{4\pi}{c}$ w/ equality when $\ell \propto -|z-w|^2$.

applied to ℓ_f, this is another manifestation of the uncertainty principle.
normalize f so that $\max \rho_f = 1$, $\max \ell_f = 0$. Then, $|\mu_{\ell_f}'(t)| \geq 4\pi \Rightarrow \mu_{\ell_f}(t) \geq -4\pi t$



Since $S^\lambda_{\rho_f} = S^{\log \lambda}_{\ell_f}$, $\mu_{\rho_f}(\lambda) = \mu_{\ell_f}(\log \lambda)$. so, this proves

$$\mathrm{Area}\ S^\lambda_{\rho_f} \geq \pi \log \lambda = \mathrm{Area}\ S^\lambda_{\rho_1}.$$

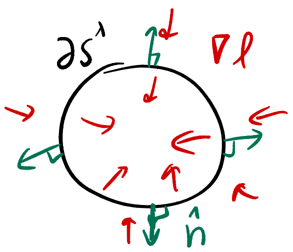
w/ equality when $\ell_f = -|z-w|^2$, or $f = \Psi_w$.

Proof of lemma: Let $S^t = S^t_\ell$, $\mu(t) = \mu_\ell(t)$. we will compute μ several ways.

(1) $\mu(t) = \frac{1}{c}\int_{\partial S^t} |\nabla \ell|\, ds$

Apply the divergence thm to the laplacian $\Delta \ell = \nabla\cdot\nabla\ell = -c$

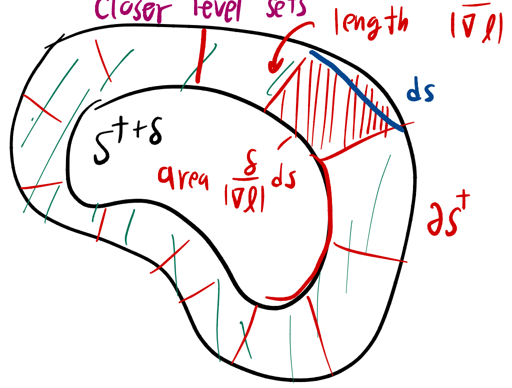
$$\mu(t) = \int_{S^t} 1\, d\mathrm{vol} = \int_{S^t} \frac{-1}{c}\nabla\cdot(\nabla \ell)\, d\mathrm{vol} \overset{\text{divergence thm}}{=} \frac{-1}{c}\int_{\partial S^t} \nabla g\cdot \hat{n}\, ds$$



Since ∂S^t is by construction a level set of ℓ, $\nabla \ell$ is parralel to $\hat{n}$. in this case, $\nabla\ell$ & $\hat{n}$ are oppisitly oriented, so $\nabla \ell\cdot\hat{n} = -|\nabla \ell|$

$\Rightarrow \mu(t) = \frac{1}{c}\int_{\partial S^t} |\nabla \ell|\, ds$

(2) $\mu'(t) = \int_{\partial S^t} \frac{1}{|\nabla \ell|}\, ds$ This is the "Coarea formula"



Consider the change in $\mathrm{Area}(S^t)$ as t increases by δ. Level sets are seperated by $\delta/|\nabla \ell|$, so the Area change is $\mu(t+\delta) - \mu(t) = -\int_{\partial S^t} \frac{\delta}{|\nabla \ell|}\, ds + O(\delta^2)$

because we look at super level sets

$\Rightarrow \mu'(t) = -\int_{\partial S^t} \frac{1}{|\nabla \ell|}\, ds$

We combine these two integrals with a clever Cauchy-Schwartz

$$\left(\int_{\partial S^t} 1\, ds\right)^2 = \left(\int_{\partial S^t} \sqrt{|\nabla \ell|}\frac{1}{\sqrt{|\nabla \ell|}}\, ds\right)^2 \overset{\text{Cauchy-schwartz}}{\leq} \left(\int_{\partial S^t} |\nabla \ell|\, ds\right)\left(\int_{\partial S^t} \frac{1}{|\nabla \ell|}\, ds\right) = -c\,\mu(t)\,\mu'(t)$$

equality when $|\nabla \ell|$ is constant.

The LHS is $\left(\int_{\partial S^t} 1\, ds\right)^2 = \left(\mathrm{Per}(\partial S^t)\right)^2$. To relate this to area...

(3) (isoperemetric inequality) $4\pi\, \mathrm{Area}(S^t) \leq \mathrm{Per}(\partial S^t)^2$ w/ equality when S^t is a disc

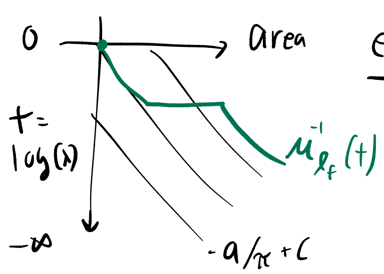
All together, $4\pi\mu(t) \leq -c\,\mu(t)\mu'(t) \Rightarrow \mu'(t) \leq -\frac{4\pi}{c}$

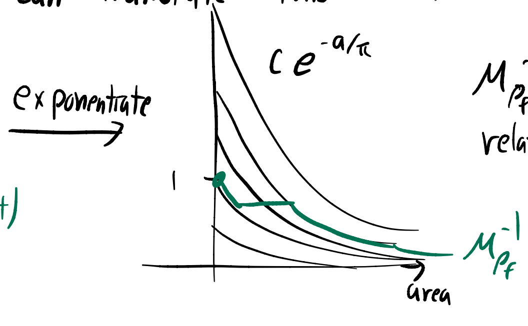
This is an equality when ∂S^t is a circle for every λ. For uniformly subharmonic functions, this implies ℓ is a scaling / translate of $|z|^2$.

Using $\mu_{P_f}(\lambda) = \mu_{\ell_f}(\log(\lambda))$ we can translate this lemma into an estimate on $\mu_{P_f}(\lambda)$

$\mu'_{\ell_f}(t) \leq -\pi$

as t gets more negative, area increases



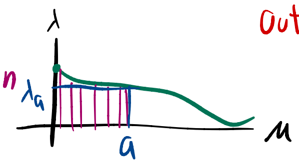


$\mu^{-1}_{P_f}(a)$ is increasing relative to $Ce^{-a/\pi}$

Proof of theorem: Let $\text{area}(\Omega)=a$, λ_a s.t $\text{Area}(S_{\rho_f}^{\lambda_a})=a$

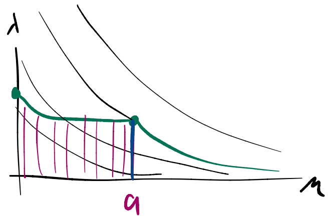
$$\frac{\int_\Omega \rho_f}{\int_{\mathbb{C}} \rho_f} \leq \frac{\int_{S_{\rho_f}^{\lambda_a}} \rho_f}{\int_{\mathbb{C}} \rho_f} = $$

Portion of Area



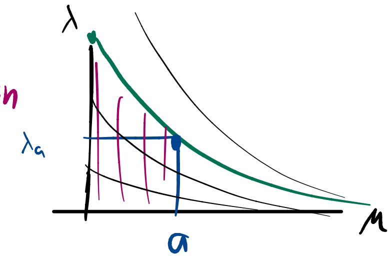
reduce L^2-mass outside $S_{\rho_f}^{\lambda_a}$

$\leq$ Portion



increase L^2-mass inside $S_{\rho_f}^{\lambda_a}$

$\leq$ Portion



$$= 1-e^{-a/\pi}$$

■

So the maximal L^2 concentration, given $\mu_{\rho_f}'(t) \leq -\pi$, occurs when $\mu_{\rho_f}'(t)=-\pi$. Thus, the optimal functions have ρ_f gaussian, or $f=\Psi_w$. The optimal sets Ω are superlevel sets of ρ_{Ψ_w}, so are balls.

Why I care: the above easily extends into $\mathbb{C}^n$, where if $\max \rho_f = 1$,

$$\text{Vol}(S_{\rho_f}^\lambda) \geq \text{vol}(S_{\rho_1}^\lambda) \quad \text{w/} \quad S_{\rho_1}^\lambda = B(-\pi\log(\lambda))$$

πr^2

I want to extend this to a symplectic measure of volume.

Define the Gromov width of $\Omega \subset \mathbb{C}^n$ to be

$$C_{Gr}(\Omega) = \sup\left\{ a\in\mathbb{R} \;\middle|\; \begin{array}{l} \exists\ \psi: B(a) \hookrightarrow \Omega \text{ embedding,} \\ \psi^* \sum dz_i\wedge d\bar{z}_i = \sum dz_i \wedge d\bar{z}_i \end{array} \right\}$$

$C_{GR}(\Omega)$ measures the largest symplectic ball which fits inside Ω

Conjecture: $C_{Gr}(S_{\rho_f}^\lambda) \geq C_{Gr}(S_{\rho_1}^\lambda) = -\pi\log(\lambda)$