Kempf-Ness Theorem thru geometric Quantization

Prelude: Groups act Twice

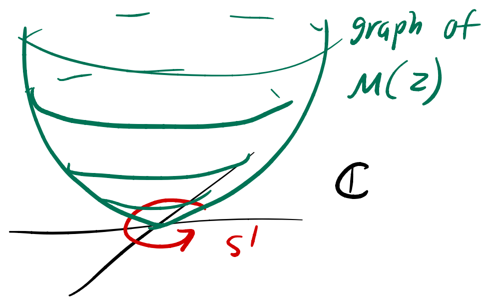
consider the perrenial example of hamiltonian group actions, $S^1 \circlearrowright \mathbb{C}^n$

$(\mathbb{C}, dz\wedge d\bar{z})$ is a symplectic manifold

$S^1 \circlearrowright \mathbb{C}$ defined by $e^{i\theta} \cdot z = e^{i\theta} z$

generated by Hamiltonian vector field $X_\mu = (-y, x)$

w/ "moment map" $\mu(z) = \frac{1}{2}|z|^2$



likewise, $S^1 \circlearrowright \mathbb{C}^2$ via $e^{i\theta}.(z_1,z_2) = (e^{i\theta}z_1, e^{i\theta}z_2)$ has moment map $\mu(z_1,z_2) = \frac{1}{2}|z_1|^2 + \frac{1}{2}|z_2|^2$

We want to take the Quotient $\mathbb{C}^2/S^1$. But this has no symplectic structure, not even infinitesimally.

$T(\mathbb{C}^2/S^1) = T\mathbb{C}^2/X_\mu$ cannot be a symplectic vector space

on symplectic spaces, directions are paired: $\omega = (dx_1 \wedge dy_1) \wedge (dx_2 \wedge dy_2)$,

w/ pairing induced by the almost complex structure $J = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}$

Motto:
In Symplectic Geometry
Groups act twice

Instead, want $T\mathbb{C}^2/\langle \underset{②}{X_\mu}, \underset{①}{JX_\mu}\rangle$: This does have an induced symplectic structure

note $JX_\mu = \nabla\mu$, so $T\mathbb{C}^2/JX_\mu = \ker d\mu \Rightarrow T\mathbb{C}^2/\langle X_\mu, JX_\mu\rangle = \overset{①}{\ker d\mu}/\overset{②}{X_\mu} = T(\mathbb{C}^2/\!/S^1)$

To globalize this construction, define $\boxed{\underbrace{\mathbb{C}^2/\!/S^1}_{\text{2 slashes}} = \underset{①}{\mu^{-1}(c)}/\underset{②}{S^1}}$ symplectic reduction

$\mathbb{C}^2/\!/S^1 = \overset{\text{hopf}}{\underset{①}{S^3}/\underset{②}{S^1}} = \mathbb{CP}^1$, w/ fubini-study form

Alternatively: $\mathbb{CP}^1 = \mathbb{C}^2 - \{0\}/\underset{①+②}{\mathbb{C}^*}$ where action of $\mathbb{C}^*$ defined by $\langle X_\mu, JX_\mu\rangle$

We encode the paired directions of the action of G w/ the action of the complexification $G_\mathbb{C}$!

$G \circlearrowright (X,\omega)$ is hamiltonian if it has a moment map $\mu: X \to \mathrm{Lie}(G)^* = \mathfrak{g}^*$: i.e,

- action of G on X generated by $V_\xi \in \Gamma(TX)$ for $\xi \in \mathfrak{g}$, & $\omega(V_\xi, \cdot) = d\langle \mu(x), \xi\rangle$
- μ is equivariant wrt coadjoint action $G \circlearrowright \mathfrak{g}^*$: $\mu_* V_\xi = ad^*_\xi$, $\quad ad^*_\xi\langle\cdot,\sigma\rangle := \langle\cdot,[\xi,\sigma]\rangle$

then $X/\!/G := \underset{①}{\mu^{-1}(0)}/\underset{②}{G}$ is a symplectic manifold, assume $G \circlearrowright \mu^{-1}(0)$ freely

for X Kahler, with $G \circlearrowright X$ Preserving the complex structure, we have

Kempf-Ness Theorem (1979) $X/\!/G = X^{ss}/G_\mathbb{C} := X/\!/\underset{①+②}{G_\mathbb{C}}$ where $X^{ss} \subset X$ open dense set of "GIT semi-stable pts", up to equivalence, which I will define later

"symplectic reduction = GIT Quotient"

Kempf-Ness, like all the best theorems is a motto: something you expect to hold in any setting where it can be stated. It suggests dual symplectic geometry & complex Geometry perspectives, extending the compact Group / complex group perspectives on Lie theory

Today I will tell you about a different motto:

Guilleman-Sternberg Conjecture (1982): $(X/\!/G)_{Quantum} = (X_{Quantum})^G$ "Quantization commutes with reduction" shortened in paper titles to "$[Q,R]=0$"

and convince you that these two mottos are the same. They were proven at the same time, in different settings, with analagous arguments.

Part 1: Geometric Quantization

Classical phase space, symplectic manifold (X,ω) $\xRightarrow{\text{Quantization}}$ Quantum state space, hilbert space $X_{Quantum}$

For any Quantization procedure:

- the vectors in $X_{Quantum}$ represent "wavefunctions" living over (X,ω)
- a G-action $G \circlearrowright (X,\omega)$ preserving ω $\Longleftrightarrow$ a linear G-action $G \circlearrowright X_{Quantum}$ "functoriality"

Guilleman-Sternberg conjecture:

Reduced Classical phase space $X/\!/G = \mu^{-1}(0)/G$ $\xRightarrow{\text{Quantization}}$ Reduced Quantum state space $(X_{Quantum})^G$ = G-fixed vectors in $X_{Quantum}$

$$(X/\!/G)_{Quantum} = (X_{Quantum})^G$$

Proven in 1982 for Kahler Quantization: (X,ω,J) Kahler manifold

Let $L \to X$ be the Prequantum line bundle:

- L holo. line bundle, $\langle\,\rangle$ hermitian metric w/ curvature equal to kahler form ω

$X_{Quantum} = H^0(X,L)$ holo. sections of prequantum line bundle

say $G \circlearrowright X$ preserves ω & J: Then $X/\!/G$ is also Kahler

define reduced prequantum bundle as $\begin{array}{c} L/\!/G \\ \downarrow \\ X/\!/G = X_0/G \end{array}$ where $\Gamma(X/\!/G, L/\!/G) = \Gamma(\mu^{-1}(0), L)^G$

sections are G-invariant sections on X_0

$\Rightarrow (X/\!/G)_{Quantum} = H^0(X/\!/G, L/\!/G)$

$L/\!/G$ carries induced holomorphic & hermitian structure

Action of G on $H^0(X,L)$:

We describe this infinitesimally, via an action of $\mathfrak{g}$ on $\Gamma(X,L)$
each $\xi\in\mathfrak{g}$, defines a differential operator $\mathcal{O}_\xi:\Gamma(X,L)\circlearrowleft$, satisfying $[\mathcal{O}_\xi,\mathcal{O}_\sigma]=\mathcal{O}_{[\xi,\sigma]}$

first guess: $\mathcal{O}_\xi \overset{?}{=} \nabla_{V_\xi}:\Gamma(X,L)\circlearrowleft$

definition of curvature

fails commutation relation!! $[\nabla_{V_\xi},\nabla_{V_\sigma}] = \nabla_{[V_\xi,V_\sigma]} - i\omega(V_\xi,V_\sigma) \neq \nabla_{V_{[\xi,\sigma]}}$

Correction: $\boxed{\mathcal{O}_\xi s = \left(\nabla_{V_\xi} + i\langle\mu,\xi\rangle\right)s}$ for $s\in\Gamma(X,L)$

$$[\mathcal{O}_\xi,\mathcal{O}_\sigma] = [\nabla_{V_\xi},\nabla_{V_\sigma}] + i[\nabla_{V_\xi},\langle\mu,\sigma\rangle] - i[\nabla_{V_\sigma},\langle\mu,\xi\rangle]$$

$[\nabla_v, f]s = (\nabla_v f)s = df(v)\cdot s$

evaluating commutator

definition of curvature
$$= \nabla_{V_{[\xi,\sigma]}} - i\,\omega(V_\xi,V_\sigma) + i\,d\langle\mu,\sigma\rangle(V_\xi) - i\,d\langle\mu,\xi\rangle(V_\sigma)$$

rearrange & recombine moment map terms

$$= \nabla_{V_{[\xi,\sigma]}} + iV_\xi\langle\mu(x),\sigma\rangle$$

push forward to $\mathfrak{g}^*$

$$= \nabla_{V_{[\xi,\sigma]}} + i\mu^*\left(\mu_* V_\xi(\langle\cdot,\sigma\rangle)\right)$$

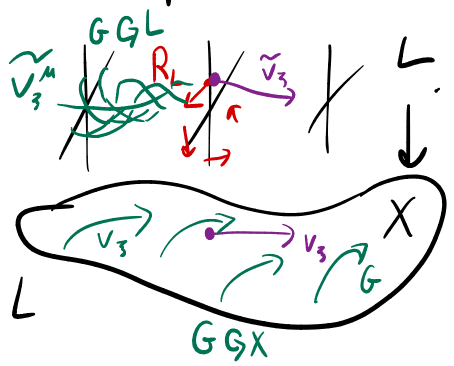
Equivariance of μ

$$= \nabla_{V_{[\xi,\sigma]}} + i\langle\mu(x),[\xi,\sigma]\rangle$$

$$= \mathcal{O}_{[\xi,\sigma]} \quad ✓$$

depends on μ

More conceptually: $\mathcal{O}_\xi$ comes from a vector field $\tilde{V}^\mu_\xi$ on the total space of L
- let V_ξ^H be the horizontal lift of V_ξ relative to ∇.
- let R_L be the generator of the S^1 action rotating the fibers of L
- Then $\tilde{V}^\mu_\xi = V^H_\xi + \langle\mu,\xi\rangle R_L$



for G simply connected, $\tilde{V}^\mu_\xi$ integrates to an action $G\circlearrowright L$ & hence to an action on sections $G\circlearrowright\Gamma(X,L)$

Fact: if $\bar\partial s=0$, then $\bar\partial\,\mathcal{O}_\xi s=0$.

So, $\mathcal{O}_\xi$ defines a representation $\mathcal{O}_\cdot:\mathfrak{g}\to\mathrm{End}(H^0(X,L))$

Since G simply connected, $\mathcal{O}_\cdot$ integrates to a representation $G\circlearrowright H^0(X,L)$

every G-invariant section $s\in H^0(X,L)^G$ restricts to a G invariant section over $\mu^{-1}(0)$

so s descends to a holomorphic section $s/\!/G\in H^0(X/\!/G, L/\!/G)$

this defines a map
$$H^0(X,L)^G \to H^0(X/\!/G,L/\!/G)$$
$$(X_{\text{Quantum}})^G\to (X/\!/G)_{\text{Quantum}}$$
$$s\mapsto s/\!/G$$

Thm (Guillemin-Sternberg): this is a bijection $H^0(X,L)^G\simeq H^0(X/\!/G,L/\!/G)$

Example: $(X,\omega)=(\mathbb{C}, dz\wedge d\bar z)$, prequantum line bundle $\begin{matrix} L\cong\mathbb{C}\times\mathbb{C} \\ \downarrow \\ \mathbb{C}\end{matrix}$ has hermitian metric $\langle s_1,s_2\rangle(z)= s_1\bar{s}_2 e^{-|z|^2/2}$

$S^1 G\, \mathbb{C}$ defined by $e^{i\theta}\cdot z = e^{i\theta}z$, moment map $\mu(z)=\frac{1}{2}|z|^2$

S^1 action on $\Gamma(\mathbb{C},L)$ generated by $\theta_1 s = \nabla_{X_\mu}s + i\mu s$

Choose trivialization $e(z)=1$ of L. In these coords, $\nabla = d+\omega$

$\omega = \partial\log|e|^2 = \partial\log e^{-|z|^2/2} = -\partial\mu$ The chern connection happens to involve the moment map in this case

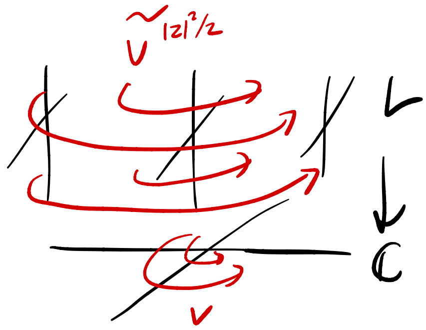
So $\nabla_{X_\mu}s = ds(X_\mu) + \underline{\partial\mu(X_\mu)}\cdot s$. To apply moment map condition to $\partial\mu$,

write ∂ using $d=\partial+\bar\partial$ and $-iJd=\partial-\bar\partial$, w/ $Jd\mu(v)=d\mu(Jv)$

$$\begin{aligned}\underline{\partial\mu}(X_\mu) &= \tfrac{1}{2}\big(d\mu(X_\mu) - i\,d\mu(JX_\mu)\big) && \text{moment map condition}\\ &= \tfrac{1}{2}\big(\cancel{\omega(X_\mu,X_\mu)} - i\,\omega(X_\mu,JX_\mu)\big) && \text{use compatible metric}\\ &= \tfrac{-i}{2}|JX_\mu|^2 = \tfrac{-i}{2}|\nabla\mu|^2\\ &= \tfrac{-i}{2}\big(|\nabla\tfrac{1}{2}(x^2+y^2)|^2 = \tfrac{-i}{2}\big(|x\partial_x+y\partial_y|^2\big) = \tfrac{-i}{2}(x^2+y^2) = -i\mu\end{aligned}$$

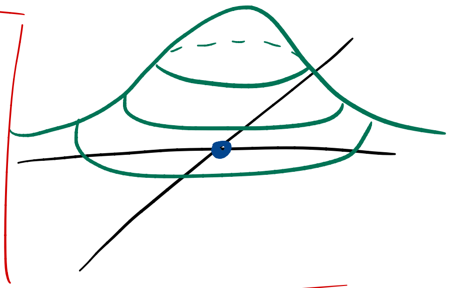
$$\Rightarrow\ \theta_1 s = ds(X_\mu) + \cancel{\partial\mu(X_\mu)s} + \cancel{i\mu s} = X_\mu(s)$$

associated to vector field $\tilde{V}^{|z|^2/2} = (X_\mu, 0)$ on $L=\mathbb{C}\times\mathbb{C}$



S^1 action $e^{i\theta}\cdot s(z) = s(e^{i\theta}z)$

$H^0(\mathbb{C},L)^{S^1}$ consists of rotation-invariant holomorphic functions: The only such function is $\boxed{s(z)=a,\ a\in\mathbb{C}}$ $\dim H^0(\mathbb{C},L)^{S^1}=0$ graph of $|s|^2=e^{-|z|^2}$:



suppose instead we used $\mu(z)=\frac{1}{2}|z|^2+k$, $k\in\mathbb{Z}$

Then $\theta_1 s = \nabla_{X_\mu}s + i\frac{|z|^2}{2}s + iks$ $\tilde{V}^{|z|^2/2+k} = (X_\mu, kR_L)$

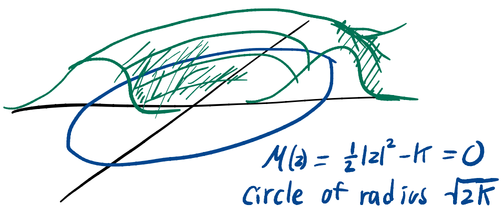
$= X_\mu(s) + iks$

Then S^1 action is $e^{i\theta}\cdot s(z) = e^{ik\theta}s(e^{i\theta}z)$

$k\geq 0 \Rightarrow$ invariant holomorphic sections are $s(z)=az^k$

Graph of $|z^k|^2 = |z|^{2k}e^{-|z|^2}$

$k<0 \Rightarrow$ no invariant holo sections



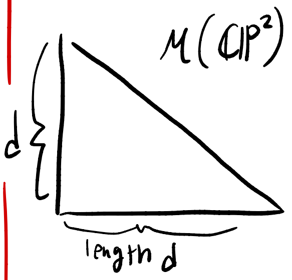
To see $s(x)$, we graph their norm $|s|^2$

- wrt hermitian metric $e^{-|z|^2}$, polynomials are localized!
- invariant holo functions concentrate around zeros of moment map

Example: $(\mathbb{CP}^2, d\,\omega_{FS})$ with $L = \mathcal{O}(d)$ $d \in \mathbb{N}$

$\mathbb{CP}^2$ admits $U(1)^2$ action $(e^{i\theta_1}, e^{i\theta_2}).[z_0 : z_1 : z_2] = [z_0 : e^{i\theta_1} z_1 : e^{i\theta_2} z_2]$

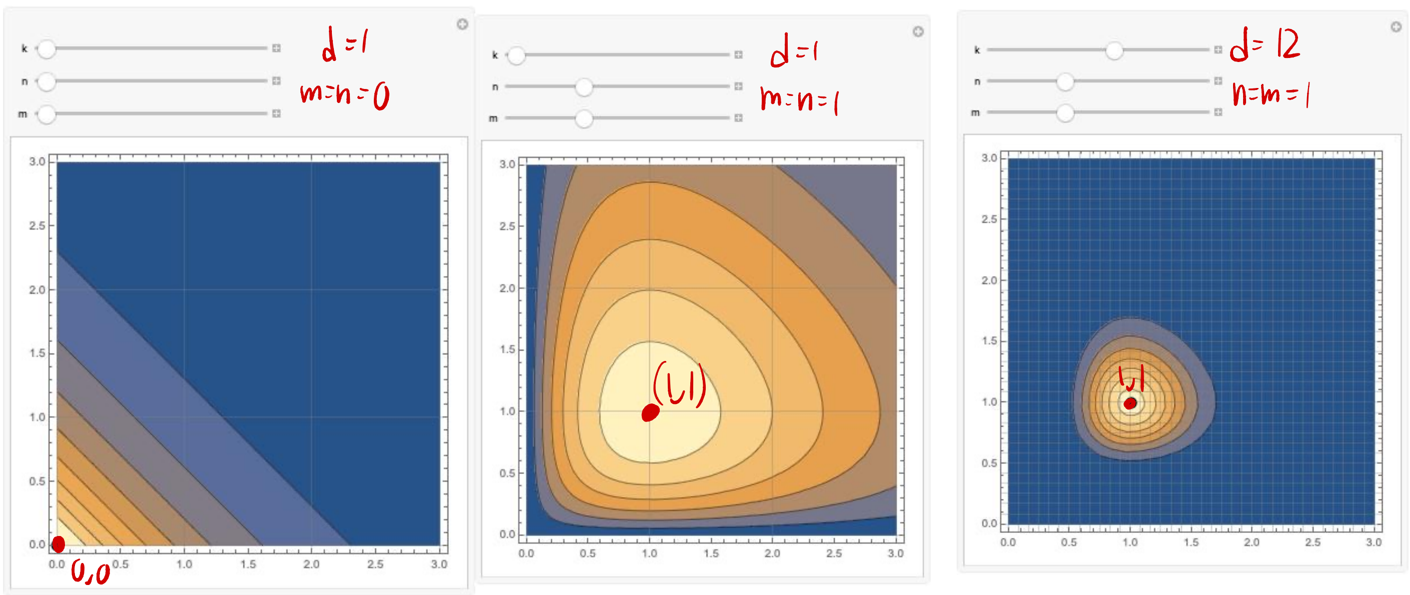
The moment map $\mu : \mathbb{CP}^2 \to \mathbb{R}^2$ is defined up to translation in $\mathbb{R}^2$



$H^0(\mathbb{CP}^2, \mathcal{O}(d)) =$ homogenous degree $d+1$ polynomials in x, y, z

consider a monomial $S_{m,n} = x^m y^n z^{(d+1)-(m+n)} \in H^0(\mathbb{CP}^2, \mathcal{O}(d))$

the norm $|S_{m,n}|^2 : \mathbb{CP}^2 \to \mathbb{R}$ is invariant under $U(1)^2$, so it descends to $: \triangle \to \mathbb{R}$. The Graphs look like:



$|S_{m,n}|^2$ is localized in moment map near (n,m)

in fact, $\langle S_{m,n} \rangle = H^0(\mathbb{CP}^2, \mathcal{O}(d))^G$ for moment map $\mu - (n,m)$

Lessons from these examples:

- if $s \in H^0(M,L)^G$, $|s|^2(x)$ seems to concentrate around $\mu^{-1}(0)$

- Different choices of μ for same group action
  $\iff$ G-invariant sections have different "weights"

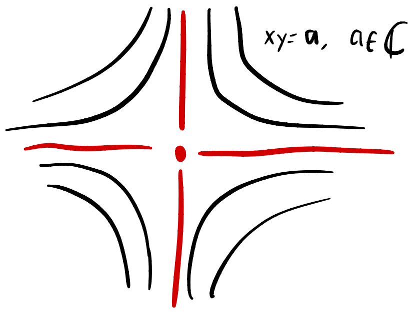
Part 2: Geometric Invariant Theory (GIT)

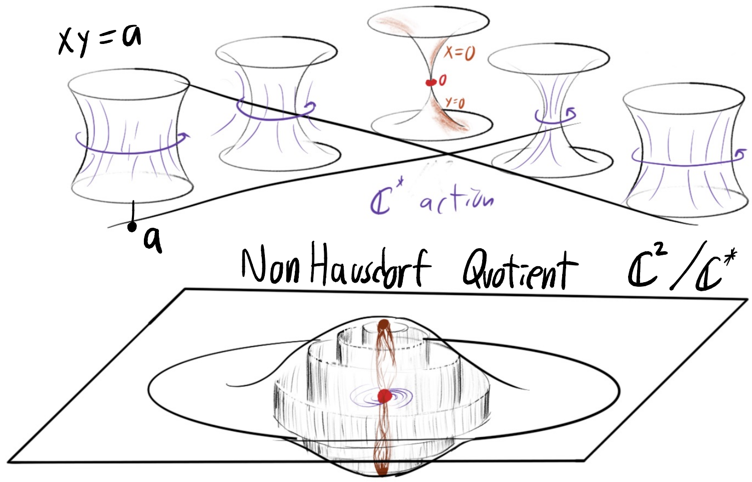
Now we move to the complex story: How do we form a Quotient $X/G_{\mathbb{C}}$?
Since $G_{\mathbb{C}}$ not compact, we don't expect to get a good Quotient space

example: $G_{\mathbb{C}} = \mathbb{C}^*$, $\mathbb{C}^* \circlearrowright \mathbb{C}^2$ via $t.(x,y) = (tx, t^{-1}y)$

complex picture:

orbits of $\mathbb{C}^*$ (real locus)





Quotient is good outside of the locus $xy=0$: $\mathbb{C}^*$ acts freely, so
$\mathbb{C}^2 - \{xy=0\}/\mathbb{C}^* = \mathbb{C} - \{0\}$. This is open dense inside quotient $\mathbb{C}^2/\!/\mathbb{C}^* = \mathbb{C}$

Idea: define functions on Quotient should be invariant functions upstairs
"Invariant theory"

affine GIT: say an algebraic variety V has ring of functions R, $G_{\mathbb{C}} \circlearrowright V$ algebraically

Define functions on $V/\!/G_{\mathbb{C}}$ as $G_{\mathbb{C}}$-invariant functions on V

$V = \operatorname{spec} R$ "build variety out of ring of functions"

$V/\!/G_{\mathbb{C}} = \operatorname{Spec} \underbrace{R^{G_{\mathbb{C}}}}$ Ring of $G_{\mathbb{C}}$-invariant functions

The construction $V = \operatorname{Spec} R$ encodes the structure of V irrespective of it's embedding into $\mathbb{C}^n$. To repoduce the embedding, write $R = \dfrac{\langle x_1, \dots, x_n\rangle}{\langle r_1, \dots, r_k\rangle}$ $\dfrac{\text{generators}}{\text{relations}}$

then R is a quotient of a polynomial ring $\dfrac{\mathbb{C}[x_1, \dots, x_n]}{\langle r_1, \dots, r_k\rangle}$ $r_i \in \mathbb{C}[x_1, \dots, x_n]$

so $V = \operatorname{Spec}\left(\dfrac{\overbrace{\mathbb{C}[x_1, \dots, x_n]}^{\mathbb{C}\text{-functions on } \mathbb{C}^n}}{\langle r_1, \dots, r_k\rangle}\right) \underset{\text{as a complex manifold}}{\simeq} \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid r_1(\vec{x}) = \dots = r_k(\vec{x}) = 0\}$

Choice of generators of $R \Leftrightarrow$ write V as zeros of polynomial

Projective GIT: a projective variety is $V \subset \mathbb{CP}^n$. Lets describe its functions

functions on $\mathbb{CP}^n$ = polynomials $f: \mathbb{C}^{n+1} \to \mathbb{C}$ s.t $f(\lambda \vec{x}) = \lambda^{d_i} f(x)$ d_i degree of f_i

$f \in H^0(\mathbb{CP}^n, \mathcal{O}(d)) = H^0(\mathbb{CP}^n, \mathcal{O}(1)^d)$ degree d homogenous polynomials

e.g $f(x,y) = a_0 x^d + a_1 x^{d-1} y + \dots + a_d y^d$ each term has same degree d

Ring of functions $\bigoplus H^0(\mathbb{CP}^n, \mathcal{O}(d))$ is graded by degree

$\Rightarrow$ Ring of functions $R^\bullet$ for projective variety $V \subset \mathbb{CP}^n$ is also graded

write $V = \operatorname{Proj}(R^\bullet)$: V has the structure of a projective variety, w/o choosing embedding into projective space

To pick an embedding, write $R^\bullet = \mathbb{C}[x_0, \dots, x_n]/f_1, \dots, f_k$ x_i have degree 1, f_i are homogenous $\Rightarrow \operatorname{Proj}(R^\bullet) = \bigcap_i f_i^{-1}(0) \subset \mathbb{CP}^n$

Then $V/\!/G_{\mathbb{C}} = \mathrm{Proj}(R^{\bullet\, G_{\mathbb{C}}})$

Recall Kodaira embedding: $\begin{matrix} L \\ \downarrow \\ X \end{matrix}$ holo. line bundle defines a map to projective space

$i: X \to \mathbb{P}(H^0(X,L)^*)$, $x \mapsto ev_x$ $\quad ev_x(s) = s(x)$ in coords, choose basis $s_0, \dots, s_n \in H^0(X,L)$. Then, $x \mapsto [s_0(x) : \cdots : s_n(x)]$

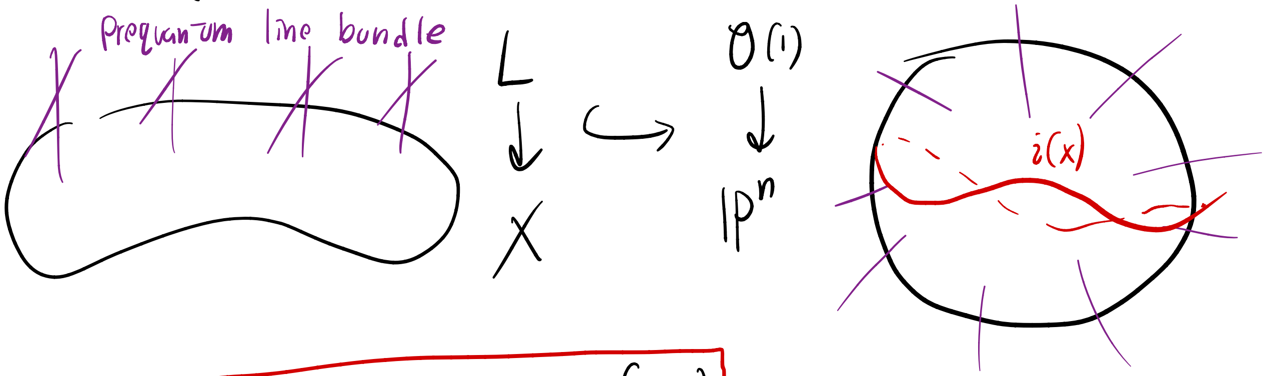
Kodaira embedding thm: if L has positive curvature, then $\exists d$ s.t $X \to \mathbb{P}(H^0(X,L^d)^*)$ is an embedding

For example, if L is prequantum line bundle, $F_h = \omega > 0$, so L ample

$\Rightarrow$ Quantizable Kahler manifolds are projective, and $L = i^* \mathcal{O}(1)$

The "ring of functions" on X is $R(X) = \bigoplus_{d \geq 0} H^0(X, L^d)$, graded by power of prequantum line bundle

$\Rightarrow X = \mathrm{Proj}(R(X))$



GIT Quotient $\boxed{X/\!/G_{\mathbb{C}} := \mathrm{Proj}(R(X)^{G_{\mathbb{C}}})}$

explicitly, $X/\!/G_{\mathbb{C}}$ has "Kodaira embedding" $X/\!/G_{\mathbb{C}} \to \mathbb{P}(H^0(X,L)^{G_{\mathbb{C}}\,*})$, $x \mapsto [s_0(x) : \cdots : s_k(x)]$, s_i basis of $H^0(X,L)^{G_{\mathbb{C}}}$

Kempf-Ness (again):

$$\underset{\text{symplectic}}{X/\!/G} = \underset{\text{GIT}}{X/\!/G_{\mathbb{C}}}$$

$$\mathrm{Proj}\left(\bigoplus_d H^0(X/\!/G, L/\!/G^d)\right) = \mathrm{Proj}\left(\bigoplus_d H^0(X, L^d)^{G_{\mathbb{C}}}\right)$$

$$\Rightarrow \quad H^0(X/\!/G, L/\!/G) = H^0(X,L)^{G_{\mathbb{C}}}$$

Guilleman sternberg (again):

$$(X/\!/G)_{\text{Quantum}} = (X_{\text{Quantum}})^G$$

$$H^0(X/\!/G, L/\!/G) = H^0(X,L)^G$$

not $G_{\mathbb{C}}$ (yet)

2.1 What's Geometric about GIT?

want to relate points in $X/\!/G_{\mathbb{C}}$ w/ orbits in X

$X/\!/G_{\mathbb{C}}$ can only hope to "see" points in X where $H^0(X,L)^{G_{\mathbb{C}}}$ is nontrivial...

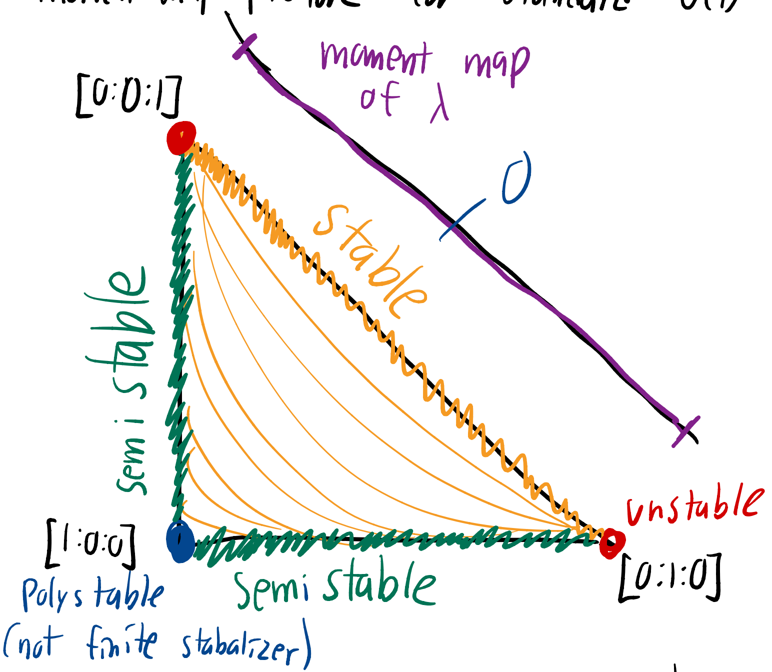
- $x \in X$ is semistable (ss) if $\exists$ $s \in R(X)^{G_{\mathbb{C}}}_{>0}$ (nonconstant, $G_{\mathbb{C}}$ invariant function on X) s.t $s(x) \neq 0$

  The semistable locus $X^{ss} = \{x \in X \mid x \text{ ss}\} \subset X$ is the part of X "visible" to $X/\!/G_{\mathbb{C}}$
- x is unstable (u) if it is not ss, $X^u = X - X^{ss}$
- x is stable (s) if $\underbrace{G_{\mathbb{C}} \cdot x \text{ is closed in ss}}_{\text{important}}$, & $\underbrace{x \text{ has finite stabalizer}}_{\text{technical !!}}$

$X^s \subset X^{ss} \subset X$

Example: let $X = \mathbb{P}^2$, $G_{\mathbb{C}} = \mathbb{C}^\times$, $\lambda . [z_0 : z_1 : z_2] = [z_0 : \lambda z_1 : \lambda^{-1} z_2]$

moment map picture for standard $U(1)^2$ action:



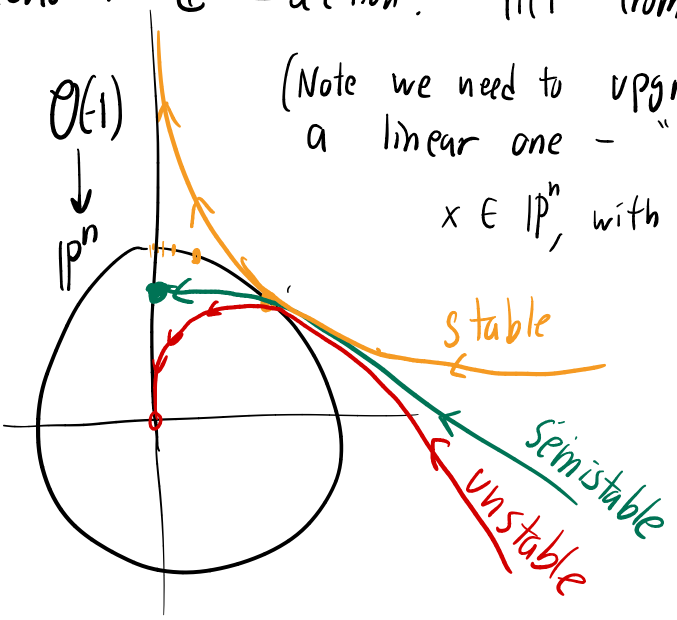
Thm (Mumford): $X /\!/ G_{\mathbb{C}} = (X^{ss}/G_{\mathbb{C}})/\sim$ where we impose "orbit equivelence"
$G_{\mathbb{C}}x \sim G_{\mathbb{C}}y$ if $\overline{G_{\mathbb{C}}x} \cap \overline{G_{\mathbb{C}}y} \cap X^{ss} \neq \emptyset$

2 slashes!

e.g above, the two semistable orbits & [1:0:0] are identified

Hilbert - Mumford Crieterion: dynamic condition for stability

consider $\mathbb{C}^*$-action: lift from $\mathbb{P}^n$ to $\mathrm{Tot}(\mathcal{O}(-1)) = \mathbb{C}^{n+1} - \{0\}$



(Note we need to upgrade from a projective representation to a linear one - "choice of linearization")

$x \in \mathbb{P}^n$, with lift $\tilde{x}$, is

Stable if $\lim_{\lambda \to 0} \|\lambda . \tilde{x}\| = \infty$
(orbit is closed, so gives good topology on Quotient)

Semistable if $\lim_{\lambda \to 0} \|\lambda . \tilde{x}\| \in (0, \infty)$
(limits of stable orbits)

unstable if $\lim_{\lambda \to 0} \|\lambda . \tilde{x}\| = 0$

we can measure the behievor of the orbit from the weight of $\mathbb{C}^\times$ acting on the fiber at the limit point.

Theorem: $x \in X$ is semistable for G iff it is semistable for all 1-parameter subgroups $\mathbb{C}^* < G$

Part 3: Proving Quantization commutes w/ Reduction

Outline of proof

$$H^0(X,L)^G \underset{(1b)}{=} H^0(X,L)^{G_{\mathbb{C}}} \xrightarrow[r]{\simeq} H^0(X^{ss},L)^{G_{\mathbb{C}}} \xrightarrow[(4)]{\simeq} H^0(X/\!/G, L/\!/G)$$

(1) extend G-action on X to $G_{\mathbb{C}}$-action

(1a) show that the action of $G_{\mathbb{C}}$ is spanned by X_μ & $\nabla\mu$

(1b) lift G-action on L to $G_{\mathbb{C}}$ action s.t $H^0(S,L)^G = H^0(S,L)^{G_{\mathbb{C}}}$

(2) show that the $G_{\mathbb{C}}$-orbit of $\mu^{-1}(0)$ is the stable locus X^s

(3) Prove that restriction $H^0(X,L)^{G_{\mathbb{C}}} \xrightarrow{\sim} H^0(X^s,L)^{G_{\mathbb{C}}}$ is an isomorphism

(4) prove that the Quotient map $H^0(X^s,L)^{G_{\mathbb{C}}} \to H^0(X/\!/G, L/\!/G)$ is an isomorphism

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(1) extending G to $G_{\mathbb{C}}$: we work infinitesimally. $\mathrm{Lie}\, G_{\mathbb{C}} = \underline{g}^{\mathbb{C}} \simeq \underline{g} \oplus i\underline{g}$

Define $G_{\mathbb{C}} \circlearrowright X$ via $V_{i\xi} = J V_\xi$ for $i\xi \in i\underline{g}$.

(1a) denote $\mu^\xi = \langle \mu, \xi \rangle$. $V_\xi = X_{\mu^\xi}$, so $V_{i\xi} = J V_\xi = \nabla \mu^\xi$

The complex Grap action is generated by the $\underline{\text{Gradient flow}}$ of the moment map

(1b) lift $V_{i\xi}$ to act on $\Gamma(L)$ in the natural way:

$\mathcal{O}_{i\xi}\, s = i\, \mathcal{O}_\xi s$ for s holomorphic. We want this written as a operator

Since s holomorphic, $\nabla_\cdot s \in \Omega^{1,0}(X)$. So, $i\nabla_\cdot s + \nabla_{J\cdot} s = 0 \Rightarrow \nabla_{JV_\xi} s = -i\nabla_{V_\xi} s$

$\mathcal{O}_{i\xi} s = i(\nabla_{V_\xi} s + i\mu^\xi s) = -(\nabla_{JV_\xi} s + \mu^\xi s) = -(\nabla_{V_{i\xi}} s + \mu^\xi s)$

in particular, if $s \in H^0(X,L)^G$, $\mathcal{O}_\xi s = 0 \ \forall \xi \in \underline{g}$ $\boxed{H^0(X,L)^G = H^0(X,L)^{G_{\mathbb{C}}}}$

so $\mathcal{O}_{i\xi} s = 0 \ \forall i\xi \in i\underline{g} \Rightarrow s$ is $G^{\mathbb{C}}$-invariant!

Now we collect facts about invariant sections. let $s \in H^0(X,L)^G$

- $\underline{V_{i\xi} |s|^2 = -2\mu^\xi |s|^2}$ Fundamental computation

  $V_{i\xi} |s|^2 = V_{i\xi}\langle s,s \rangle = 2\langle \nabla_{V_{i\xi}} s, s\rangle$ as ∇ is a metric connection

  s is G-invariant, hence $G_{\mathbb{C}}$-invariant, so $0 = \mathcal{O}_{i\xi} s = -(\nabla_{V_{i\xi}} s + \mu^\xi s)$

  and $\nabla_{V_{i\xi}} s = -\mu^\xi s$. Plugging this in, $2\langle \nabla_{V_{i\xi}} s, s\rangle = -2\mu^\xi |s|^2$.

• The maximum of $|s|^2$ occurs along $\mu^{-1}(0)$ (assuming $s \neq 0$)

Like we observed emphenically!

say x_0 is a global maximum of $|s|^2$. Then $V|s|^2(x_0)=0$ for any V. In particular,

$$0 = V_{i\xi}\langle s,s\rangle\Big|_{x_0} = -2\mu^3\,|s|^2\Big|_{x_0}$$

Since $s \neq 0$, $|s(x_0)|^2 = \max|s(x_0)|^2 > 0$. so, the above equality gives $\mu^3(x_0)=0$.

• Along any $G_{\mathbb{C}}$-orbit $G_{\mathbb{C}}x$, consider the function $\psi_s(g) = |s(gx)|^2$

if $k \in G$, then $|s(kx)|^2 = |s(x)|^2$. The compact part preserves norm. So, $\psi(g)$ descends

to $\psi_x : G\backslash G_{\mathbb{C}} \to \mathbb{R}$ Recall $G\backslash G_{\mathbb{C}} \simeq \mathrm{Exp}(i\mathfrak{g}) \overset{\text{ditto.}}{\simeq} \underline{\mathfrak{g}}$

Then ψ_x is <u>concave</u>

This is another consequence of our fundamental computation

$T\, G\backslash G_{\mathbb{C}} \simeq i\underline{\mathfrak{g}}$. compute the second derivative in direction $i\xi$:

$$V_{i\xi}^2|s|^2 = V_{i\xi}(-2\mu^3|s|^2) = -2\left(\nabla\mu^3(\mu^3)\,|s|^2 + \mu^3\cdot\mu^3|s|^2\right)$$
$$= -2\left(|\nabla\mu^3|^2|s|^2 + |\mu^3|^2|s|^2\right) \leq 0$$

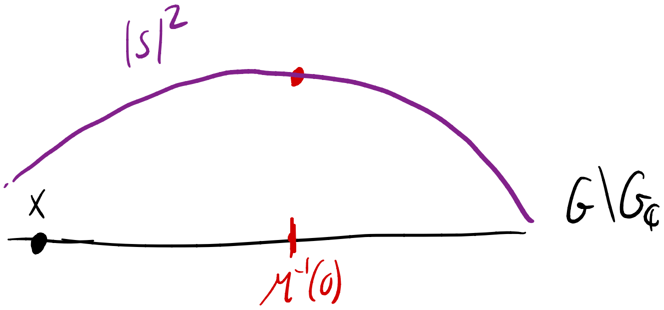
② relationship with (semi) stable locus

The concavity of ψ_s implies, along $G_{\mathbb{C}}x$, $|s|^2$ has at most 1 critical pt.

from the fundamental computation, these critical points are exactly $G_{\mathbb{C}}x \cap \mu^{-1}(0)$

$\Rightarrow$ each $G_{\mathbb{C}}$ orbit intersects $\mu^{-1}(0)$ @ at most 1 G-orbit!

Options:

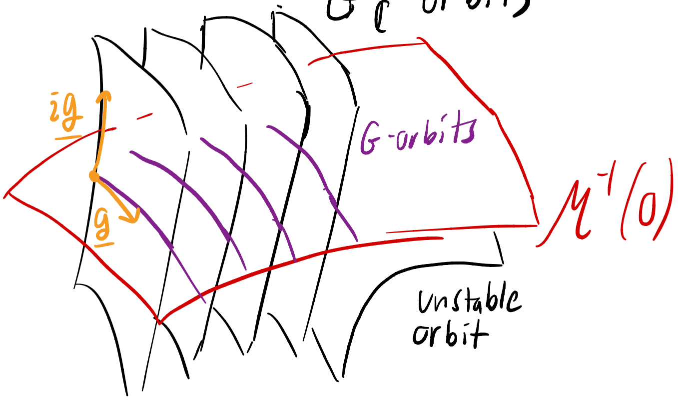


- 1 critical pt
- $G_{\mathbb{C}}x$ intersects $\mu^{-1}(0)$
- x is stable (assuming $G \circlearrowright \mu^{-1}(0)$ free)

- no critical pts (horizontal asymtote)
- $\overline{G_{\mathbb{C}}x}$ intersects $\mu^{-1}(0)$
- x is semistable but not stable

- $|s|^2 = 0$ on orbit
- if this is true for all $s \in H^0(X,L)^G$, x is unstable

So x s.s $\iff$ $\overline{G_{\mathbb{C}} x} \cap \mu^{-1}(0) \neq 0$



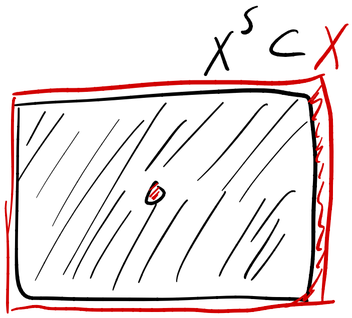
$\mu^{-1}(0)$ is slice for action of $i\underline{\mathfrak{g}}$

④ $H^0(X^s, L)^{G_{\mathbb{C}}} \xrightarrow{q} H^0(X/\!/G, L/\!/G)$ via the quotient map q.

q is surjective: for $s_0 \in H^0(X/\!/G, L/\!/G) \cong H^0(\mu^{-1}(0), L)^G$, want to extend to $s \in H^0(G_{\mathbb{C}}\mu^{-1}(0) = X^s, L)^{G_{\mathbb{C}}}$. define this using action $\theta_{i\mathfrak{g}}$. The resulting section is unique, b.c $G_{\mathbb{C}}$ acts freely on X^s

q is injective: wTS that for $s \in H^0(X^s, L)^{G_{\mathbb{C}}}$, $s \neq 0$, $r(s) \neq 0$. This is true b.c $|s|^2$ achieves its maximum on $\mu^{-1}(0)$

③ $H^0(X,L)^{G_{\mathbb{C}}} \underset{r}{\overset{\sim}{\to}} H^0(X^s, L)^{G_{\mathbb{C}}}$



r is injective because $X^s \subset X$ is dense open

r surjective: take $s \in H^0(X^{ss}, L)^{G_{\mathbb{C}}}$. we wish to extend s to $\tilde{s}$

Lemma: X^u is complex codimension ≥ 1. in particular, it is contained in a divisor D

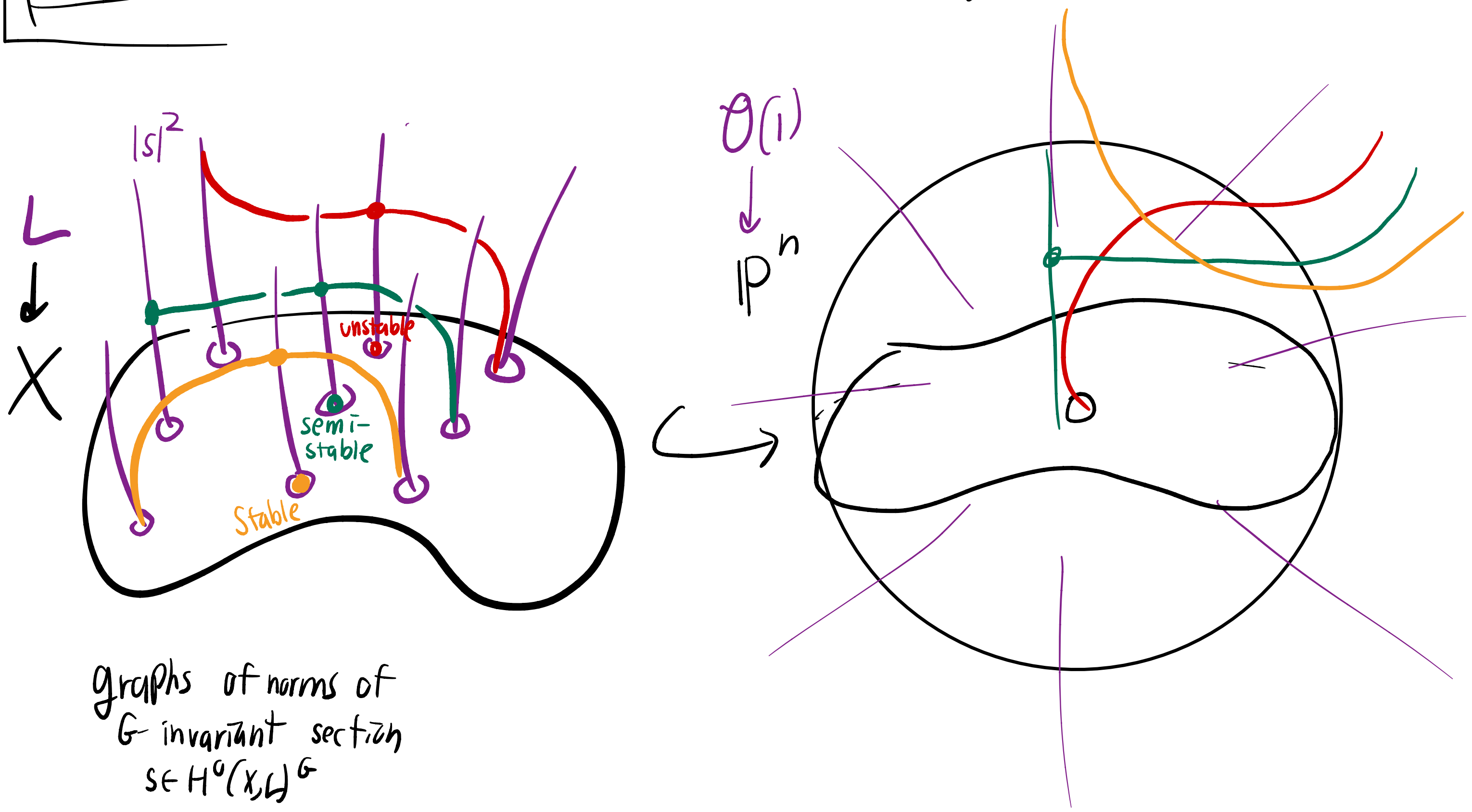
assumming the lemma, we know by construction that $|s|^2$ is bounded. so it cannot have a pole at D. By the Riemann extention theorem, s extends smoothly to $\tilde{s} \in H^0(X,L)^{G_{\mathbb{C}}}$, s.t $r(\tilde{s}) = s$.

Proof of lemma:

it suffices to find a single non-0 function $s \in H^0(X,\mathcal{L})^G$. Indeed, $D = s^{-1}(0)$ is a divisor which necessarily contains X^u

Guilleman-Sternberg argue for $\dim H^0(X,L)^G \neq 0$ using fourier integral operators to estimate the rank of the Szego kernel - gross.

We could alternativly use Kodaira embedding to make X a projective variety. Then we are in the setting of classic GIT. we want to show $X /\!/^{GIT} G_{\mathbb{C}} = \mathrm{Proj}(\oplus H^0(X,\mathcal{L})^G)$ is nonempty. By GIT, $X/\!/G_{\mathbb{C}} \neq \emptyset$ if X has a semistable point. by our computation above, we know X has a semistable pt b.c $\mu^{-1}(0)$ is nonempty. ∎



graphs of norms of G-invariant section $s \in H^0(X,L)^G$

$$|s|^2 \longleftrightarrow \frac{1}{|v|^2} \text{ in lift}$$

$$L = i^*\mathcal{O}(1) \longleftrightarrow \mathcal{O}(-1)$$

Kempf-Ness proved: $v \in \mathbb{P}(V)$ is stable under $G \circlearrowright V$ iff the orbit attains the infimum of $|v|^2$. $\Leftrightarrow$ zero of moment map in $(\mathbb{P}^n, \omega_{FS})$

C.f "an orbit is stable if the g-invariant holo. section $|s|^2$ attains its supremum