Riemann-
Roch

Motivating Problem

- holomorphic fns not localized – "sees" whole space
  - Goal: probe topology w/ holo/mero fns

X compact Riemann surface ($\dim_{\mathbb{C}} X = 1$)

Q: # independent holo. fns? A: 1 (constant)

Q: # independent mero. fns? A: ∞

Q: # independent mero. fns w/ prescribed poles/0s?

A: finite & interesting...

Divisors

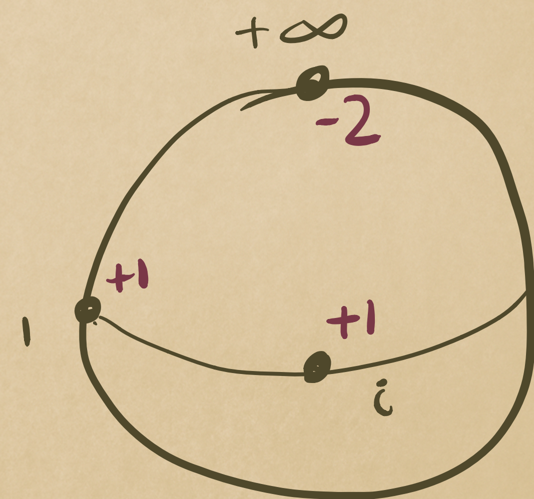
f mero on X: X curve $\Rightarrow$ $\overset{\text{zeros}}{f^{-1}(0)}$, $\overset{\text{poles}}{\left(\frac{1}{f}\right)^{-1}(0)}$ set of pts

Def: Divisor (Δ) a formal sum of pts of X

↳ $\Delta = \sum m_i P_i$, $m_i \in \mathbb{Z}$, $P_i \in X$, sum is finite

Divisor of f: add 0s/poles w/ multiplicity given by order

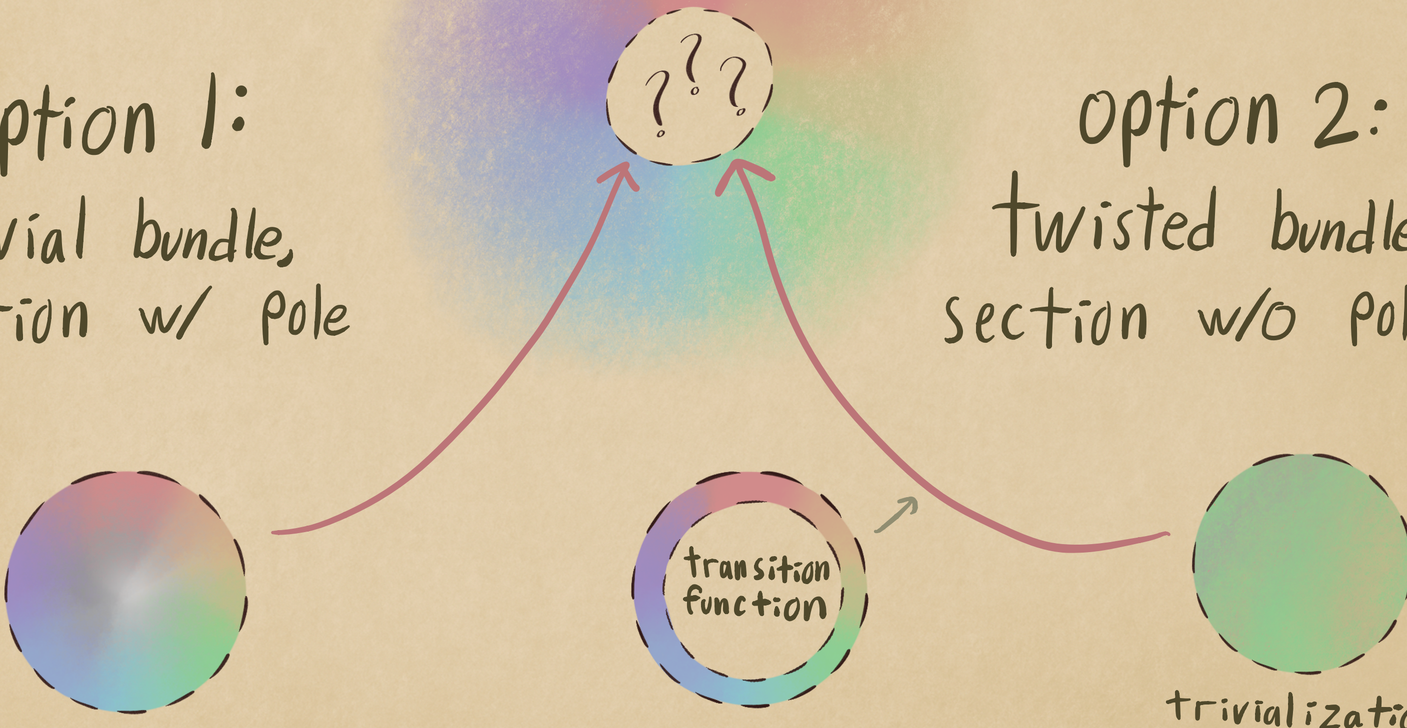
$$f(z) = (z-1)(z-i)$$



Line bundle of Divisor

option 1:
trivial bundle,
section w/ pole

option 2:
twisted bundle,
section w/o pole



- section w/ pole/zero ⇔ twisted section w/o pole/zero

- start w/ trivial bundle $\mathcal{O}$:

  $\hookrightarrow$ twist all points in Δ, get $\mathcal{O}(\Delta)$

  – holo. section of $\mathcal{O}(\Delta) \Longleftrightarrow$ section of $\mathcal{O}$ w/ Divisor Δ

- Twist from each zero (pole) increments (decrements) $c_1(\mathcal{O}(\Delta))$

  $\hookrightarrow$ $ch_1(\mathcal{O}(\Delta)) = \sum m_i$

Implicit description of $\mathcal{O}(\Delta)$:

$\mathcal{O}(\Delta)$ = sheaf of mero fns f w/ $\mathrm{div}(f) + \Delta \geq 0$

if $\Delta = \sum m_i p_i$, f @ p_i has lowest order $\geq -m_i$

f "better behaved" than Δ

locally free rank 1 $\mathcal{O}$-module $\Rightarrow$ line bundle!

Deriving R-R

Split up Δ into Δ^+ & Δ^-: suffices to show for $\Delta^- = \sum_{\leq 0} m_i p_i$

$$0 \to \mathcal{O} \to \mathcal{O}(\Delta^-) \to \mathcal{J} \to 0$$

$\mathcal{O}(\Delta^-)$ & $\mathcal{O}$ only differ @ pts in $\Delta^- \Rightarrow \mathcal{J}$ skyscraper sheaf

$\mathcal{J}_{p_i} = \mathbb{C}^{m_i}$: corresponds to $\sum_{m_i < k < 0} c_k z^k$ of $\mathcal{O}(\Delta^-)$

- X has genus g:

  $H^1(X) = H^{0,1}(X) \oplus H^{1,0}(X) = \mathbb{C}^{2g} \Rightarrow H^{0,1}(X) = H^1(\mathcal{O}) = \mathbb{C}^g$
- X compact $\Rightarrow$ holomorphic fns constant $\Rightarrow H^0(\mathcal{O}) = \mathbb{C}$
- $\mathcal{J}$ has support of dimension $0 \Rightarrow H^1(\mathcal{J}) = 0$

Deriving R-R (2)

$$0 \to \mathcal{O} \to \underset{\displaystyle E}{\overset{E}{\overset{\parallel}{\mathcal{O}(\Delta)}}} \to \mathcal{S} \to 0$$

$$0 \to H^0(\mathcal{O}) \to H^0(\mathcal{O}(\Delta)) \to \mathbb{C}^{\sum m_i} \to H^1(\mathcal{O}) \to H^1(\mathcal{O}(\Delta)) \to \underset{\displaystyle 0}{\underset{\parallel}{H^1(\mathcal{S})}}$$

$$h^0(\mathcal{O}(\Delta)) + \underset{\displaystyle g}{\underset{\parallel}{h^1(\mathcal{O})}} = \underset{\displaystyle 1}{\underset{\parallel}{h^0(\mathcal{O})}} + \underset{\displaystyle c_1(E)}{\underset{\parallel}{\sum m_i}} + h^1(\mathcal{O}(\Delta))$$

$$h^0(E) - h^1(E) = c_1(E) + 1 - g$$

R-R Interpetation

Serre Duality:

$H^1(E) = H^0(X, \Omega^1 E)$

$= H^0(X, K^* \otimes E)$

E-valued 1-forms

$$h^0(E) = \underbrace{c_1(E) + 1}_{(1)} - (\underset{(2)}{g} - \underset{(3)}{h^1(E)})$$

(1) $h^0(\mathcal{O}(\Delta))$ wants to be $\sum m_i + 1$: on $\mathbb{C}$, for $m_i = 1$ poles @ distinct pts $p_1, \ldots, p_n$, $f = a_0 + \sum_{i=1}^{n} \frac{a_i}{z - p_i}$

(2) to extend local meromorphic fns to global is a cohomology problem; obstruction is $H^{0,1} \cong \mathbb{C}^g$

(3) is correction to (2), cohomology classes which do not affect $\{p_i\}$ (vanish on $\{p_i\}$)

$$\overbrace{\underbrace{h^0(E)-h^1(E)}_{\text{Analytic}}}=\overbrace{\underbrace{c_1(E)+1-g}_{\text{topological}}}$$

$$0\to\mathcal{O}\to\Omega^0(E)\xrightarrow{\bar\partial_E}\Omega^{0,1}(E)\to 0$$

index $\bar\partial_E :=$ dim ker $\bar\partial_E$ − dim coker $\bar\partial_E$

ker $\bar\partial_E$ = holomorphic sections of E $= H^0(E)$ ✓

coker $\bar\partial_E = H^1(\mathcal{O})$, by L.E.S ✓

$$\Omega^0(E)\xrightarrow{\bar\partial_E}\Omega^{0,1}(E)\to H^1(\mathcal{O})\to \cancel{H^1(\Omega^0(E))} = 0$$

index $\bar\partial_E = c_1(E)+1-g$

Index Theorems

- R-R is prototypical index theorem:
- Vector bundle V on surface:

$$h^0(V) - h^1(V) = ch_1(V) + \underline{\mathrm{rank}(V)}(1-g)$$

- Hirzebruch-Riemann-Roch: X compact, complex

$$\sum (-1)^i h^i(V) = \int_X ch(V)\, td(X)$$

★ Atiyah-Singer index thm: D elliptic, X cmpt

$$\underbrace{\text{index } D}_{\text{analysis}} = \underbrace{\int_X ch(D)\, td(X)}_{\text{topology}}$$

Fin

Sources:

- Complex analytic and differential geometry, Demailly, section 6.10
- Riemann Surfaces, Donaldson, section 12.1