Lie thm

-Elliot Kienzle

Def Derived series of Lie algebra $\underline{g}$:

$$\mathcal{D}^1\underline{g} = [\underline{g},\underline{g}] \qquad \mathcal{D}^k\underline{g} = [\mathcal{D}^{k-1}\underline{g}, \mathcal{D}^{k-1}\underline{g}]$$

$\underline{g}$ is *solvable* if $\exists k$ s.t $\mathcal{D}^k\underline{g} = 0$

Key example: upper triangular matricies

$$\begin{array}{ccccccccc}
\underline{g} & & \mathcal{D}^1\underline{g} & & \mathcal{D}^2\underline{g} & & & & \mathcal{D}^k\underline{g} \\
\begin{pmatrix} * & * & * & * \\ & * & * & * \\ 0 & & * & * \\ & & & * \end{pmatrix} & \xrightarrow{[,]} & \begin{pmatrix} 0 & * & * & * \\ & 0 & * & * \\ 0 & & 0 & * \\ & & & 0 \end{pmatrix} & \xrightarrow{[,]} & \begin{pmatrix} 0 & 0 & * & * \\ & 0 & 0 & * \\ 0 & & 0 & 0 \\ & & & 0 \end{pmatrix} & \xrightarrow{[,]} & \cdots & \xrightarrow{[,]} & 0
\end{array}$$

fact: $\underline{g}_{s.s.} = \underline{g}/\mathrm{Rad}(\underline{g})$ is semisimple

maximal solvable ideal

moral: solvable $\approx$ non-semisimple (abelenian + nilpotent)

$$\begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$$

each $\mathcal{D}^k\underline{g}$ is an ideal: solvable $\Rightarrow$ many ideals

Weyl Decomposition: $\underline{g} \cong \underline{g}_{ss} \oplus \underline{g}_{sol}$, $\underline{g}_{sol} \cong \mathrm{Rad}(\underline{g})$

To classify all Lie algebra reps, suffices to classify s.s & solvable

Thm (Lie) complex irreps of complex solvable Lie algebras are 1-dimensional

in sequal, field = $\mathbb{C}$ (or agebraically closed)

Equivently, every rep $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ fixes a 1-D subspace of V. or...

Equiv. Thm every solvable $\mathfrak{g} < \mathfrak{gl}(V)$ has a common e.vect: $v \in V$ s.t v is e.vect of X $\forall X \in \mathfrak{g}$

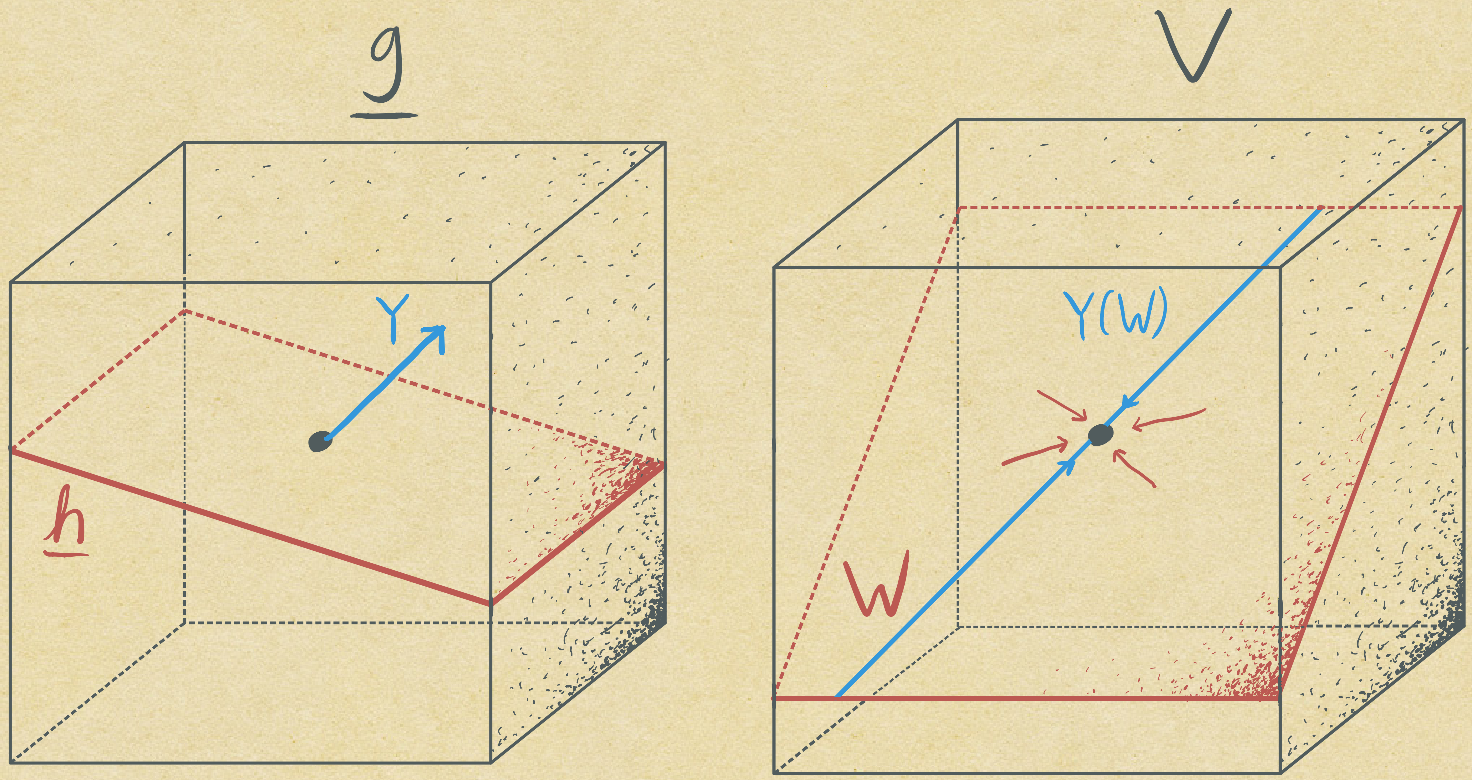
Proof concept:

Since $\mathfrak{g}$ solvable, we have descending chain of ideals

$$0 = \mathcal{D}^k\mathfrak{g} \triangleleft \mathcal{D}^{k-1}\mathfrak{g} \triangleleft \cdots \triangleleft \mathcal{D}^1\mathfrak{g} \triangleleft \mathfrak{g}$$

- build ideal $\mathfrak{h}$ sequentally according to this chain
- Track simultaneous e.space of $\mathfrak{h}$: $W := \{v \in V \mid Xv = \lambda(X)v \ \ \forall X \in \mathfrak{h}\}$ $\subseteq V$ $\lambda \in \mathfrak{h}^*$
- Suffices to check $\dim W \geq 1$ always: think $\lambda = 0$ for most $\mathfrak{h}$

$\hookrightarrow$ show every addition to $\mathfrak{h}$ has e.vec in W



Proof of Lie's thm: $\underline{g} < \underline{gl}(V)$ solvable

induction on dimension of $\underline{g}$:

$\underline{g}$ has some codimension 1 ideal $\underline{h}$:

$\underline{g}/\mathcal{D}'\underline{g} = \underline{g}/[\underline{g},\underline{g}]$ is abelian $(\forall x,y,\ [[x,y]]=[0])$

take any codim 1 ideal of $\underline{g}/\mathcal{D}'\underline{g}$:

preimage under $\underline{g} \to \underline{g}/\mathcal{D}'\underline{g}$ is codim 1 ideal of $\underline{g}$

note $\underline{h}$ is solvable: $\mathcal{D}^k\underline{h} < \mathcal{D}^k\underline{g} = 0$

Define simultanious e.space $W := \{v \in V \mid Xv = \lambda(X)v \ \forall X \in \underline{h}\}$

by inductive hypothesis, $\dim W > 0$ $\qquad \lambda \in \underline{h}^*$

lemma W fixed by Y $\forall Y \in \underline{g}$

take $w \in W$. $Y(w) \in W \Rightarrow X(Y(w)) = \lambda(X)Y(w)$ $\qquad \underline{h}$ ideal $\Rightarrow [X,Y] \in \underline{h}$

(*) $X(Y(w)) = Y(X(w)) + [X,Y](w) = \lambda(X)Y + \lambda([X,Y])w$

$\to Y$ fixes $W \iff \lambda([X,Y]) = 0$

Show $\lambda([X,Y]) = 0$:

define $U = \text{span}(w, Y(w), Y^2(w), \dots, Y^k(w))$ $\qquad$ k is largest s.t $Y^{i<k}(w)$ are L.I

lemma: $\underline{h}$ fixes U:

induction base case (*): $\underline{h}Y(w) \subset \text{span}(Y(w), w)$ -

$X(Y^k(w)) = Y(X(Y^{k-1}(w)) + [X,Y]Y^{k-1}(w)$

$\hookrightarrow \underline{h}(Y^k(w)) \subset Y(\underline{h}(Y^{k-1}(w))) + \underline{h}Y^{k-1}(w) \subset U$ (inductive hypothesis)

in praticular, $X(Y^k(w)) \in \lambda(X)Y^k(w) + \text{span}(w, \dots, Y^{k-1}(w))$

matrix form of X in basis $Y^i(w)$: $\begin{pmatrix} \lambda(x) & & * \\ & \lambda(x) & \\ 0 & & \ddots \ \lambda(x) \end{pmatrix}$

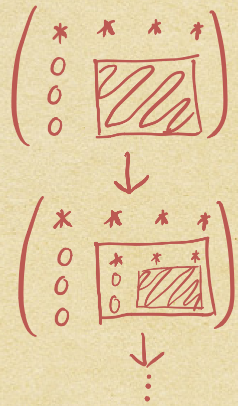
note $\text{tr}_U(X) = \lambda(x) \cdot \dim U$ $\qquad$ upper-Δ

$\lambda([X,Y]) = \frac{1}{\dim V} \operatorname{tr}([X,Y]) = 0$

Y fixes $W \Rightarrow Y|_W$ has eigenvector (say, v)

$v \in W \Rightarrow v$ is common e.vector for $Y \oplus \underline{h} = \underline{g}$

Cor Any rep of solvable Lie alg. has a basis where it's upper-triangular



- Choose that e.vec as first basis element
- Quotient by e.space: get $n-1$ D solvable algebra
- Induct

all solvable algs behave like upper-$\triangle$!

Fin

Main source: Fulton-Harris, ch. 9