Higgs bundles

Summary:

1) geometric structures $\Rrightarrow$ higgs bundles
2) higgs bundles $\Rrightarrow$ geometric structures
3) general higgs theory

$$\mathrm{Def}_{(G,X)}(\Sigma) = \{\text{marked } (G,X)\text{-structures}\}$$

Riemann surface

Geometry

(G,X) - Atlas

Graph:
flat (G,X) -bundle w/
transverse section

Topology

Developing pair (dev, h)

$\tilde{\Sigma} \xrightarrow{dev} X \qquad \pi_1(\Sigma) \xrightarrow{h} G$

eqivariant

G-bundles

hol

holonomy principle
"almost"

Analysis?

Higgs bundles

$\mathrm{Hom}(\pi_1(\Sigma), G)/\mathrm{Inn}(G)$
charecter vareity

Algebra

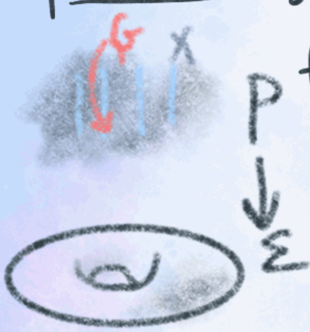
want: analytic description of {flat G-bundles}

PDE w/ 1 solution per flat G-bundle: extremizing problem!

Gauge structure $\overset{\text{over } \Sigma}{:}$ associated to $\underset{\text{"Gauge grp"}}{G}$, X, $\underset{\text{grp action}}{(X \curvearrowleft G)}$

atlas $U_i \xrightarrow{\psi_i} X$, transition fn. $U_i \cap U_j \xrightarrow{\varphi_{ij}} G$ $\quad \varphi_{ij}\psi_i(x) = \psi_j(x)$

pointwise geometric structure, localizes global symmetry

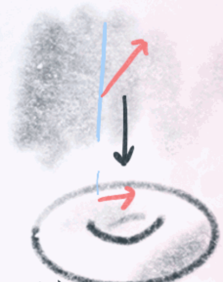


fiber bundle $P \xrightarrow{\pi} \Sigma$, $\pi^{-1}(x) = X$

ex: Principal bundle: $(G \curvearrowleft \cdot G)$-structure

vector bundle: $(V \curvearrowleft O(n))$-structure

$(X \curvearrowleft G)$ Connection: lift $T\Sigma \to TP$

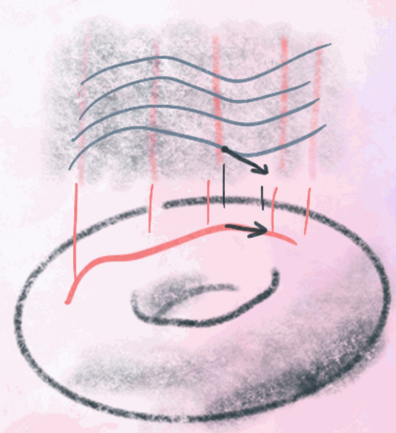


$A = \underset{\text{trivial part}}{d} + \phi \qquad \phi \in \underset{\text{section}}{\Gamma}(T^*\Sigma \otimes (TX \curvearrowleft DG))$

Principal bundle: $X = G \quad (TX \curvearrowleft DG) = (\mathfrak{g} \curvearrowleft Ad) \quad = \Omega^1(\Sigma, Ad(P))$

Curvature: $F_A = dA + A \wedge A$, $\qquad \underset{\Downarrow}{\Gamma(\mathfrak{g} \curvearrowleft Ad)}$ $v_1 \wedge v_2 := [v_1, v_2]$

$F_A \in \Omega^2(\Sigma, Ad(P))$



Flat G-bundle = $\{P, A\}$ s.t $F_A = 0$

Riemann-Hilbert: {flat connection}

$$\updownarrow$$

$$\{\rho : \pi_1(\Sigma) \to G\} / \mathrm{Inn}(G)$$

character variety!

want: analytic description of {flat G-bundles}

PDE w/ 1 solution per flat G-bundle: extremizing problem!

P principal G-bundle, G reductive, K maximal compact

s.s + abelenian

$(G/_K \circlearrowleft G)$ structure: reduce gauge grp to K $\begin{pmatrix} G = GL(n,\mathbb{R}) \\ K = O(n) \end{pmatrix}$

"metrics" (e.g $GL(n,\mathbb{R}) \to O(n)$ is metric)

$GL(n,\mathbb{C}) \to U(n)$ herm. metric)

flat (P,A) w/ $\rho : \pi_1(\Sigma) \to G$ reductive

$$s \in \Gamma(G/_K \circlearrowleft G) \qquad Ds \in \Omega^1(\Sigma, Ad(P))$$

tr : $\underline{g} \otimes \underline{g}^* \to \mathbb{C}$

Harmonic 'metric': $E(s) = \int_\Sigma |Ds|^2 = \int_\Sigma \text{tr } Ds \wedge * Ds$ critical point

'twisted harmonic map'

hodge star: conformal invariant

$\Rightarrow$ well defined

laplacian!

Euler-lagrange: $\nabla E = 0 \Rightarrow d * d\, s = 0$

Donaldson '86 Corlette '88

Thm for ρ irriducible, $\exists!$ harmonic metric

Pf flow according to $\dot{s} = \nabla E|_s$

Aside: A-priori estimates $|\nabla s| < C$, $s_t \in$ compact ρ irriducible

G/K negative sectional curvature

say $\rho \Leftrightarrow V$, $V = E \oplus F$: only get convergence within E, F

cross-component may not converge :(

hermitian metric

take $G = GL(n,\mathbb{C})$ $K = U(n,\mathbb{C})$ $h \in \Gamma\left(GL(n)/U(n) \circlearrowleft GL(n)\right)$

work on $(V, GL(n))$ w/ connection D

reduce to (V, U) w/ h: $D = D_h + \Psi$

connection on (V,U)

$\in \Omega^1(ad(V)) \cong \Omega^1(End(V) \circlearrowleft conj.)$

holo antihol

$\Omega^1 \cong \Omega^{1,0} \oplus \Omega^{0,1}$

$D = D_h^{1,0} + D_h^{0,1} + \phi^{1,0} + \phi^{0,1}$

Ψ is section for h

defines holomorphic structure on V

Take D flat, Ψ harmonic:

covariant derivative

- $D_h(*\Psi) = 0$ harmonic!
- $dD + D \wedge D = 0$

$\Rightarrow$ • $F_{D_h} + \Psi \wedge \Psi$

• $D_h \Psi = 0$

Hitchins eqs

$F_{D_h} + \Psi^{1,0} \wedge \Psi^{1,0} = 0$

$D_h(*\Psi^{1,0}) = 0$

$\Rightarrow D_h^{0,1}\Psi^{1,0} = 0$

$(\underbrace{V, D_h^{0,1}}_{\text{holomorphic V.B}}, \Psi^{1,0})$ $\Omega^{1,0}(End(V))$

D_h is chern connection

unique connection respecting

- complex on Σ
- holo on V
- herm inner prod h

Higgs Bundle: (P, A, ϕ) $\Omega^{1,0}(ad(P))$

hitchin's eqs: $$F + \phi \wedge \phi = 0$$ $$D_A^{0,1} \phi = 0$$

covariant derivative

irriducible flat bundle $\longrightarrow$ stable higgs bundle

$\updownarrow$ $\qquad\qquad$ $\updownarrow$

unique harmonic 'metric' $\longleftarrow$ Unique solution to hitchin eqs

up to gauge transform!

implied by hitchin eqs.

Stability: $(G = GL(n, \mathbb{C}))$ $\quad V$ holo bundle, $W \subset V$

V stable if $\forall W$ $\quad \frac{\deg(W)}{rk(W)} < \frac{\deg(V)}{rk(V)}$ $\quad \deg(V) = c_1(V) = \deg(\det(V))$

ϕ-invariant!! $\quad$ "slope" $\quad$ normalized deg

"charge density" $-\mu = -\frac{\deg}{rk}$ $\quad$ charge, mass $\quad$ deg = topological curvature

goal: minimize $-\mu$ = ammount of $-$ curvature

'Particle' V splits to W, V/W if it lowers $-\mu$

stable $\Rightarrow$ no splitting!

$$0 < \deg(Hom(W,V)) = d(V \otimes W^*) = d(V)\underbrace{r(W^*)}_{= r(W)} + \underbrace{d(W^*)}_{= -d(W)} r(V)$$

$$\Rightarrow d(W)r(V) < d(V)r(W) \Rightarrow d(W)/r(W) < d(V)/r(V)$$ unstable!

$\exists$ sections of $Hom(W,V) \overset{\sim}{\Leftrightarrow}$ unstable $\quad$ continous gauge freedom!

how to use higgs bundles to make geometric structure?

helps us construct transverse sections

Surface geom strs

Conformal

$X = \mathbb{C}P^1$ $\quad G = \underset{\text{mobius}}{PSL(2,\mathbb{C})}$

hyperbolic

$X = H$ $\quad G = \underset{\text{isometry}}{PSL(2,\mathbb{R})}$

use Higgs bundle to prove milnor-wood, uniformization, & teichmuller thm

$\pi \xrightarrow{\text{fuschian}} PSL(2,\mathbb{R})$ spin structure $\Rightarrow$ lift to $SL(2,\mathbb{R}) \subset SL(2,\mathbb{C}) \to$ flat rk 2 bundle

take E deg 0 rk 2 bundle $\quad SL(2,\mathbb{C}) \to SL(2,\mathbb{R})$

say E <u>real</u>: preserved under $(E,\phi) \overset{\sigma}{\mapsto} (E, -\phi)$ mod gauge equiv (involution)

$\phi = 0 \Rightarrow$ flat connections $(SU(2))$ $\quad \phi \neq 0$: connection A reduces to $U(1)$:

$\phi = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \xrightarrow{\sigma} \begin{pmatrix} a & -b \\ -c & -a \end{pmatrix}$ $\quad$ want $\sigma^* \phi = -\phi$

$\Rightarrow a = 0$

hence, $E = L_1 \oplus L_2$. but, $\deg(E) = 0$, $\deg(L_1 \oplus L_2) = \deg(L_1) + d(L_2) = 0$

$d(L_1) = -d(L_2) \Rightarrow L_2 = L_1^*$ $\qquad E = L \oplus L^*$

then $\phi = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$

(poly) stable higgs

WLOG $\deg L > 0 \Rightarrow E$ not stable bundle

$\Rightarrow \phi$ <u>must</u> not preserve $L^* \Rightarrow c \neq 0$

$\phi = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ $c \neq 0$ $c \in H^0(X, \mathrm{Hom}(L, L^{-1}) K)$

$\qquad = d(KL^{-2}) > 0$

$\Rightarrow dL \leq \frac{1}{2} dK = \frac{1}{2} \chi(\Sigma) = g-1$

take ϕ <u>maximal</u> $d(L) = g-1$

Then ϕ <u>fuschian</u> corresponds to fuschian representation

why? well...

take $L = K^{1/2}$ some L w/ $L \otimes L = K$ corresponds to spin structure, lift $PSL \to$ ~~SO~~? $SL(2,\mathbb{R})$

$E = K^{1/2} \oplus K^{-1/2}$

$c \in H^0(KL^{-2} = 1)$ so c trivial! line bundle

use gauge transform take $c \to 1$ $\phi = \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$

$b \in H^0(\mathrm{Hom}(L^{-1}, L) K) = H^0(K^2)$ quadradic differential parametrizes higgs bundles

<u>$b=0$</u>: uniformizing ~~connection~~ representation every degree for $E = L \oplus L^{-1}$ gives connected component of character vareity - euler #

$b \neq 0$: get family of g_a metrics by classification,

every higgs bundle $= \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix} \Rightarrow$ every metric conj to one g_a

curvature of g_a is -1 Teichmuler space parametrizes equiv class of metrics w/ curvature -1