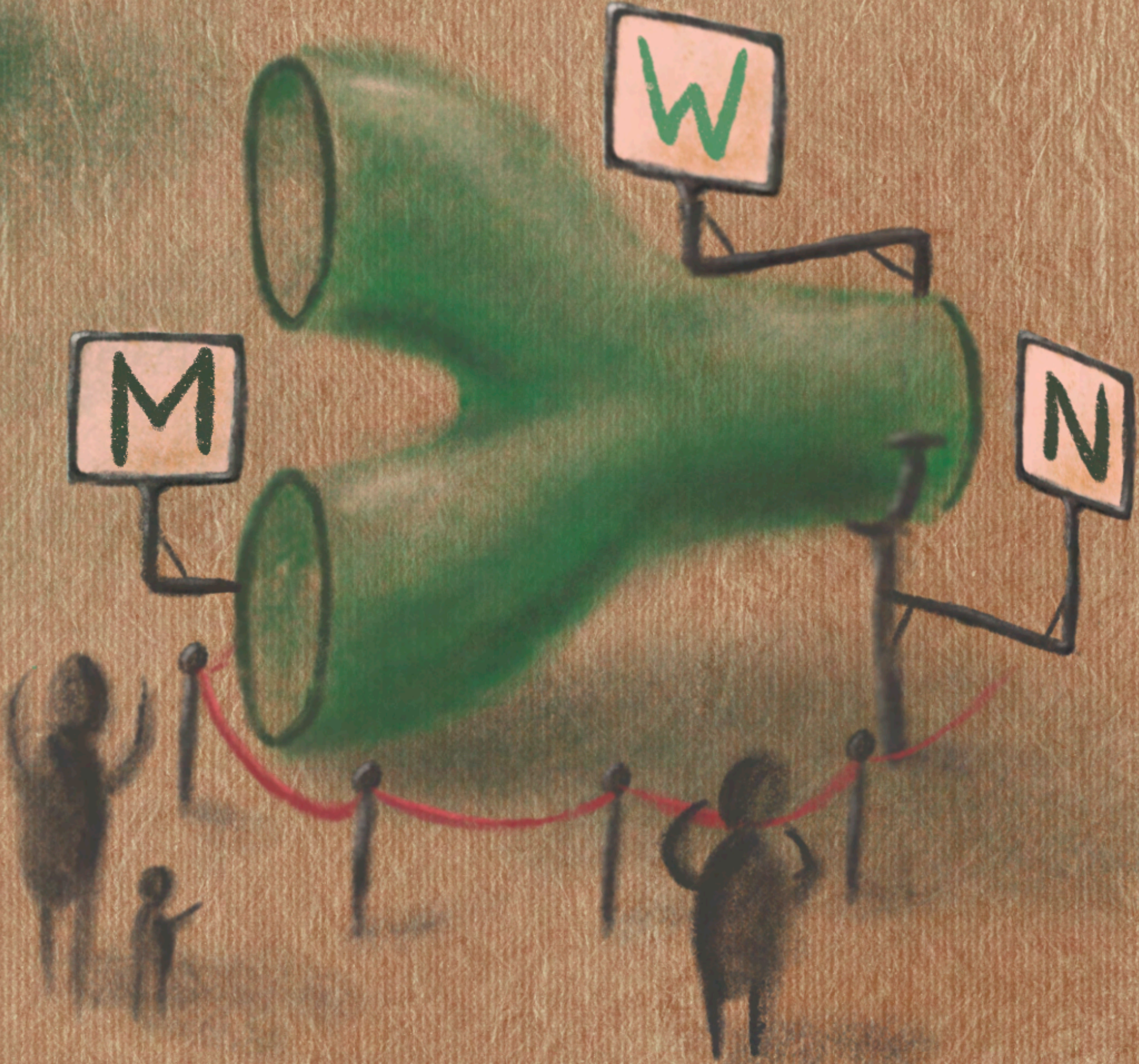
Cobordisms & Thom spectra

Cobordisms

$$M \sqcup N = \partial W$$

boundry coboundry

M & N cobordant




Equivlence Relation
Reflexive
Symmetric
Transitive
M
M
M
N
N
M
M
M'
M''
$\partial(M \times [0,1]) = M \sqcup M$
$M \sim N \Leftrightarrow N \sim M$
$M \sim M'$ & $M' \sim M''$
$\Rightarrow M \sim M''$

Ring Structure

addition: disjoint union

$M \sim N \ \& \ M' \sim N' \Rightarrow M \sqcup M' \sim N \sqcup N'$

$\Rightarrow [M] + [M'] = [M \sqcup M']$ well defined!

abelian!

$[M] + [M] = 0$

$\Rightarrow$ vector space over $\mathbb{F}_2$

Graded by dimension: Ω_n

Multiplication: cartesian product

Ω_*

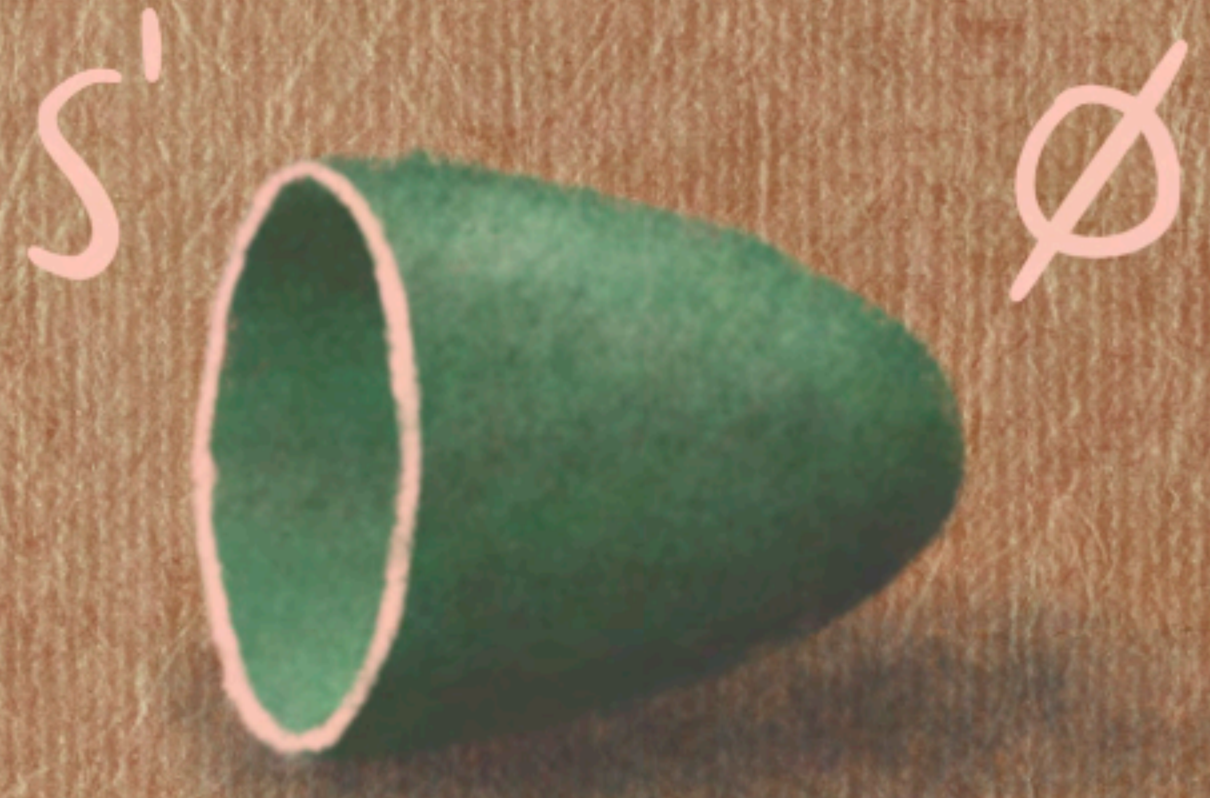
cobordism ring!

Examples

0

$$0 \leftarrow \Omega_0 = \mathbb{Z}_2 \rightarrow 1$$

1



$S^1 \sim \emptyset$

closed 1-manifold $\cong \sqcup S^1$

$\Downarrow$

$$\Omega_1 = 0$$

2: $\Omega_2 = \mathbb{Z}_2$

surface classification:

$$\mathbb{RP}^2 \# \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$$

WTS $\begin{cases} \mathbb{RP}^2 \text{ not boundry} \\ \mathbb{RP}^2 \# \mathbb{RP}^2 \text{ boundry} \end{cases}$

suppose $\partial M = \mathbb{RP}^1$

$2M = M \sqcup M / \partial M$ closed

closed, odd-dim $\to 0$

$$\cancel{\chi}(2M) = 2\chi(M) - \chi(\partial M) \Rightarrow \chi(\partial M) = 2\chi(M)$$

meyer-vatoris

even!

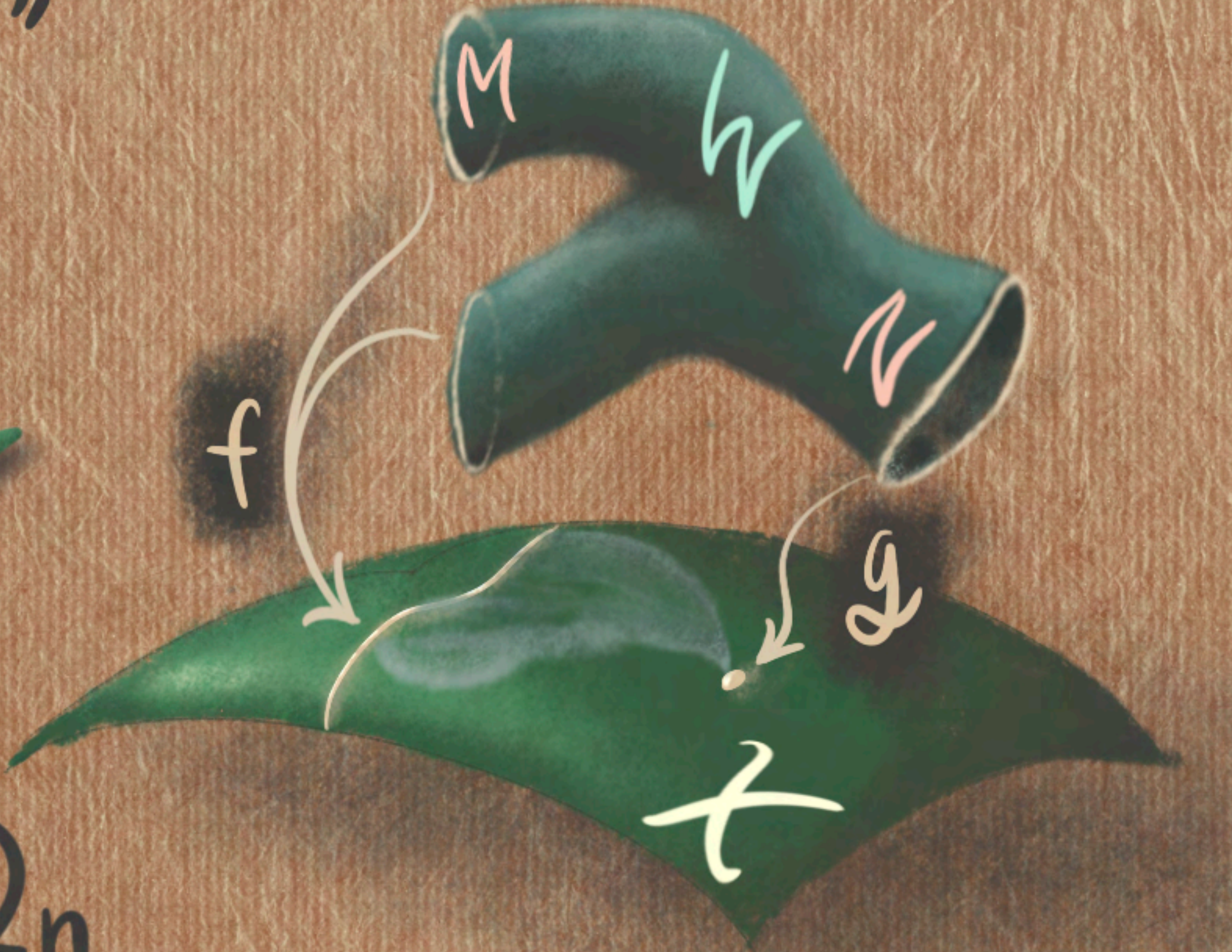
But, $\chi(\mathbb{RP}^2) = 1$ odd!

ℝP²
#
ℝP²
Klien bottle
∂

$\Omega_n(X)$ "cobordisms over X"

$\mathrm{Top} \to \mathrm{Grp}$

functorial: ✓ homotopy invariant: ✓



$\Omega_n(pt) = \Omega_n$

$\Omega_n(X)$ is bigger than $\Omega(pt)$

↳ module over $\Omega(pt)$

Generalized Homology Theory!!

Eilenberg-Steenrod axioms: functoriality homotopy Exactness Excision

$\underbrace{\text{Exactness} \quad \text{Excision}}_{\text{cobordisms over pairs } (X,A)}$

Dimemension: $\Omega_*(pt) \neq 0$

Proof: find spectrum...

Cob-ordism

suspension isomorphism

$$\begin{tikzcd} W \arrow[d] & & \Sigma W \arrow[d] \\ X & \Rightarrow & \Sigma X \end{tikzcd} \qquad \Omega_n(X) \cong \Omega_{n+1}(\Sigma X)$$

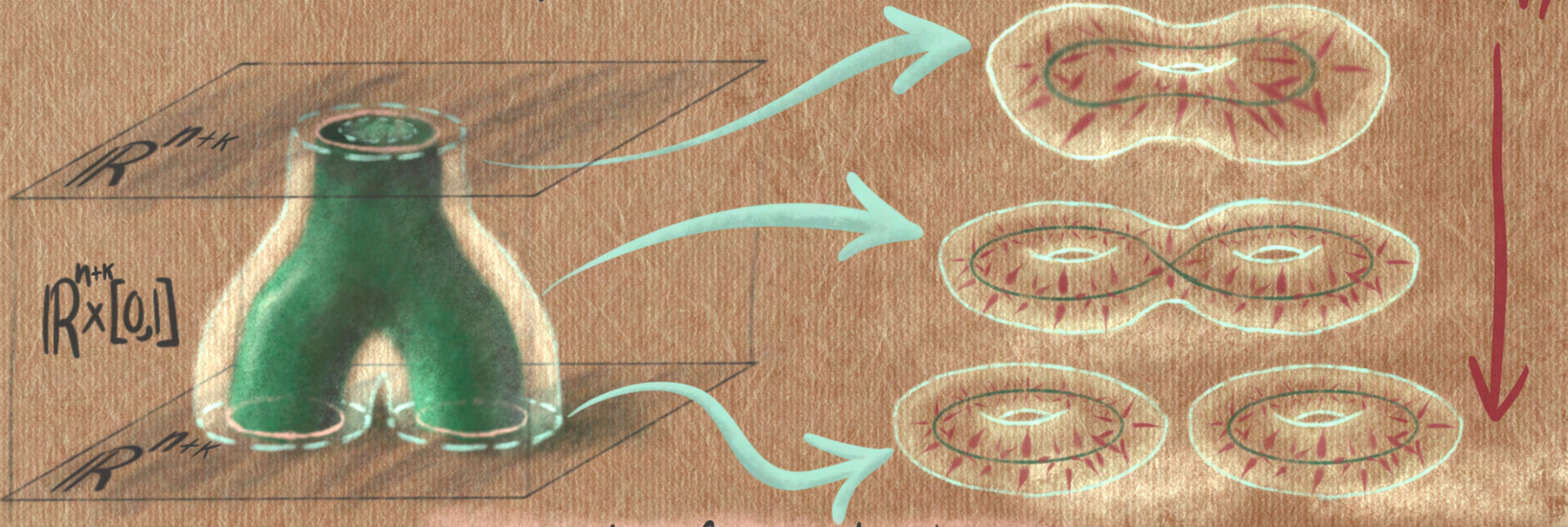

Who cares about cowbordisms?
Build manifolds (like me!)
Surgery theory
h-cobordism Thm

Cobordism. Invariant.

embed $M^k \hookrightarrow \mathbb{R}^{n+k}$: assign pt $\rightarrow$ displacent from M

outside of small nbhd: say distance $=\infty$

homotopy class is cobordism invariant!



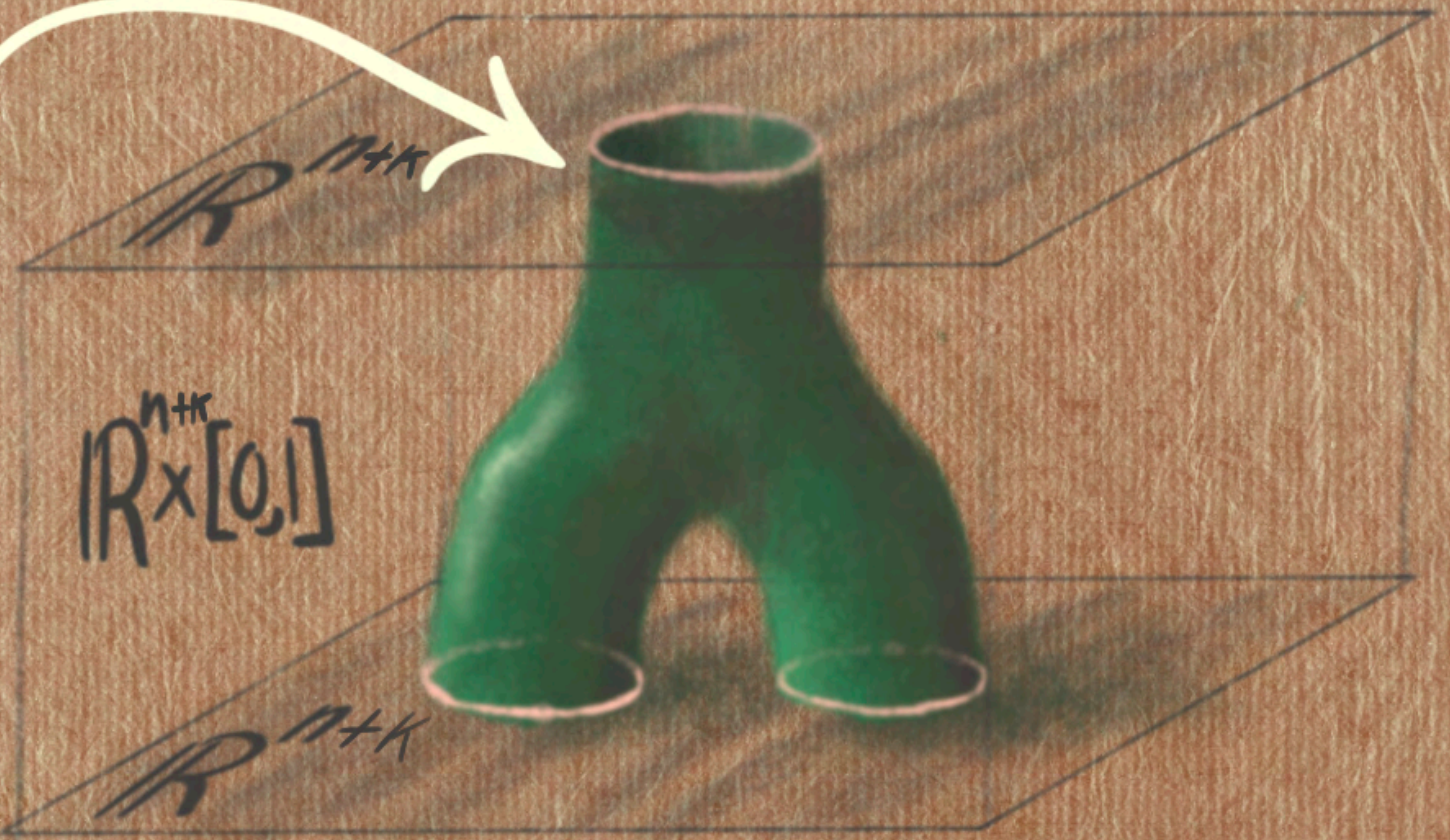
totally classifies cobordisms!!

Differential topology

Whitney embeding Thm:

↳ all mflds embed in some $\mathbb{R}^{n+k}$

all cobordisms can embed like



Tubular nbhd Thm:

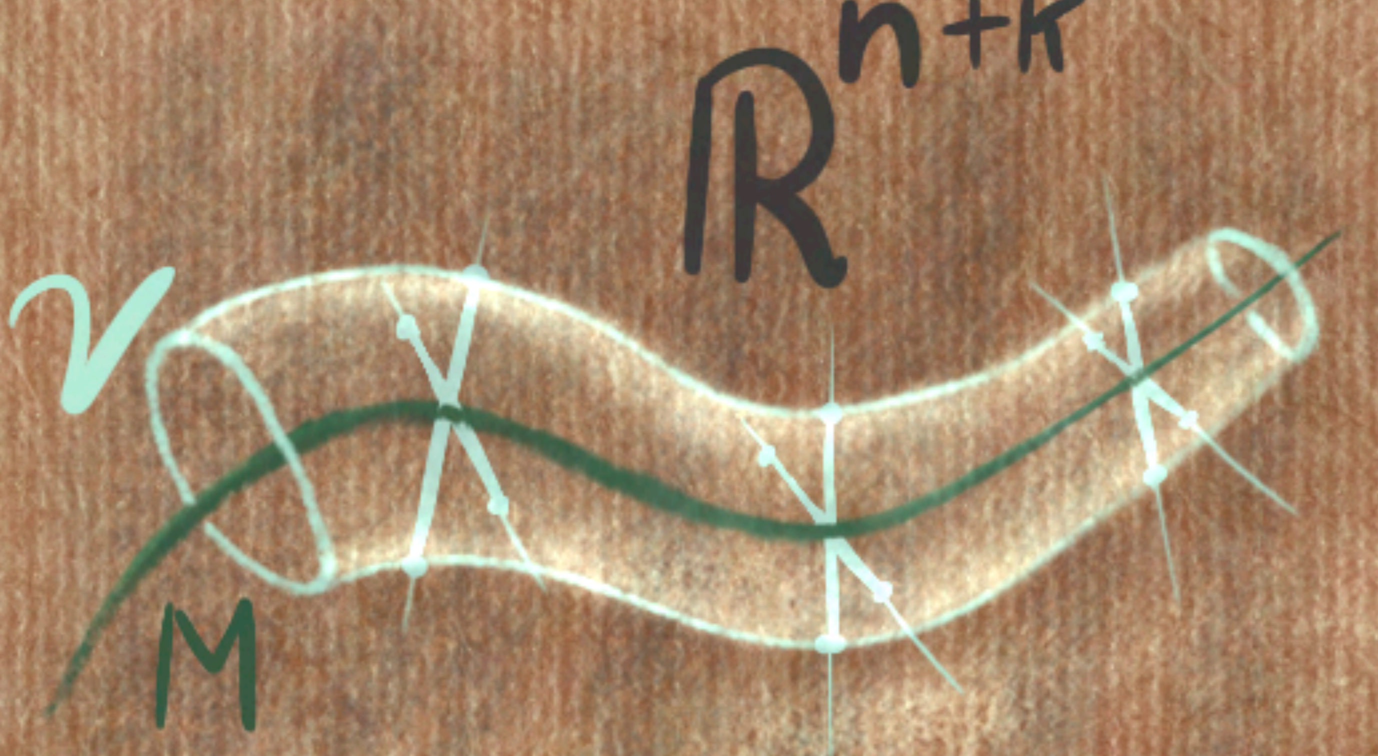
$M \xhookrightarrow{i} \mathbb{R}^{n+k}$ Normal bundle $\nu = TM^{\perp} \subset T\mathbb{R}^{n+k}$

$\nu \longrightarrow M$

i extends to embedding

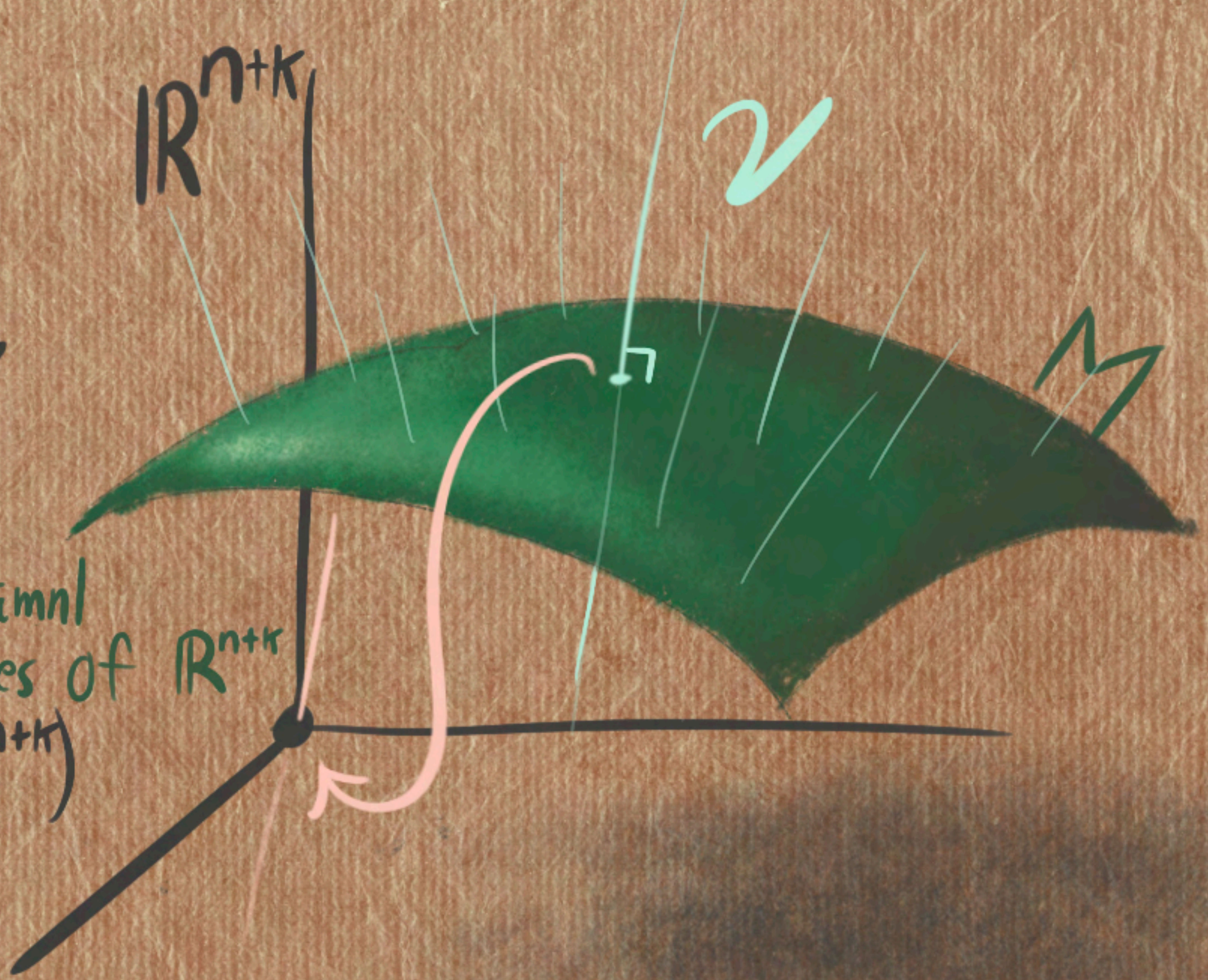
$$\nu \hookrightarrow \mathbb{R}^{n+k}$$

$$\begin{tikzcd} \nu \arrow[dr, dashed, hook] & \\ M \arrow[u, "\text{zero-section}"] \arrow[r, hook, "i"'] & \mathbb{R}^{n+k} \end{tikzcd}$$



normal bundle w/ tubular nbhd
assigns pt to 'displacement' from M

$\Rightarrow$ Want homotopy classification for ν



for $M \hookrightarrow \mathbb{R}^{n+k}$, $\nu_p \subset T_p\mathbb{R}^{n+k} \cong \mathbb{R}^{n+k}$

k dimnl subspaces of $\mathbb{R}^{n+k}$

ν_p canonically associated to pt in $Gr(k, \mathbb{R}^{n+k})$

$\Rightarrow$ map $f: M \to Gr(k, \mathbb{R}^{n+k})$

Tautological k-bundle:

$$\varepsilon^k \longrightarrow Gr(k, \mathbb{R}^{n+k})$$

fiber @ p = subspace for p

$$\nu_p = f^*\varepsilon^k_{f(p)} \;\forall p \Rightarrow \nu = f^*\varepsilon^k$$

Universal bundle

every k-bundle is pullback of $\varepsilon^k \to Gr(k,\mathbb{R}^{n+k})$ for some n

Glue all $Gr(k,\mathbb{R}^{n+k})$ together:

$\mathbb{R}^{n+k} \hookrightarrow \mathbb{R}^{n+k+1}$ induces $Gr(k,\mathbb{R}^{n+k}) \hookrightarrow Gr(k,\mathbb{R}^{n+k+1})$

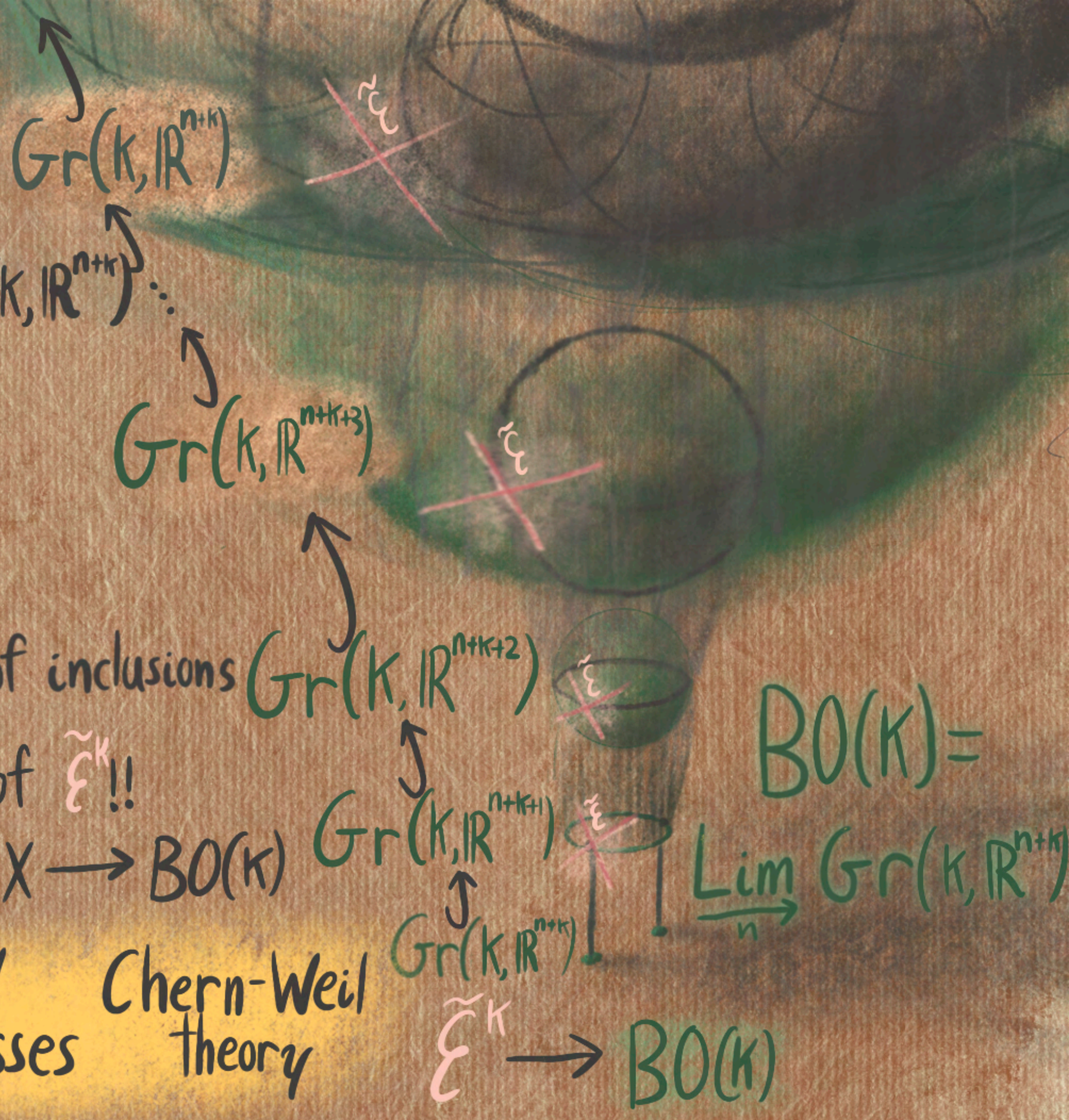
$BO(k) :=$ telescoped mapping cylinder of inclusions

<u>every</u> k-bundle is a pullback of $\tilde{\varepsilon}^k$!!

induced by classifying map $f: X \to BO(k)$

invariants of $[X, BO(k)] \leftrightarrow$ invariants of V Chern-Weil theory

cohomology of $BO(k)$ = charactistic Classes



Collapse everything outside ν
(set distance to 'infinity')

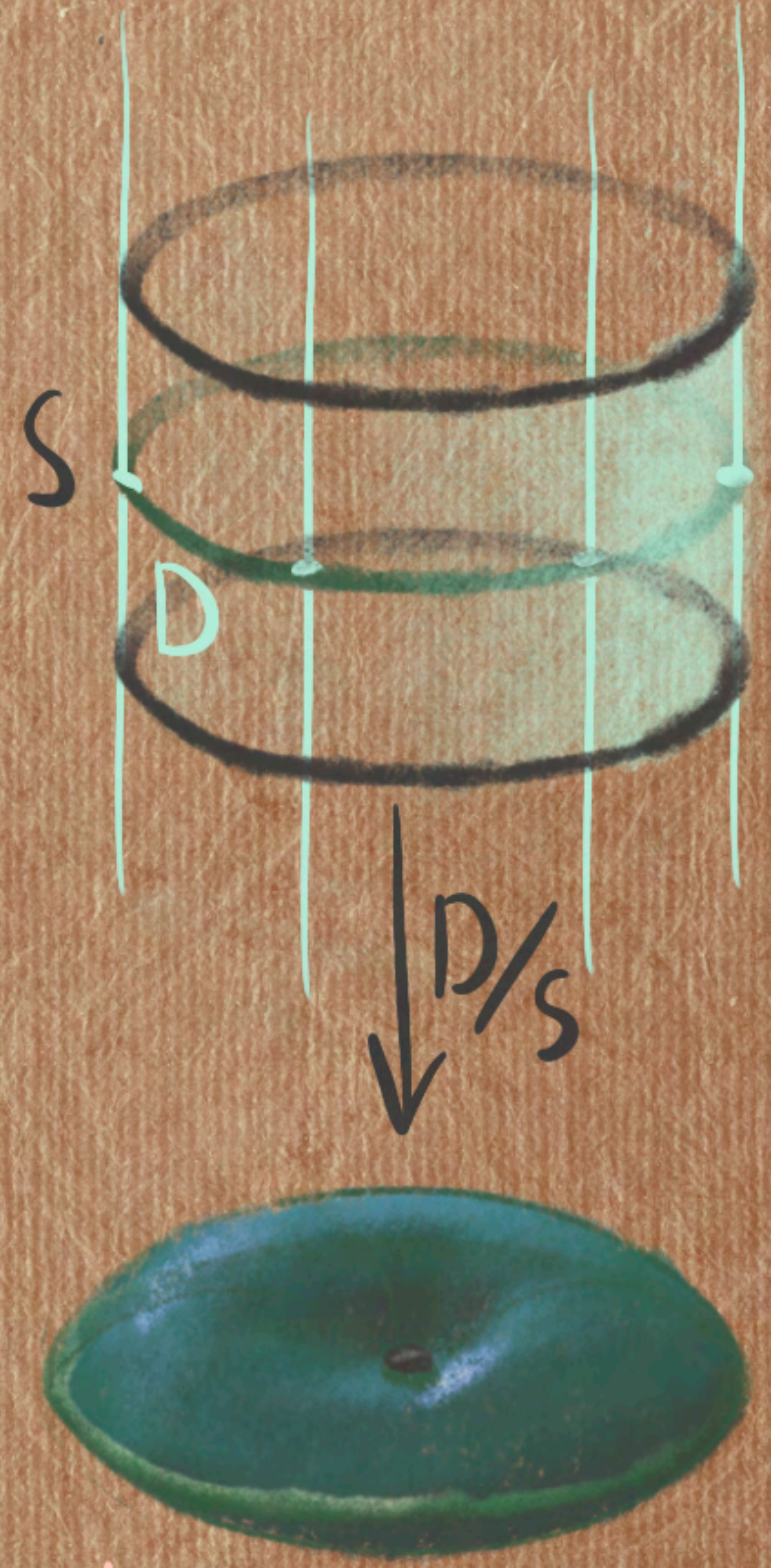
$\Rightarrow$ 1-pt compactification of ν

Thom space $Th(\nu) = \nu_+$

1-pt compactification is functorial:

$$\begin{tikzcd} \nu \arrow[r] \arrow[d] & \tilde{\varepsilon}^k \arrow[d] \\ Th(\nu) \arrow[r] & Th(\tilde{\varepsilon}^k) = MO(k) \end{tikzcd}$$

$$\nu \xhookrightarrow[\text{open}]{} \mathbb{R}^{n+k} \Rightarrow \underset{\substack{\shortparallel \\ S^{n+k}}}{\mathbb{R}^{n+k}_+} \longrightarrow \underset{\substack{\shortparallel \\ Th(\nu)}}{\nu_+}$$



Thom space

Pontryagin Thom construction

$$\infty \quad \mathbb{R}^{n+k^{+}} \cong S^{n+k} \xrightarrow[\text{collapse map}]{\substack{\text{P.T} \\ \text{pontryagin-Thom}}} Th(\nu) \xrightarrow[\text{map}]{\text{classifying}} MO(n)$$

Stable normal bundles

$M^k \hookrightarrow \mathbb{R}^{n+k}$ n shouldn't matter!! as long as it's big enough

trivial line bundle

$\mathbb{R}^{n+k} \hookrightarrow \mathbb{R}^{n+k+1}$ induces $\nu \longrightarrow \nu \oplus 1$

$n \gg 1 \Rightarrow$ space of embedings is connected $\Rightarrow$ all normal bundles iso. "stable normal bundle"

$(X \times I)_+ = X_+ \wedge I_+ = X_+ \wedge S^1 = \Sigma X \Rightarrow Th(\nu \oplus 1) = \Sigma\, th(\nu)$

$\nu \longrightarrow \nu \oplus 1 \Rightarrow BO(k) \rightarrow BO(k+1)$

Thom's Theorem:

$$\begin{tikzcd}
\mathbb{R}^{n+k+1}_{+} = S^{n+k+1} \arrow[r] \arrow[d, leftrightarrow, "\cong"] & Th(\nu \oplus 1) \arrow[r] \arrow[d, leftrightarrow, "\cong"] & MO(k+1) \arrow[d, leftrightarrow, "\nu \to \nu \oplus 1"] \\
\Sigma S^{n+k} \arrow[r] & \Sigma Th(\nu) \arrow[r] & \Sigma MO(k)
\end{tikzcd}$$

$\Sigma MO(k) \longrightarrow MO(k+1) \Rightarrow$ pre-spectrum!

$\pi_k(MO) = \varinjlim \pi_{n+k}(MO(n))$ well defined

$$\Omega_k \cong \pi_k(MO)$$

$$\begin{tikzcd}
\Omega_k \arrow[r, bend left, "\alpha"] \arrow[r, phantom, "\alpha\beta = \beta\alpha = id"] & \pi_k(MO) \arrow[l, bend left, "\beta"]
\end{tikzcd}$$

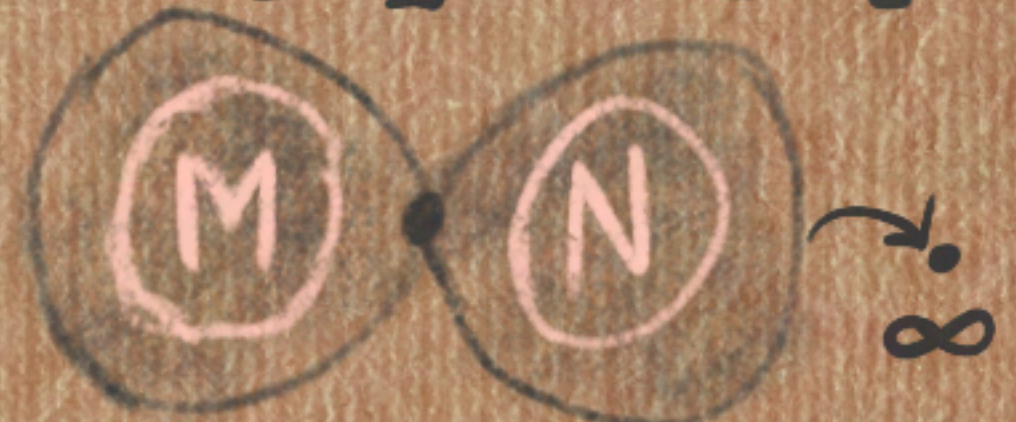
$\alpha: \Omega_k \longrightarrow \pi_k(MO)$

cobordism: $\partial W = M_0 \sqcup M_1$ $\quad E(\nu_{M_i}) = E(\nu_W)|_{\mathbb{R}^{n+k}\times i}$ $\quad i\in\{0,1\}$

P.T(M_i) is P.T$(W)|_{S^{n+k}\times i}$ $\Rightarrow$ W gives homotopy P.T$(M_0)\to$ P.T(M_2)

Group homomorphism:

$\alpha[M\sqcup N] = \alpha[M]+\alpha[N]$



$\beta : \pi_k(MO) \longrightarrow \Omega_k$

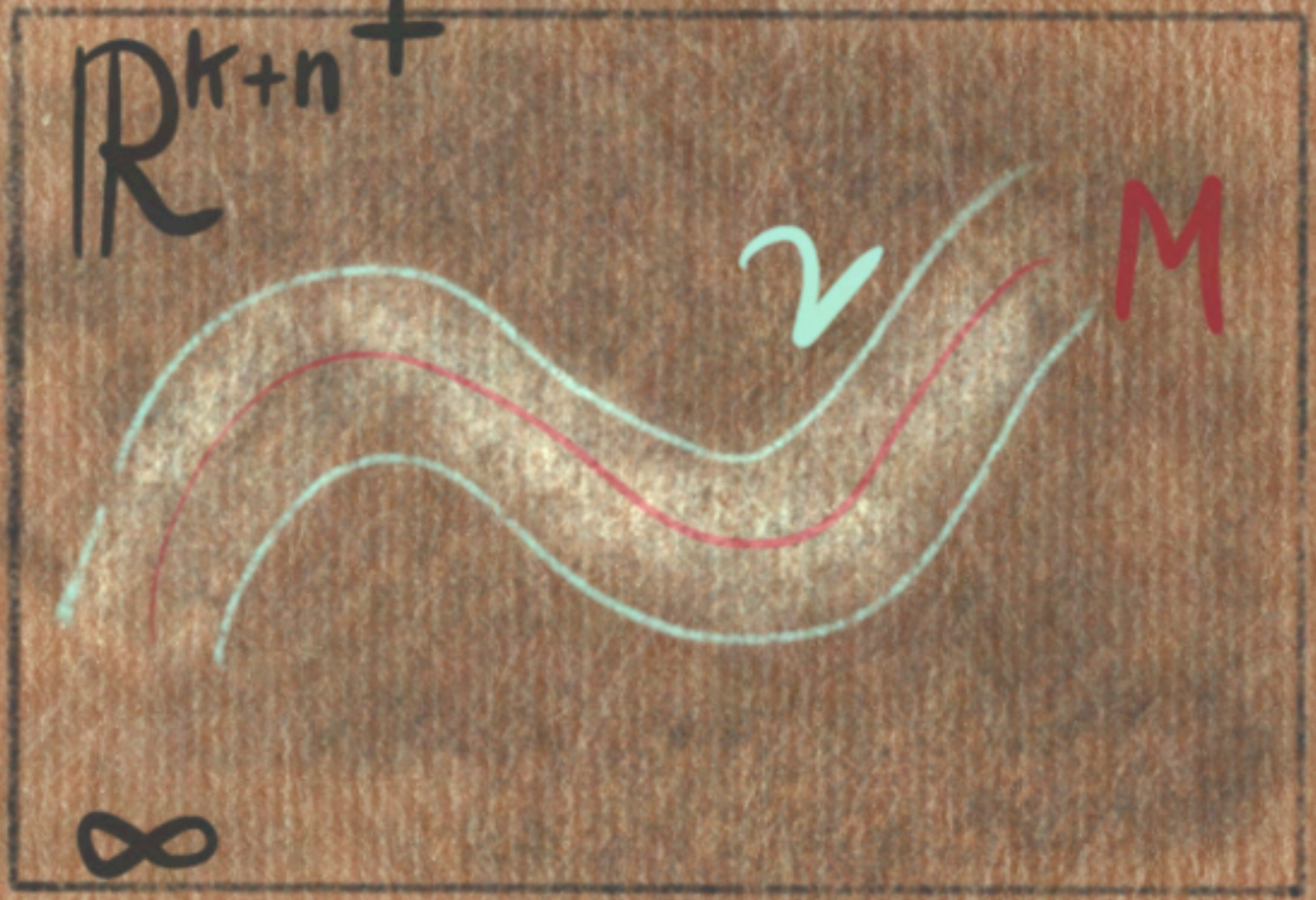
realize $f : S^{k+n} \to MO$ w/ $M \hookrightarrow \mathbb{R}^{n+k}$ & P.T map

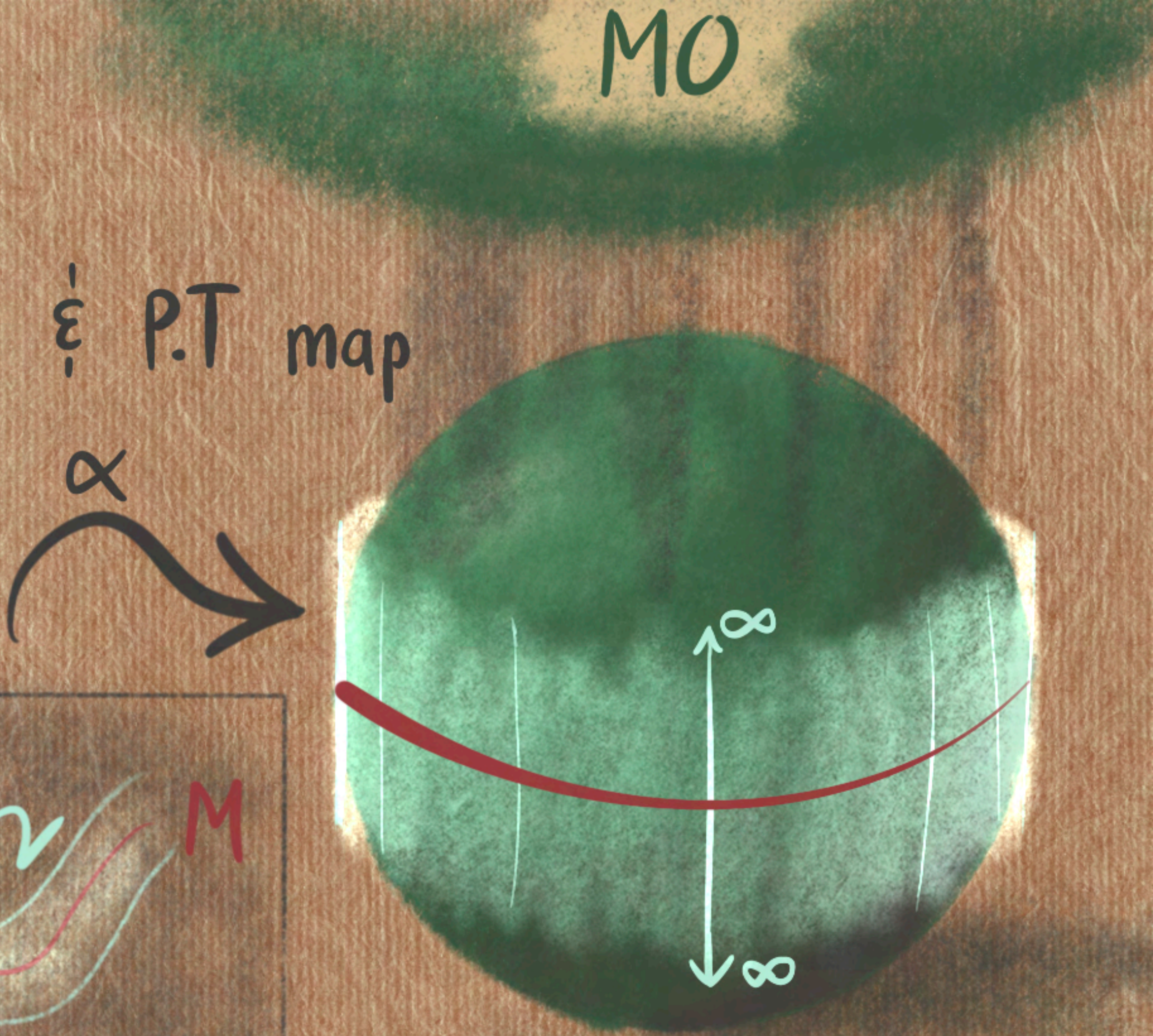
What does $\alpha[M] : S^{k+n} \to MO$ look like?

0-section only contains M

$\nu \to$ fibers over M

$\partial\nu \to \infty$ in MO





$\beta: \pi_k(MO) \longrightarrow \Omega_k$

When does $f: S^{n+k} \longrightarrow MO$ come from P.T?

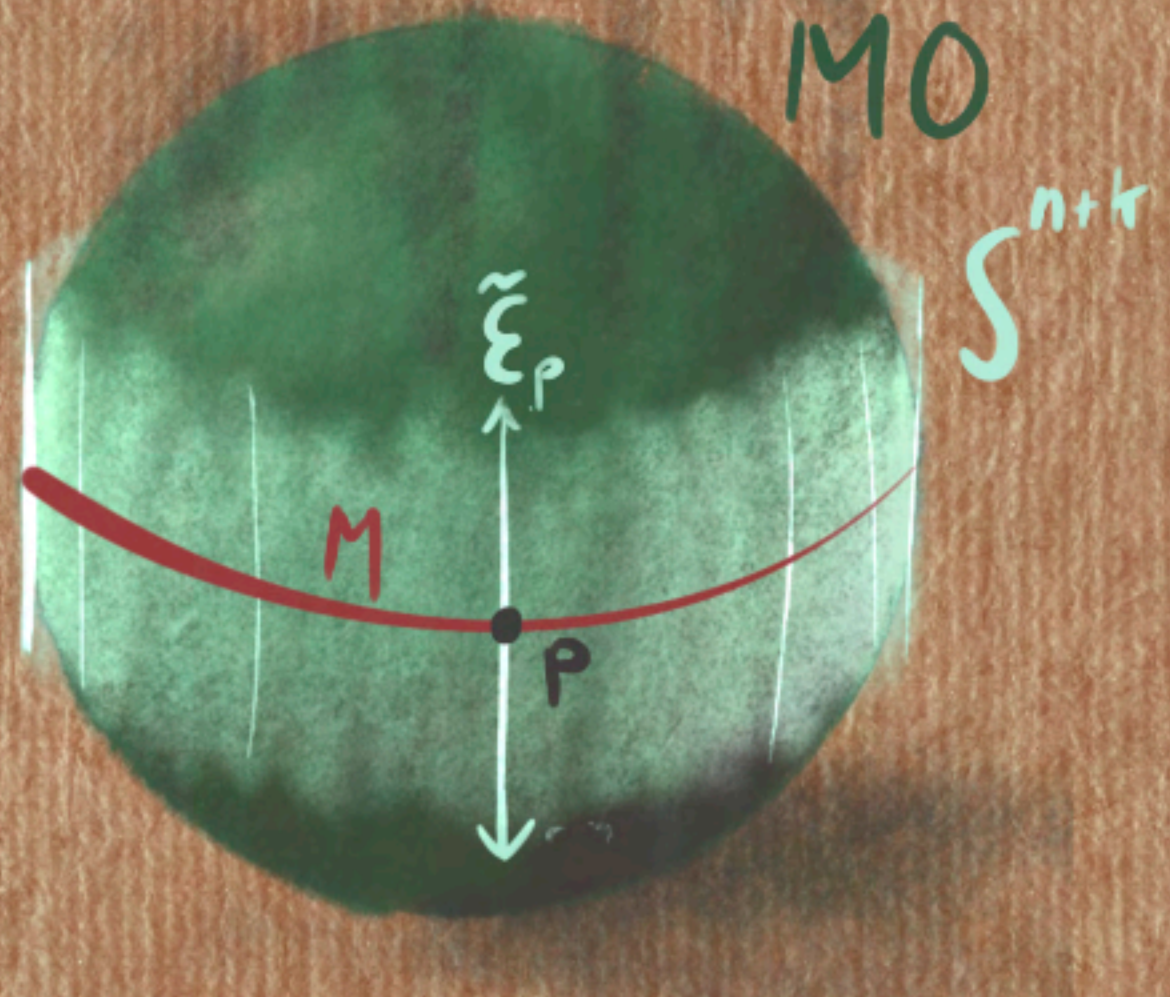
0-section of $\tilde{\xi}^k$

Take $S^{n+k} \supset M = f^{-1}(\theta)$ say S^{n+1} intersects θ transversly: $T_p\theta \oplus T_pS^{n+k} = T_pMO$

normal bundle

dimension count: $T_pS^{n+k} = T_pM \oplus \tilde{\xi}_p$

So, P.T∘classifying sends $p \in M$ to $p \in Gr$ & sends $T_pM^\perp$ to $\tilde{\xi}_p$. i.e, it sends M to f!



S^{n+k} compact $\Rightarrow$ lies in some $Th(Gr(K, \mathbb{R}^{n+k}))$

Thom transversality transversality is generic!

$\Rightarrow \exists\, \check{f} \in [f]$ w/ $\alpha[f^{-1}(\theta)] = [f]$

Q.E.D!

Thom spectrum of X

$$MO_k(X) = MO(k) \wedge X_+$$

$$\Omega_k(X) = \lim_{n\to\infty} \pi_{n+k}(MO_n(X))$$

$$= \pi_k(MO(X)$$

General cobordism Theory

theory	spectra
cobordism	MO
oriented cobordism	MSO
spin cobordism	$MSpin$
framed cobordism	MFr
cowbordism	MOO

Cobordisms w/ extra structure

Tangental structures: lifts $\begin{tikzcd} & B \arrow[d, "f"] \\ M \arrow[r] \arrow[ur, dotted] & BO(k) \end{tikzcd}$

e.g oriented cobordism: For $\partial W = M \sqcup N$, demand W have orientation s.t $\partial W = M - N$

Framed cobordism: n pointwise-L.I sections of ν (i.e - trivialization)

$\Rightarrow$ tubular nbhd $E(\nu) = M \times D^n$; $Th(\nu) = M \times S^n$

pontryagin-Thom map: $\mathbb{R}^{n+k+} \longrightarrow M \times S^n \Rightarrow S^{n+k} \rightarrow S^n$

defines element of π_n^s!

$\Omega_k^{Fr} \cong \pi_k^s$!