A & B Models: The story of mirror symmetry

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A Side

Physics motivation

B Side

Curve Counting / Gromov-Witten invariants

Quantum Cohomology

Frobenius manifolds

Landau-Ginzburg model

Singularity Theory

variation of semi-∞ Hodge Structures

Mirror Symmetry!

Physics Background

Def: Physics is the subfield of math studying integrals of the form $\int e^{-S(\phi)} D\phi$

- QFT defined by choice of $S(\phi)$ ("action")

- Operators: $\langle \mathcal{O} \rangle = \int \mathcal{O}(\phi)\, e^{-\frac{1}{\hbar} S(\phi)}\, D\phi$

Locality: consider $\phi \in \{ \text{Maps } \Sigma \to X \}$

Operators spanned by $\mathcal{O}_i(x)$, fns of $x \in \Sigma$

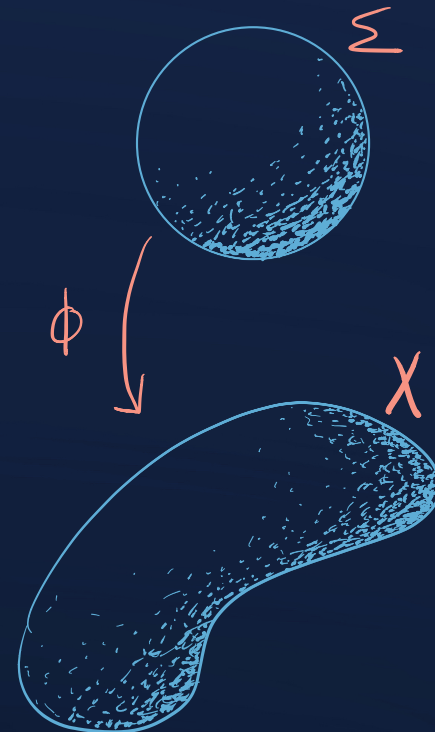
$$\mathcal{O}_i(x)\mathcal{O}_j(y) = \sum f_{ij}{}^{k}(x,y,z)\, \mathcal{O}_k(z)$$

Encodes entire QFT!

Nonlinear σ-Model:

- 2D theory on Riemann surface Σ:

$$Z = \int_{\phi:\Sigma\to X} e^{-\frac{1}{\hbar}\underbrace{\int_\Sigma \|d\phi\|^2}_{S(\phi)}}$$



- Introduce supersymmetry:

super manifold

$$\Sigma\underbrace{(x_1, x_2)}_{\text{coordinates}} \Rightarrow \tilde{\Sigma}\underbrace{(x_1, x_2, \theta^+, \theta^-, \bar{\theta}^+, \bar{\theta}^-)}$$

x_i: commuting (bosons)

$\theta^\pm, \bar{\theta}^\pm$ anticommuting (fermions)

"$N=2$ extended supersymmetry"

Topological Theories

- Twist the Theory:
  1) promote fermions to bosons
     - 2 ways to do this:
       "A-twist" and "B-twist"

     * see appendix A @ end of slides for more info
  2) Integrate them out
- Now action is topological!
  - Doesn't depend on metric of Σ or X
- Topological theories $\Rightarrow$ topological operators
  - $\mathcal{O}_i$ is position independent!

$$\mathcal{O}_i \mathcal{O}_j = \sum C_{ij}{}^{k} \mathcal{O}_k$$

A-Twisted σ-model:

X is Kähler, w/ Kähler form ω (closed 2-form)

$$\tilde{Z} = \int_{\substack{\phi : \Sigma \to X \\ \text{holomorphic}}} e^{-\frac{1}{\hbar}\int_\Sigma \phi^* \omega}$$

$$\int_\Sigma \phi^* \omega = \underbrace{[\omega]}_{\in H_2(X)} \cdot \phi_* \underbrace{[\Sigma]}_{\in H^2(X)}$$

"Chiral Ring"

Landau-Ginzburg theories

- 2D, w/ fields in $\mathbb{C}^N$ & Potential $W:\mathbb{C}^N \to \mathbb{C}$

$$Z = \int_{\phi:\Sigma\to\mathbb{C}^N} e^{-\frac{1}{\hbar}\left(\underbrace{\int_\Sigma \|d\phi\|^2}_{\text{Kenetic}} + \underbrace{\|\nabla W(\phi)\|^2}_{\text{Potential}}\right)}$$

- B-twist, & integrate out extra variables
  - Get another topological theory!
- Each L-G theory has dual σ-model
  - same physics – same chiral rings

"Mirror Symmetry"

Q: How do we formalize this?

Modulii of CFTs:

- These are famlies of CFTs: conformally invariant
  - σ-model: any Kähler structure gives CFT
- any operator A deforms a theory:

$$S(\phi) \longmapsto S(\phi) + \int_\Sigma \delta \cdot A \qquad \delta \ll 1$$

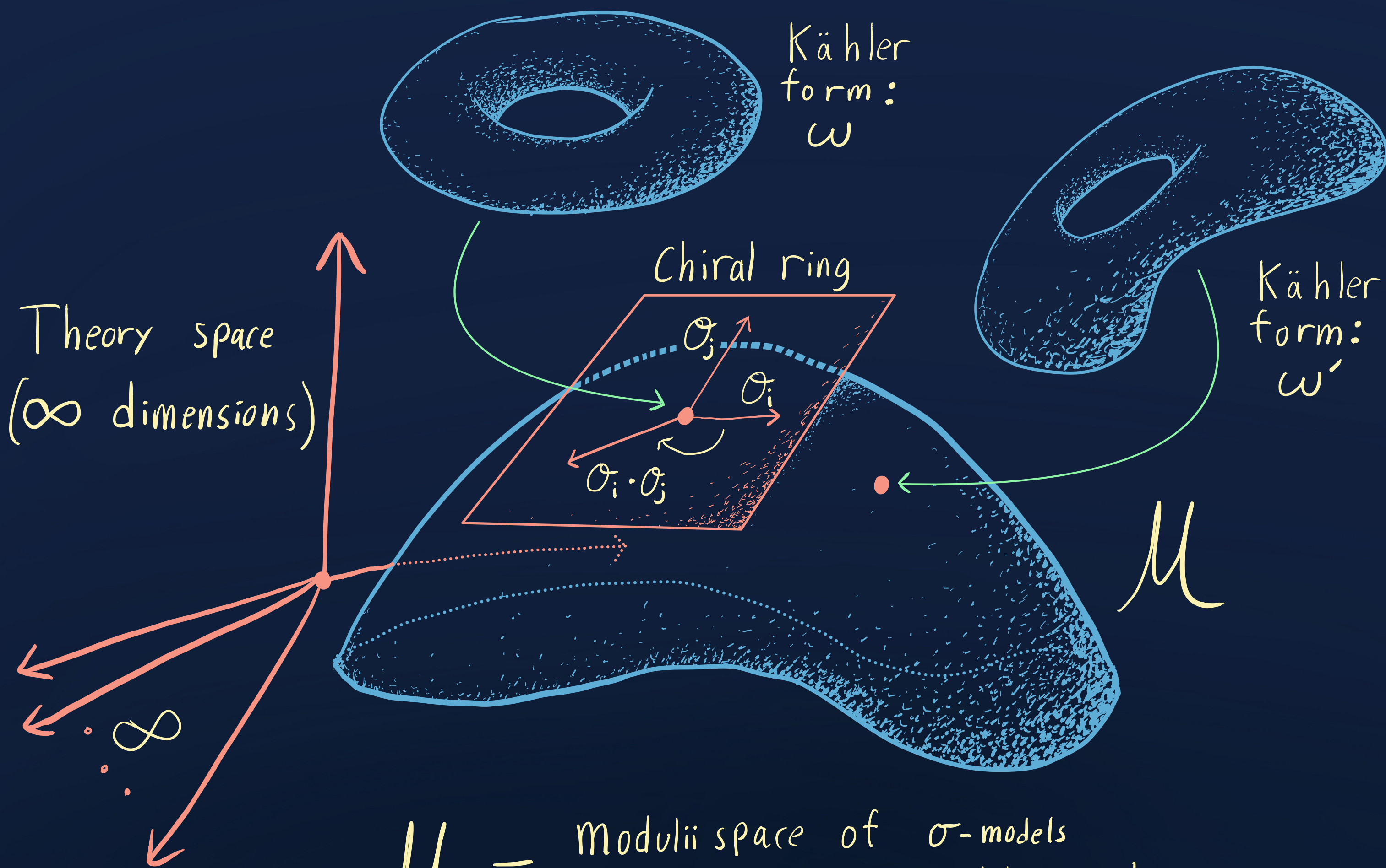
- Exactly marginal operators preserve conformal invariance
  - "Tangent vectors" to CFT family

(Maybe) Remarkable fact of 2D $N=(2,2)$ theories:

- Chiral ring $\Rightarrow$ exactly marginal
- exactly marginal $\Rightarrow$ Chiral ring (of A or B twist)

Tangent space of CFT family encodes the physics!

σ-Model modulii space:



$\mathcal{M} =$ modulii space of σ-models
modulii space of Kähler structures

Goal: state mirror symmetry

Encode the "physics" w/ a mathetical structure

Mirror Symmetry: isomorphism of σ & LG models

Physics guidance:

Look for ring structure on some tangent space

A- Side

Physics $\Rightarrow$ Curve Counting

- Want to make sense of expectation values

$$\langle \mathcal{O}_1(s_1)\dots\mathcal{O}_n(s_n)\rangle = \int_{\phi:\Sigma\to X \text{ holomorphic}} \mathcal{O}_1(\phi)\dots\mathcal{O}_n(\phi)\, e^{-\frac{1}{\hbar}\int_\Sigma \phi^*\omega}$$

$s_1,\dots,s_n\in\Sigma$

holomorphic degree

antiholomorphic degree

- operators in σ-model represented by closed p,q forms on X
  - not dependent on position in Σ (as we expected!)

$\mathcal{O}_i \leftrightarrow O_i$

- Treat $s_1,\dots,s_n$ as marked points, property of map ϕ

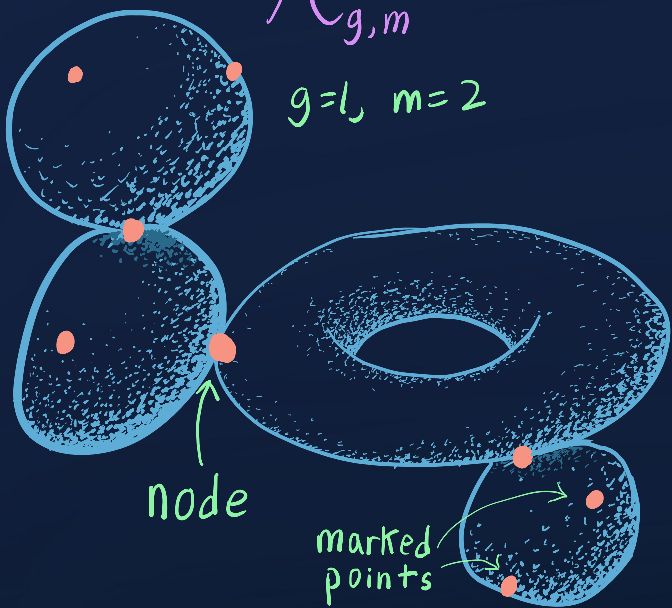
$$\int_{\phi:\Sigma\to X}\mathcal{O}_1(\phi) \longrightarrow \int_{\phi:(\Sigma,\{s_i\})\to X} ev^*(O_i)$$

- Split by homology class of the curve $\phi_*[\Sigma]\in H^2(X,\mathbb{Z})$

$$\langle \mathcal{O}_1\dots\mathcal{O}_n\rangle = \sum_{\beta\in H^2(X,\mathbb{Z})} e^{-\frac{1}{\hbar}\int_\beta\omega}\int_{\substack{\phi:\Sigma\to X \text{ holomorphic}\\ \phi_*[\Sigma]=\beta}} ev_1^*(O_1)\wedge\dots\wedge ev_n^*(O_n)$$

Moduli Space of Stable Maps

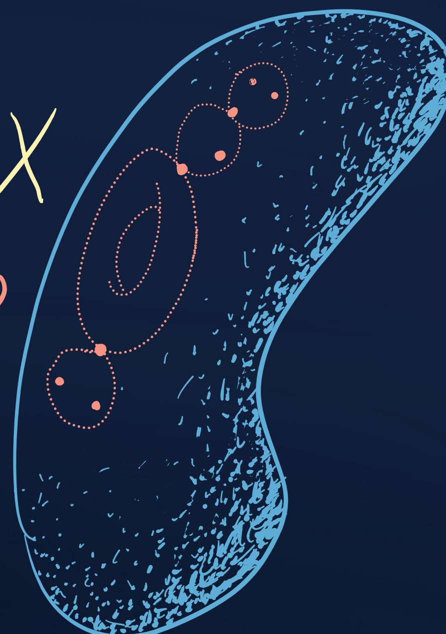
$\overline{\mathcal{M}}_{g,m}$



$\overline{\mathcal{M}_{g,m}(X,\beta)}$

$$\Sigma \xrightarrow{\overset{\text{holomorphic}}{\phi}} X$$

$$[\Sigma] \stackrel{\phi_*}{\longmapsto} \beta$$



nodal stable marked curves

"Stable": finite automorphisms

Gromov-Witten invariants

- evaluation map $ev_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \longrightarrow X$
  - sends the i^{th} marked point to it's image

for $\{\alpha_1, \dots, \alpha_n\} \in H^\bullet(X)$, $\beta \in H_2(X, \mathbb{Z})$,

$$\boxed{\langle \alpha_1, \dots, \alpha_n \rangle_g^\beta := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} ev_1^*(\alpha_1) \smile \cdots \smile ev_n^*(\alpha_n)}$$

Gromov-Witten invariant

If $\overline{\mathcal{M}}_{g,n}(X,\beta)$ not smooth, choose your own homology class to integrate against (the hard part)

Gromov-Witten Potential

- Fix genus $=0$ (don't know why), construct generating fn:

$$\phi(\underset{H^\bullet(X)}{\underset{\cap}{\gamma}}) = \sum_m \sum_{\beta \in H_2} \frac{e^{-\frac{1}{\hbar}\int_\beta \omega}}{m!} \langle \overbrace{\gamma, \dots, \gamma}^{m} \rangle_{g=0}^{\beta}$$

physically,

$$\Phi(\gamma) = \sum_m \frac{1}{m!} \langle \overbrace{\gamma \cdots \gamma}^{m} \rangle$$

is the Free Energy

- Does this converge? Don't care.
  - make into formal series: $e^{-\frac{1}{\hbar}\int_\beta \omega} \Rightarrow q^\beta$

formal variables satisfying

$$q^{\beta_1} \cdot q^{\beta_2} = q^{\beta_1 + \beta_2}$$

$\beta_1, \beta_2 \in H^2(X, \mathbb{Z})$

- still don't know if q^β coefficients converge:

$T_0, \dots, T_n$ basis of $H^\bullet(X)$ $\quad$ $\{y_0, \dots, y_n\}$ formal coords on $H^\bullet(X)$

Gromov-Witten potential: $\quad \Phi := \phi(\sum y_i T_i) \in \mathbb{C}[[q^\beta]][[y_0, \dots, y_n]]$

- This expansion contains every $g=0$ GW invariant!

Quantum Cohomology

- metric on $H^\bullet(X)$: $g(T_i, T_j) = \int_X T_i \cup T_j$.
- Quantum Cup Product $*_\gamma$:

$$g(T_i *_\gamma T_j, T_k) = \sum \partial_i \partial_j \partial_l \phi(\gamma) \int_X T_l \cup T_k$$

$$= \sum_m \sum_\beta \frac{q^\beta}{m!} \langle T_i, T_j, T_k, \underbrace{\gamma, \dots, \gamma}_{m} \rangle^\beta_{g=0}$$

$$\boxed{T_i *_\gamma T_j := \sum \partial_i \partial_j \partial_l \phi(\gamma) g^{lk} T_k}$$

For later: $g(T_i *_\gamma T_j, T_k) = g(T_i, T_j *_\gamma T_k)$

Technical note: $T_i \cup T_j = (-1)^{\deg T_i + \deg T_j} T_j \cup T_i$ Supercommunative

From now on, only consider even cohomology

Quantum Cohomology as Deformation

- recall $q^\beta = e^{-\frac{1}{\hbar}\int_\beta \omega}$
- ω is Kähler, so is positive: $\int_{\beta\neq 0} \omega > 0$, $\int_{\beta = 0} \omega = 0$
- "Semiclassical limit": $\hbar \to 0$ "large volume limit"
  - formally, take $q^\beta \to 0$ for $\int_\beta \omega > 0$

$$\langle \alpha_1, \ldots, \alpha_n \rangle_0^{\beta=0} = \begin{cases} \int_X \alpha_1 \smile \cdots \smile \alpha_n & \text{if } n = \dim_{\mathbb{C}} X \\ 0 & \text{otherwise} \end{cases}$$

$$g(T_i *_\gamma T_j, T_k) = \sum_n \frac{1}{n!} \langle T_i, T_j, T_k, \gamma, \ldots, \gamma \rangle_{g=0}^{\beta=0}$$

$$= \langle T_i, T_j, T_k \rangle_{0,3}^0 = \int_X T_i \smile T_j \smile T_k$$

$$\Rightarrow T_i * T_j \to T_i \smile T_j \text{ as } \hbar \to 0 \; !$$

Associativity of Quantum Product:

$$(T_i * T_j) * T_k = T_i * (T_j * T_k)$$

$$\Big\Downarrow \text{ Plug in } \Phi$$

$$\sum_{a,b} (\partial_i \partial_j \partial_a \Phi)\, g^{ab} (\partial_k \partial_l \partial_b \Phi)$$

$$\|$$

$$\pm \sum_{a,b} (\partial_i \partial_k \partial_a \Phi)\, g^{ab} (\partial_l \partial_j \partial_b \Phi)$$

"WDVV Equation"

(Witten-Dijkgraaf-Verlinde-Verlinde)

Recursive formula for GW invariants on $\mathbb{CP}^2$

$$N_d = \langle [\mathbb{CP}^1] \rangle_{g=0,\, n=3d-1}^{\beta = d\cdot[\ell]}$$

counts # degree d rational curves thru 3d-1 generic points

WDVV gives:

Theorem (Konstevich 1994):

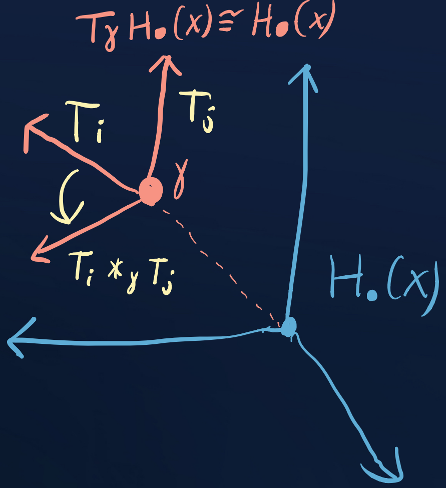
$$\boxed{N_d = \sum_{\substack{d_A + d_B = d \\ d_A, d_B \geq 1}} N_{d_A} N_{d_B} \left(d_A^2 d_B^2 \binom{3d-4}{3d_A-2} - d_A^3 d_B \binom{3d-4}{3d_A-1} \right)}$$

Algebra of Quantum Cohomology

- under $*_\gamma$, $H^\bullet(X)$ is a unitial, associative algebra w/ metric g satisfying $g(A *_\gamma B, C) = g(A, B *_\gamma C)$
- These are called Frobenius Algebras!
- Paramertized by $\gamma \in H_\bullet(X)$

recall goal: want product on the tangent space

Luckily, $T_\gamma H_\bullet(X) \cong H_\bullet(X)$!



- Treat quantum cup product as frobenius algebra on $TH_\bullet(X)$

Frobenius Manifolds

- Manifolds w/ locally constant frobenius algebra structure on TM (exactly what we were looking for!)

Def: A Frobenius manifold is a mfld M w/
- a flat metric g (w/ levi-civita connection ∇)
- a associative Algebra product $*_x$ on T_xM, w/ unit e_x

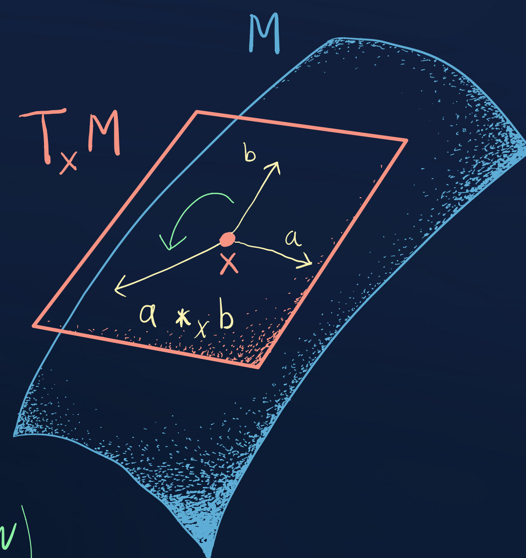
satisfying compatability conditions:
- $g(a *_x b, c) = g(a, b *_x c)$
- $\nabla e = 0$

($\Rightarrow \mathcal{A}(a,b,c) := g(a*b,c)$ is symmetric tensor)

w/ (local) potential fn $\Phi: M \to \mathbb{C}$ s.t

$$\mathcal{A}(a,b,c) = \partial_a \partial_b \partial_c \Phi$$

(Think GW potential)



Connection on Quantum Cohomology

- Take formal coordinates $\{y_1, \dots, y_r\}$ of $H^2(X)$
  Associated to basis $\{T_1, \dots, T_r\}$ of $H^2(X)$
- $\nabla^{\hbar}$ is connection on $TH^{2\bullet}(X)$:

$$\nabla^{\hbar} = d + \frac{1}{\hbar} \sum_i^r \frac{dy_i}{y_i} \left(p^i \circ - \right)$$

$$\nabla^{\hbar}_X Y = \nabla_X Y + \frac{1}{\hbar} X \circ Y$$

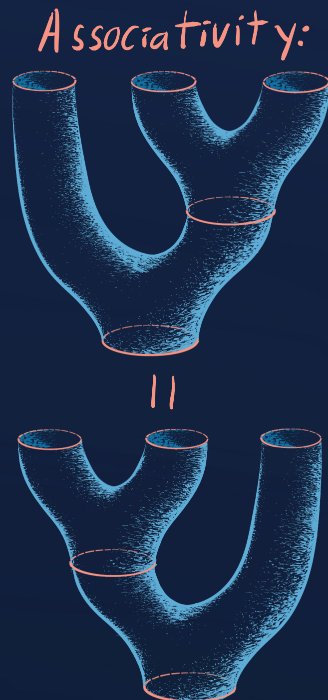
trivial flat connection on $TH^{2\bullet}(X)$

"Dubrovin connection"

Frobenius Manifolds as 2D TQFTs

Axiomatic approach to 2D TQFTs:

- Hilbert space = 1-D manifold
- Operators = 2-D connecting mfld
  - "Cobordism"
- Operator algebra = Frobenius algebra



- Frobenius algebra is <u>essence</u> of 2D TQFT
- Frobenius manifold is <u>essence</u> of 2D TQFT family
  - Each pt on mfld is a TQFT
  - Transfer between TQFTs w/ Levi-Civita connection

Modulii space of Kähler mflds

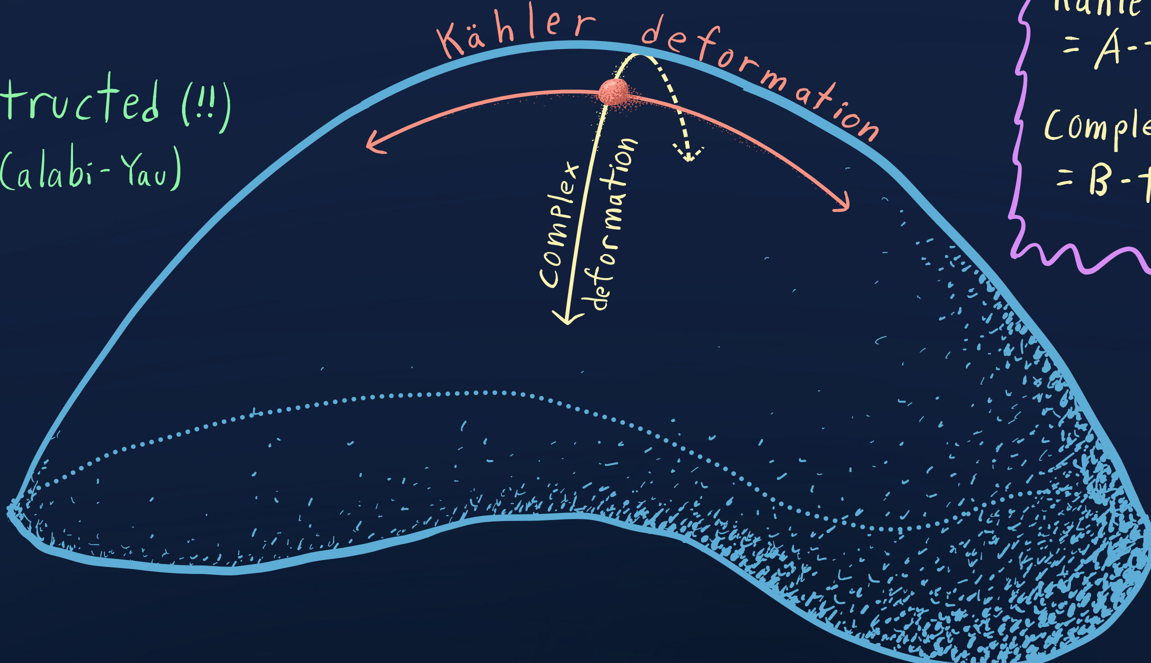
- σ-model described by Kähler target space (X,ω)
  - Kähler form $\omega \in \Omega^{1,1}(X,\mathbb{R})$ & complex structure $J \in TX \otimes T^*X$
- deformations split according to these:

Kahler deformations
= A-twisted chiral ring

Complex deformations
= B-twisted chiral ring

unobstructed (!!)
(for X Calabi-Yau)

$\mathcal{M}$



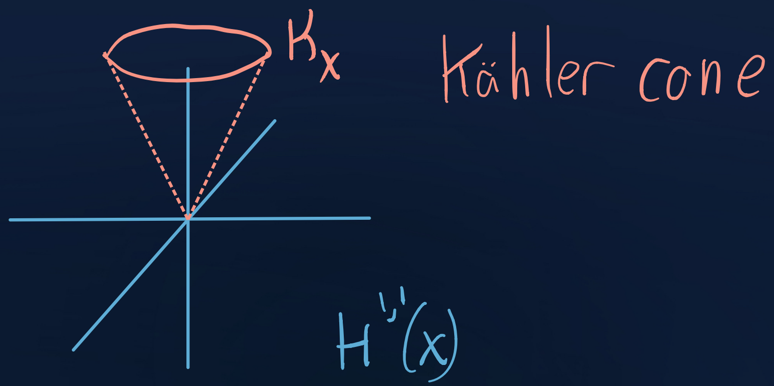
$$T_p\mathcal{M} \cong T_p\mathcal{M}_{\text{kähler}} \oplus T_p\mathcal{M}_{\text{complex}}$$

Kähler deformations

- Kähler form $\omega \in H^{1,1}$, positive $\left(\int_\beta \omega \geq 0 \quad \forall \beta \in H_2(X,\mathbb{Z})\right)$
  - space of Kähler forms is a cone in $H^{1,1}$: K_X:
- Physical theories are more general: need complexified
  - allow "B-fields": $\{B + i\omega \mid B \in H^2(X,\mathbb{R})\}$

$$\mathcal{M}_{\text{Kahler}} = \{H^2(X,\mathbb{R}) + iK_X\}/H^2(X,\mathbb{Z})$$

$$T\mathcal{M}_{\text{Kahler}} \cong H^2(X,\mathbb{R}) \quad \text{what we want}$$



A-Side Story:

σ-model partition function

$\Downarrow$

Gromov-Witten invariants

$\Downarrow$

Quantum cohomology

deformation of regular cohomology!

$\Downarrow$

Frobenius manifold

$\parallel$

modulii space of Kähler deformations

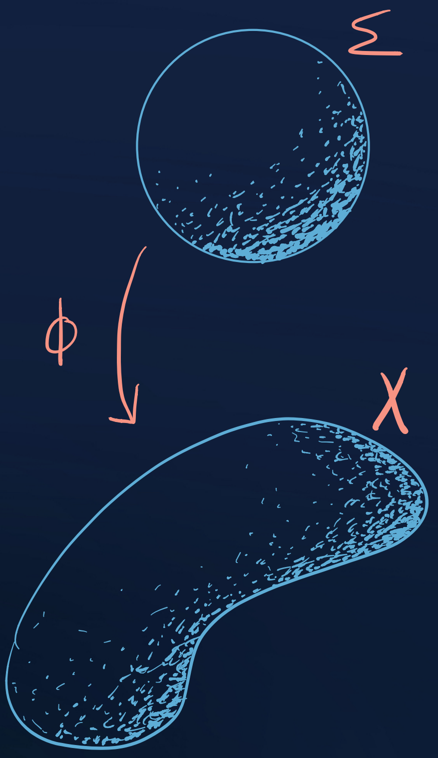
B-Side

Landau-Ginzburg Models:

$$Z = \int_{\phi:\Sigma \to X} e^{-\frac{1}{\hbar}\left(\underbrace{\int_\Sigma \|d\phi\|^2}_{\substack{\text{Kenetic} \\ (\sigma\text{-model})}} + \underbrace{\|\nabla W(\phi)\|^2}_{\text{Potential}}\right)}$$

1) Introduce supersymmetry
2) B-twist (fermions $\Longmapsto$ bosons)
3) Integrate out twisted fields

Get topological theory!



Renormalization:

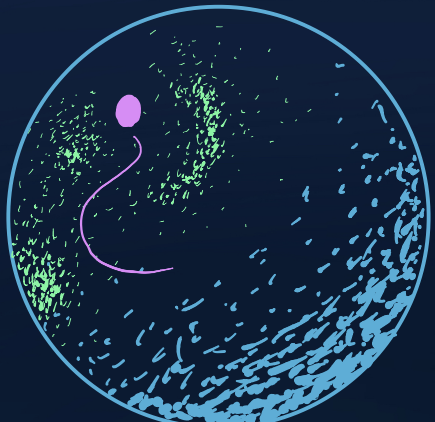
- Physical world is low energy: observed theorys are effective
- Renormalization: Change of effective theory based on energy scale
  - math: scale metric $g \mapsto \frac{1}{\lambda} g$: λ is length scale, $\lambda \sim 1/E$

$$S(\phi) = \int_\Sigma \|d\phi\|^2 + \|\nabla W(\phi)\|^2 \qquad \|d\phi\|^2 = g^{ij}\partial_i\phi\,\partial_j\phi$$

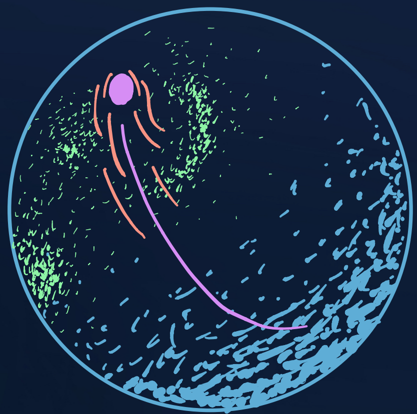
- $\lambda \ll 1 \Rightarrow \|d\phi\|^2$ dominates $\Rightarrow S(\phi)$ becomes σ- model "zoom out"
- $\lambda \gg 1 \Rightarrow \|d\phi\|^2$ drops out $\Rightarrow S(\phi)$ becomes LG model "zoom in"

Analogy: Classical Particle on manifold $\quad H = g^{ij} p_i p_j + V(x)$

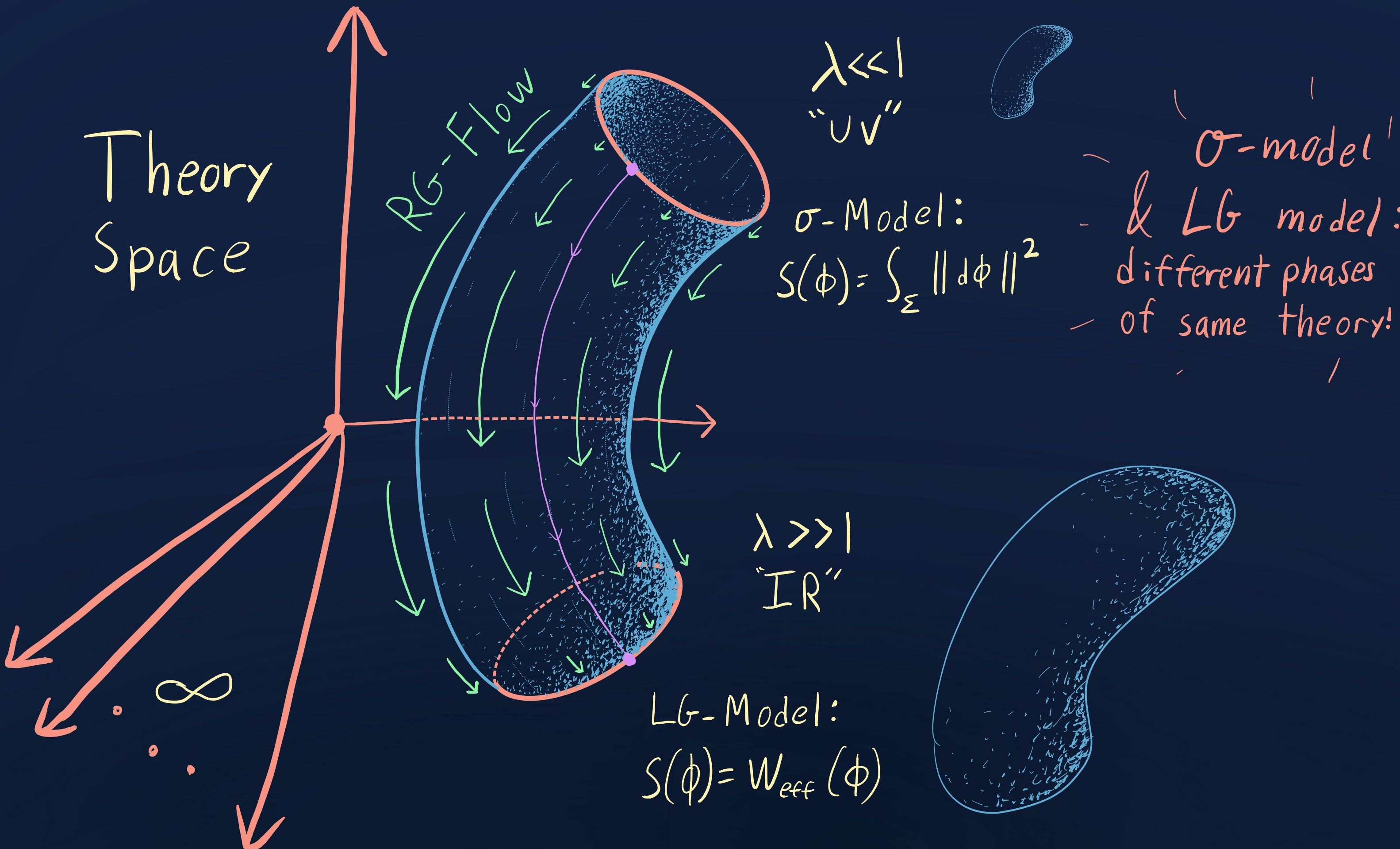
- low energy
- low velocity
- depends on potential



- high energy
- high velocity
- depends on metric



Renormalization Group Flow:



Renormalization of potential

- IR limit: very zoomed in $\Rightarrow$ looks flat
  > can treat as fields in $\mathbb{C}^n$: $(\phi_1, \dots, \phi_n)$
- Write potential w/ taylor expansion
- renormalize: fields scale like $\phi_i \mapsto \lambda^{-l_i}\phi_i$ scaling dimension
  - $\phi_1^{\alpha_1} \cdot \dots \cdot \phi_n^{\alpha_n} \mapsto \lambda^{-\sum l_i \alpha_i} \phi_1^{\alpha_1} \cdot \dots \cdot \phi_n^{\alpha_n}$
- only highest order term surives
  - quasihomogenous: $W_{eff}(\lambda^{l_1}\phi_1, \dots, \lambda^{l_n}\phi_n) = \lambda^{l} W_{eff}(\phi_1, \dots, \phi_n)$
- renormalized action: $S(\phi) \xmapsto{IR} W_{eff}(\phi_1, \dots, \phi_n)$

Goal: Classify distinct LG theorys $\Longleftrightarrow$ Classify quasihomogenous polynominals up to coordinate transform

BRST Picture of Physical Operators

- supersymmetry generated by operator Q "BRST charge"
  - 'physical operators' obey supersymmetry: $[Q, P] = 0$ graded commutator
    - $[Q, \bullet]^2 = 0$, thus forms a complex
  - Q-exact operators $[Q, \mathcal{O}]$ are trivial in path integral $\Rightarrow$ physically trivial
  - 'physical operators' are Q-closed, defined up to Q-exact: Q-cohomology
- Physical operators often cohomology groups:

σ-model: $Q = \bar{\partial}$, physical operators = complex cohomology classes $H^{p,q}$

- LG models: effective potential W
  - Q-closed: holomorphic functions of fields $\mathbb{C}[\phi_1, \dots, \phi_n]$
  - Q-exact: functions of form $v^i \cdot \nabla_i W$ $\left(\frac{\partial W}{\partial \phi_1}, \dots, \frac{\partial W}{\partial \phi_n}\right) = I_{\nabla W}$

$$\text{Chiral ring} = \text{Physical operators} = \frac{Q \text{ closed}}{Q \text{ exact}} = \boxed{\mathbb{C}[\phi_1, \dots, \phi_n] / I_{\nabla W}}$$

Q: how to interpret as a Frobenius Manifold?

(local) Singularity Theory

Goal: Classify critical points up to coordinate transform

- Consider $W : \mathbb{C}^n \to \mathbb{C}$, $W(0) = \nabla W(0) = 0$
- Treat this locally: germ $\mathcal{O}_W$ $\left(\mathcal{O}_W = \mathcal{O}_{W'} \text{ if } \exists\, \overset{0}{\underset{\in}{\ }} U \subseteq \mathbb{C}^n \text{ s.t } W|_U = W'|_U\right)$
  - space of germs of fns $\mathbb{C}^n \to \mathbb{C}$: $\mathcal{O}_n$
- Coordinate change: germ of biholomorophisms $g : \mathbb{C}^n \to \mathbb{C}^n$

  $W \mapsto W \circ g$ want $\mathcal{O}_n / \mathcal{O}_W \sim \mathcal{O}_{W \circ g}$
- infetesimal picture: $g_t : \mathbb{C}^n \to \mathbb{C}^n$

  $$W \sim W \circ g_t \Rightarrow W \sim W + \frac{d}{dt} W \circ g_t = W + \overbrace{\nabla W \cdot \frac{d}{dt} g_t}^{\text{Jacobian ideal } I_{\nabla W}\,!}$$

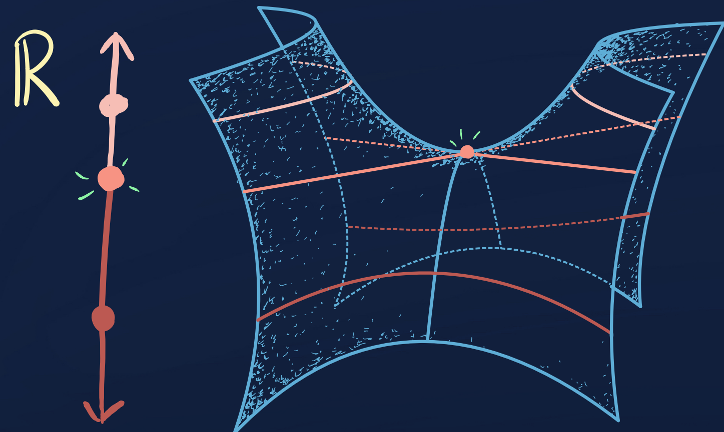
Local Ring $Q_W = \mathcal{O}_n / I_{\nabla W}$

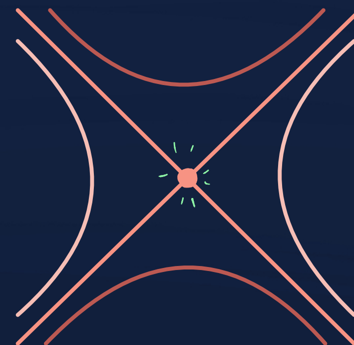
Diffeomorphism invariant of critical point!

need a more Sophisticated invariant

Picard-Lefschetz Theory:

Morse Theory: level set topology change $\Longleftrightarrow$ critical point structure





Topology change

$\Rightarrow$ crit point index 1

Complex morse theory?

homotopy classes of maps

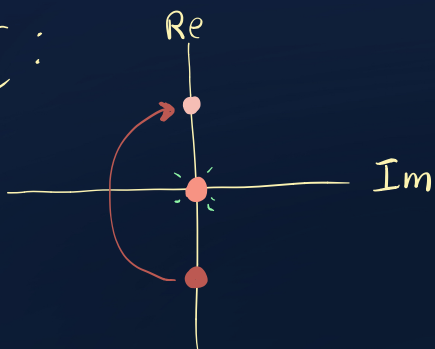
Real isolated crit points $\Longleftrightarrow [\mathbb{R}-\{0\}, \mathrm{Mfld}^n_{\mathbb{R}}]$

$\mathbb{R}-\{0\} \simeq \{-1,1\}$: action of $\pi_0(\mathrm{Mfld}^n_{\mathbb{R}})$

Complex isolated crit points $\Longleftrightarrow [\mathbb{C}-\{0\}, \mathrm{Mfld}^n_{\mathbb{C}}]$

$\mathbb{C}-\{0\} \simeq S^1$: action of $\pi_1(\mathrm{Mfld}^n_{\mathbb{C}})$

$\mathbb{C}$:



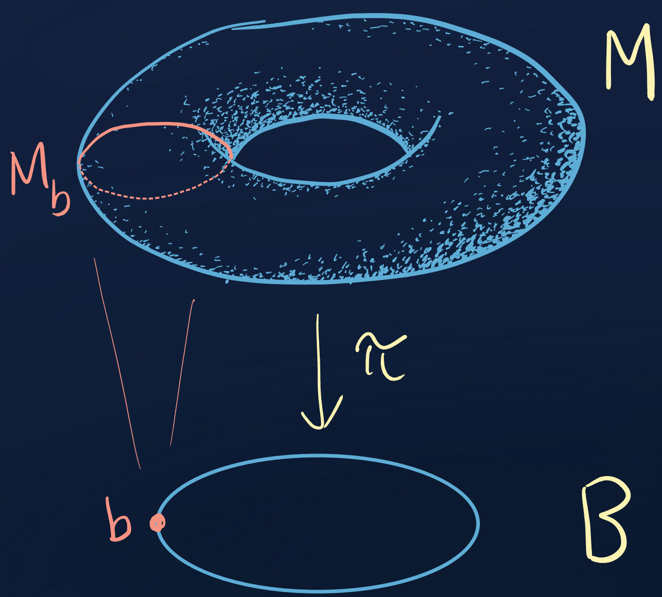
e.g: monodromy on (co)homology

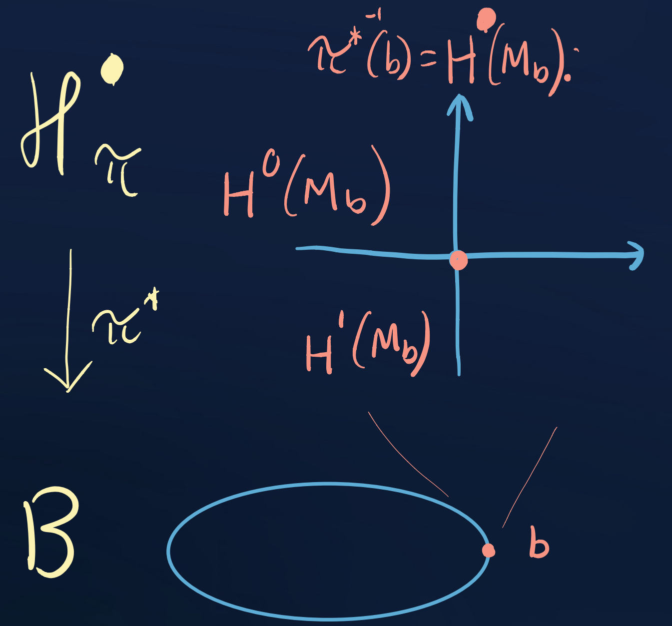
Cohomology Bundle

- Paramertized family of manifolds M_b, $b \in B$
  - i.e fibration $\pi : M \to B$ w/ fibers $M_b = \pi^{-1}(b)$
- M_b has cohomology $H^k(M_b)$

Cohomology bundle: $\mathcal{H}^k_\pi = \bigsqcup_b H^k(M_b)$, $\pi^* : \mathcal{H}^k_\pi \to B$

B connected $\Rightarrow$ M_b diffeomorphic

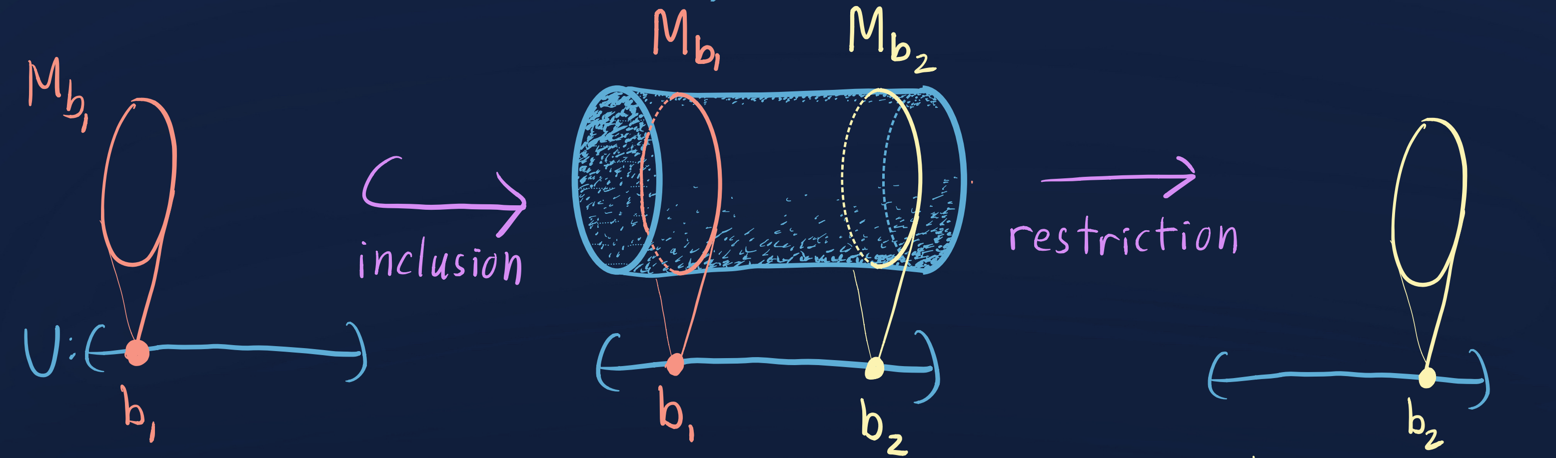




Gauss-Manin Connection

Trivialization:
$U \subset B$ contractible

$$\pi^{-1}(U) \cong U \times M_b$$



$\pi^{*^{-1}}(b_1) = H^k(M_{b_1})$

$$\pi^{*^{-1}}(U) = H^k(U \times M_b) \times U \cong H^k(M_b) \times U$$

$\pi^{*^{-1}}(b_2) = H^k(M_{b_2})$

- (locally) const. transition fns : sheaf $\Gamma(\mathcal{H}^k \pi)$ is a local system

$\Rightarrow \exists$ flat connection ∇^{GM} on $\mathcal{H}^k_\pi$ w/ locally const. horizontal sections

∇^{GM} is <u>Gauss – Manin Connection</u>

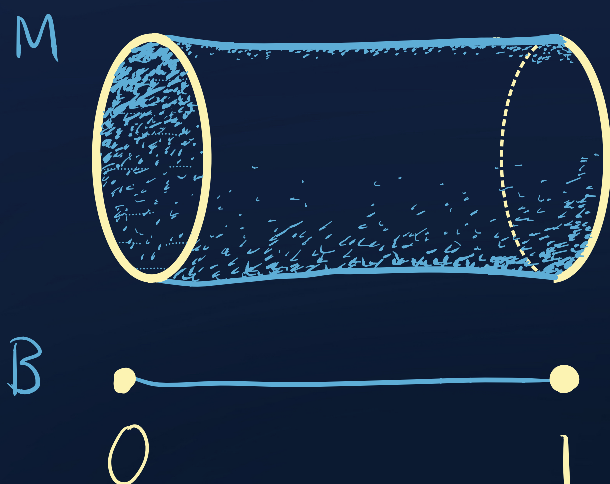
Gauss-Manin Monodromy Example

- GM connection ∇^{GM} on $\pi^*: \mathcal{H}^k_\pi \to B$ flat

  $\Leftrightarrow$ representation of $\pi_1(B)$ on $H^k(M_b)$ (monodromy)

Mapping torus for orientation-reversing circle map $\cong$ Klien bottle!

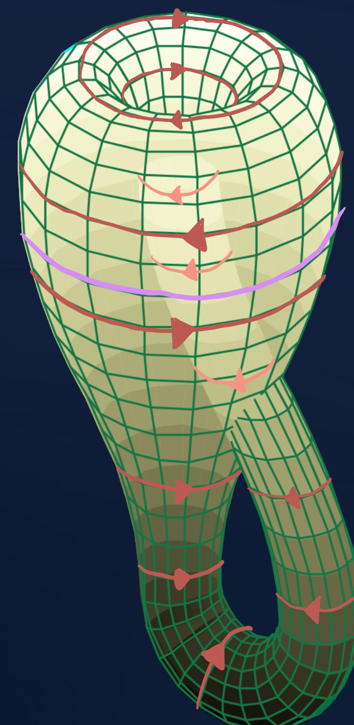
$S^1 \times [0,1]/\sim;\ (z,0) \sim (-z,1)$



Monodromy:

$\omega \in H^1(S^1)$

$\omega \mapsto -\omega$



Unfoldings

- Want to strengthen Picard-Lefschetz invariant
  - PL invariant formed from deforming W by constant
    - > What if we consider all possible deformations?
- Unfolding: deformation of W paramertized by base Λ

  $W_\lambda(x): \mathbb{C}^n \times \Lambda \to \mathbb{C}$ in practice, $\Lambda = \mathbb{C}^\mu$; $W_0(x) = W(x)$
- versal unfolding: subsumes all other unfoldings (up to coordinates)

  every $W'_{x'}(x) = W_{\theta(x')}(g_{x'}(x))$ for some
  - change of coords $g_{x'}(x)$
  - change of parameters $\theta(x')$
- For basis $\varphi_1, \dots, \varphi_\mu$ of local ring $\underbrace{\mathcal{O}/I_{\nabla W}}$ contains all infetesimally nonequivilent germs

  $W_\lambda(x) = W(x) + \sum \lambda_i \varphi_i(x)$ is versal
- $\mu = \dim \mathcal{O}/I_{\nabla W}$ is minimal dimension of versal unfolding "milnor number/multiplicity"
  - μ-dimensional versal unfolding is "miniversal"

Unfolding Example

$W(z) = z^4$

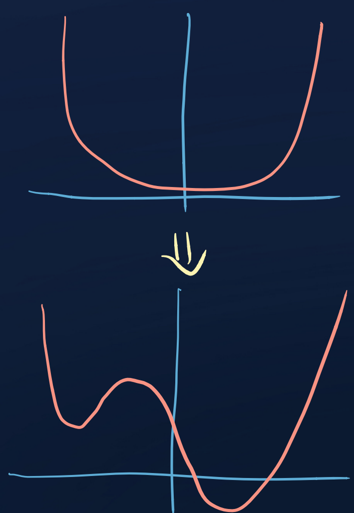
$\nabla W(z) = 4z^3$, so $I_{\nabla W} = (z^3)$

local ring: $Q_W = \mathbb{C}[z] / I_{\nabla W} = \mathbb{C}[z] / (z^3) = \text{span}\langle 1, z, z^2 \rangle$

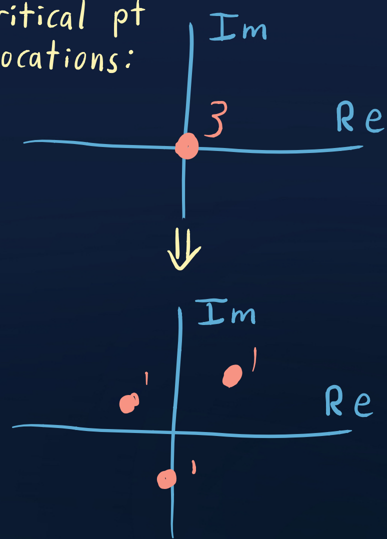
$\mu = \dim Q_W = 3$

versal deformation: $W_{\vec{\lambda}}(z) = z^4 + \lambda_2 z^2 + \lambda_1 z + \lambda_0$

$\mathbb{R}$:



critical pt locations:



Generic deformations: split critical point into μ nondegerate (morse) critical points

Milnor Fibration

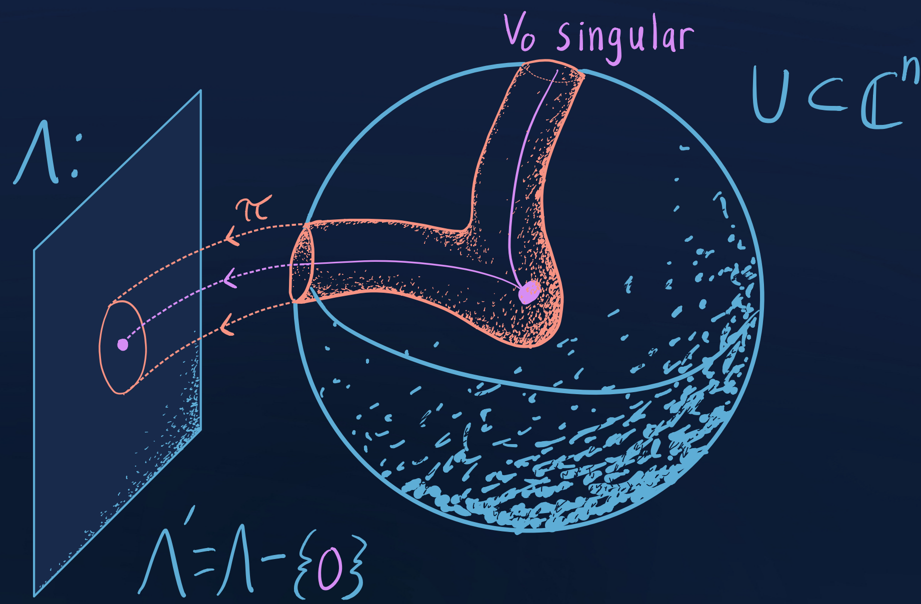
- choose $U \subset \mathbb{C}^n$ neighbourhood of isolated crit. point
- consider hypersuface $V_\lambda = W_\lambda^{-1}(0)$, $\lambda \in \mathbb{C}^\mu$
  - restrict to $\Lambda \subset \mathbb{C}^\mu$ s.t $V_{\lambda \in \Lambda}$ intersects ∂U transversely

$$\Lambda' = \{ \lambda \in \Lambda \mid V_\lambda \text{ is nonsingular} \}$$

Milnor Fibration:

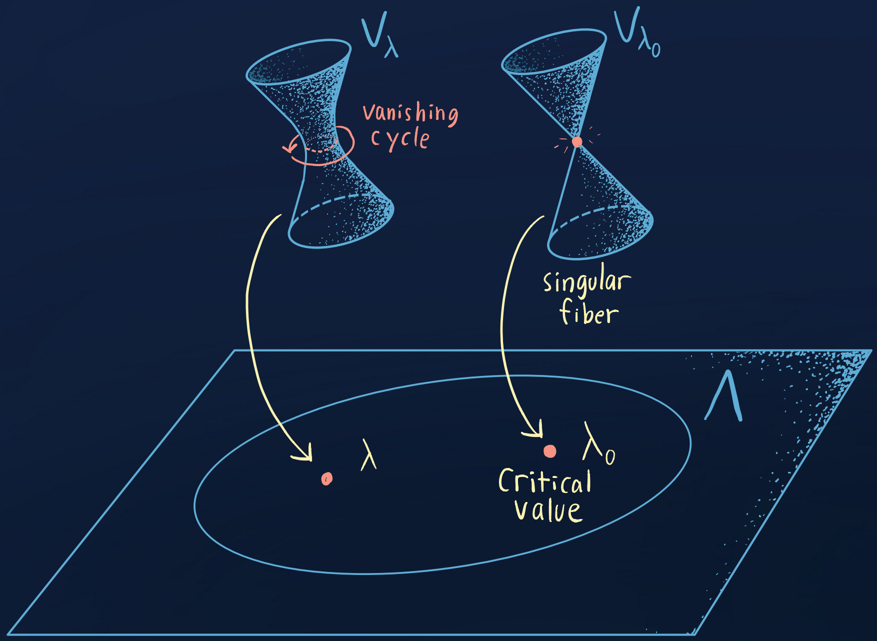
$$\pi : V_{\Lambda'} \to \Lambda'$$

$$V_{\lambda'} = \pi^{-1}(\lambda')$$



Structure of Milnor Fibration

- Milnor fibers all diffeomorphic for any deformation
- homotopy type $\underbrace{S^{n-1} \wedge \ldots \wedge S^{n-1}}_{\mu}$
- (co)homology $H^k(V_\lambda, \mathbb{R}) = \begin{cases} \mathbb{R}^\mu & k=n-1 \\ 0 & \text{else} \end{cases}$ "vanishing (co)homology"



- cycles that go to 0 at singular fiber
- monodromy is of vanishing cohomology bundle

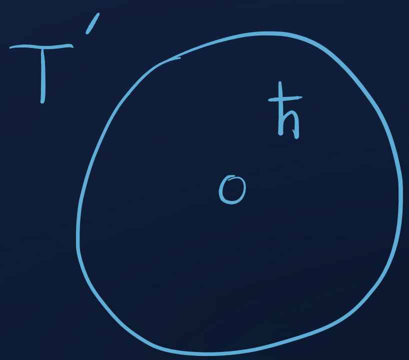
Period Map:

- $(n-1)$ form ψ on $\mathbb{C}^n \times \Lambda$
  - $\psi_\lambda = \psi|_{V_\lambda}$; ψ_λ closed, so $[\psi_\lambda] \in H^{n-1}(V_\lambda)$
  - ψ defines global section $[\psi]$ of vanishing cohomology bundle

  $[\psi]$ is period map of ψ

Residue form (or Gelfand-Leray form): For n-form ω, $\exists$ $(n-1)$ form η s.t $\eta \wedge dW = \omega$. Restriction $\omega/dW_\lambda := \eta|_{V_\lambda}$ is unique

Geometric sections are of the form $[\omega/dW]_\lambda$



- Vanishing cohomology over punctured disk T':

Analytic form of Gauss-Manin:

$$\nabla_{\partial/\partial\hbar}[\psi] = [d\psi/dW]$$

Analytic solution to Gauss-Manin:

- choose coordinates
  - pick n-forms $\omega_1, \dots, \omega_\mu$
    > $[\omega_i/dw]$ are basis of geometric sections
  - $\nabla_{\partial_\hbar}[\omega_i/dw] = \sum P_{ij}(\hbar)\,[\omega_j/dw]$
    > $P(\hbar) = [P_{ij}(\hbar)]$ holo. on T' (punctured disk), mero. on T (disk)
  - $\delta(\hbar)$ horizontal section of vanishing integer homology bundle
    > $\vec{I}(\hbar) := \left(\int_{\delta(\hbar)} \omega_1/dw, \dots, \int_{\delta(\hbar)} \omega_\mu/dw \right)$

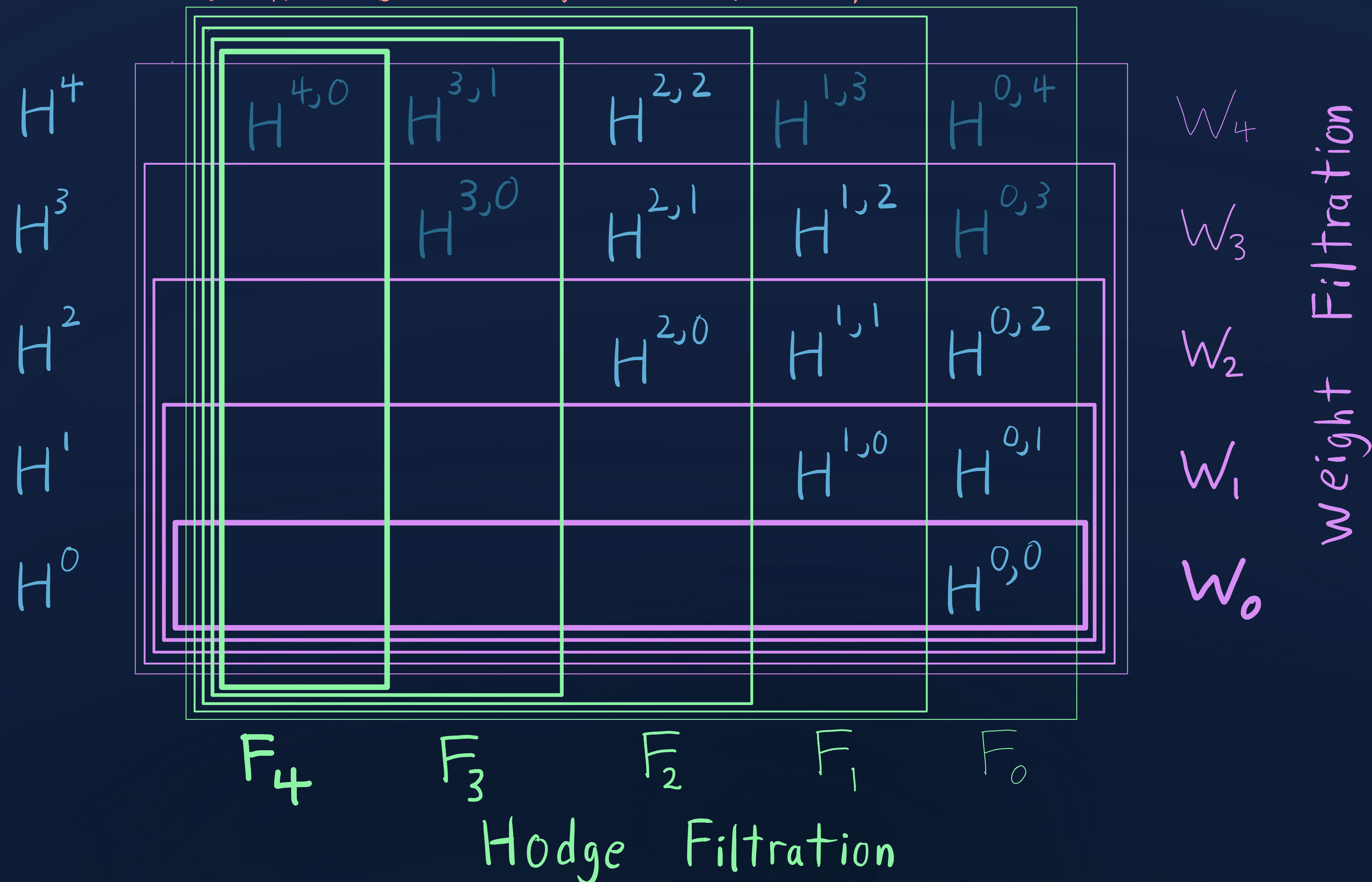
$$\Rightarrow \frac{d}{d\hbar}\vec{I} = P \cdot \vec{I} \qquad \text{Picard-Fuchs equation}$$

solution: $$\boxed{[\omega/dw] = \sum_{k \geq 0} \hbar^\alpha (\ln \hbar)^k A^\omega_{k\alpha}/k!}$$

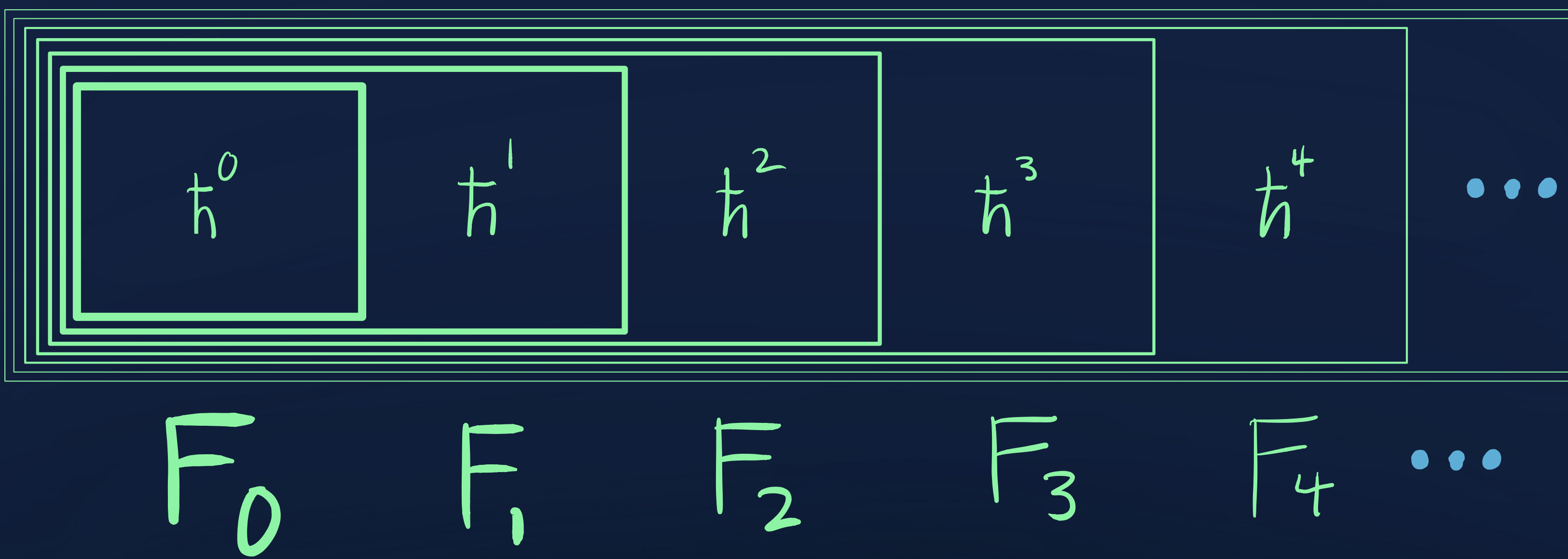
Asymtotics of cohomology near singularity $\hbar = 0$

(Mixed) Hodge Structures

*see appendix B for mixed hodge structure of a singularity



Semi-∞ Hodge Structures:



* + some other requirements

Variation of Semi-∞ Hodge Structure (VSHS)

Data: $(\mathcal{M}, \mathcal{E}, \nabla)$

- $\mathcal{M}$: Parameter space
- $\mathcal{E}$: locally free sheaf of $\mathcal{O}_{\mathcal{M}}\{\hbar\}$ modules
  - ie vector bundle w/ $\hbar$-power series coefficents
- $\nabla : \mathcal{E} \to \Omega^1(\mathcal{M}) \otimes \hbar^{-1}\mathcal{E}$ flat connection

  Griffiths transversality

- Other data: polarization $(\cdot, \cdot)_{\mathcal{E}} : \mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{O}_{\mathcal{M}}\{\hbar\}$

B-Model VSHS

LG Path integral $\int_X e^{w/\hbar}$

- Setup: LG potential $w : \underset{\mathbb{C}^n}{\overset{}{X}} \to \mathbb{C}$ (with $X \subseteq \mathbb{C}^n$) w/ versal unfolding $w_\lambda,\ \lambda \in \Lambda$
- consider new 'unfolding' $w_{\lambda,\hbar}(x) = w_\lambda(x)/\hbar \quad (\lambda,\hbar) \in \Lambda \times T'$

- fix $\lambda \in \Lambda'$: have fibration $\pi_\lambda : V \to T'$, $\pi^{-1}(\hbar) = w_{\lambda,\hbar}^{-1}(0) := V_\hbar$
  nonsingular
- induces vanishing cohomology fibration

$$\pi_* : \mathcal{H}^{n-1}_\pi \longrightarrow T', \quad \pi_*^{-1}(\hbar) = H^{n-1}(V_\hbar)$$

- Analytic solution of Gauss-Manin: $\Omega \in \Omega^n(X)$

$$[\Omega/dw] = \sum_{\kappa \geq 0} \hbar^\alpha (\ln \hbar)^\kappa A^\omega_{\kappa\alpha}/\kappa! \in \Gamma(\mathcal{H}^{n-1}_\pi)$$

basis of flat sections

$$\overset{\text{formally}}{\Longrightarrow} H^{n-1}(w_\lambda^{-1}(0))\{\hbar\}$$

- $\mathcal{M} = \Lambda_{\text{(formal)}}$
- $\mathcal{E} = H^{n-1}(w_\lambda^{-1}(0))\{\hbar\}$
- $\nabla = \nabla^{GM}$

A-Model VSHS

Quantum cohomology: new product on $H^\bullet(X)$:

Paramertized (formally) by $\gamma \in H^\bullet(X)$

$$*_\gamma : H^\bullet(X) \times H^\bullet(X) \to H^\bullet(X)\{\hbar\}$$

Deformed (formally) by $\hbar$

- $\mathcal{M} = H^\bullet(X)$ (formal)
- $\mathcal{E} = H^\bullet(X) \otimes \mathcal{O}_{\mathcal{M}}\{\hbar\}$

normal flat connection on $H^\bullet(X)$

- $\nabla_X Y(\gamma) = \nabla_{0\,X} Y(\gamma) + \frac{1}{\hbar} X *_\gamma Y$

(flat connection from Frobenius mfld structure)

note: For trivial tangent bundle w/ usual flat connection,
Product on $T\mathcal{M} \Longleftrightarrow$ new flat connection

*missing technical details

VSHS as Moving Subspace

VSHS $(\mathcal{M}, \mathcal{E}, \nabla)$: suppose $\mathcal{M}$ simply connected

$$\mathcal{H} = \{ s \in \Gamma(\mathcal{E} \otimes \mathcal{O}_{\mathcal{M}}\{\hbar, \hbar^{-1}\}) \mid \nabla s = 0 \}$$

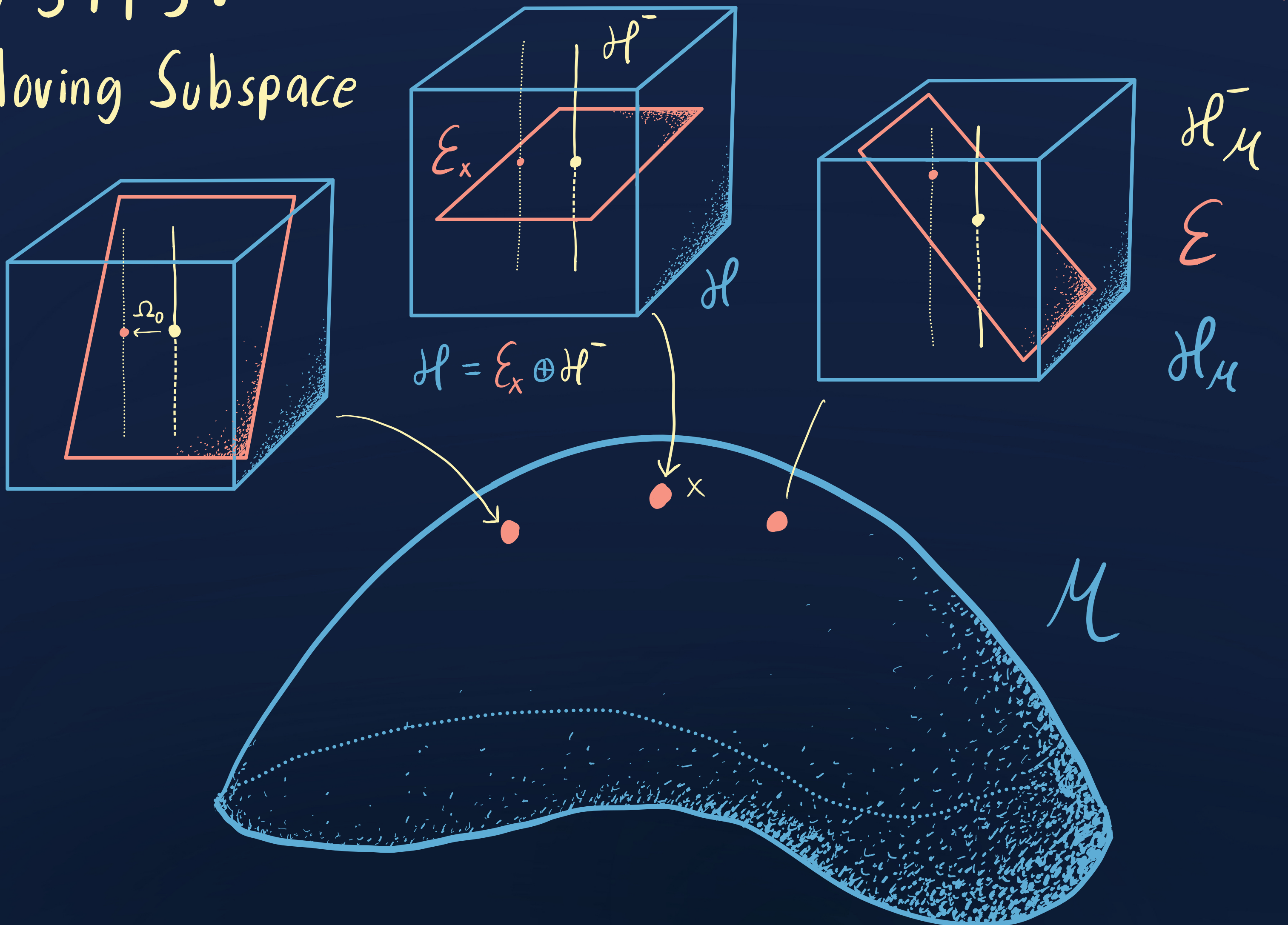
laurent series of flat sections

- ∇ flat & $\pi_1(\mathcal{M}) = 0 \Rightarrow \mathcal{H} \cong \mathcal{E}_x \otimes \mathbb{C}\{\hbar, \hbar^{-1}\}$ adds $\hbar^{-1}$ terms to $\mathcal{E}_x$
- $\mathcal{E}_x \cong \{ s \in \mathcal{H} \mid s(x) \in \mathcal{E}_x \}$: extend $\mathcal{E}_x$ by flat sections
  - $\mathcal{E}_x$ chooses subspace of $\mathcal{H}$ w/ positive $\hbar$ coefs
- $\mathcal{E}$ is subbundle of trivial bundle $\mathcal{H}_{\mathcal{M}} := \mathcal{H} \times \mathcal{M}$
- Oppisite subspace: $\mathcal{H}^- \subset \mathcal{H}$

$\forall x : \mathcal{H} = \mathcal{H}^- \oplus \mathcal{E}_x$

  - $\mathbb{C}\{\hbar^{-1}\}/\mathbb{C}$ submodule: only $\hbar^{n<0}$ terms
  - $\mathcal{H}^-_{\mathcal{M}} = \mathcal{H}^- \times \mathcal{M}$ subbundle of $\mathcal{H}_{\mathcal{M}}$

VSHS: Moving Subspace



VSHS $\Rightarrow$ Frobenius manifold !!

$\mathcal{O}_{\mathcal{M}}(\hbar)$ sheaf, flat connection, Polarization, metric, Product on $T_x\mathcal{M}$

$$(\mathcal{M}, \mathcal{E}, \nabla, (-,-)_{\mathcal{E}}) \Rightarrow (\mathcal{M}, g, *_x)$$

base mfld

- $\hbar^0$ terms: $\mathcal{E}/\hbar\mathcal{E} \overset{\text{locally}}{\cong} \mathcal{E} \cap \hbar\mathcal{H}^- \overset{\text{locally}}{\cong} \hbar\mathcal{H}^-/\mathcal{H}^-$ trivial!
- for $\Omega_0 \in \hbar\mathcal{H}^-$, $(\Omega_0 + \mathcal{H}^-) \cap \mathcal{E}_x$ is 1 pt: (draw)

semi-∞ period map: $\Psi: \Omega_0 \longmapsto (\Omega_0 + \mathcal{H}_-) \cap \mathcal{E} \in \Gamma(\mathcal{E})$ in $\hbar^0$ part, or "in" $\mathcal{E}/\hbar\mathcal{E}$

- Suppose $X \longmapsto \hbar \nabla_X \Psi(\Omega_0)$ gives isomorphism $T\mathcal{M} \cong \mathcal{E}/\hbar\mathcal{E}$ "miniversal"
- connection ∇ on $\mathcal{E}$ induces $\hat{\nabla}$ on $\hbar\mathcal{H}^-/\mathcal{H}^- \cong T\mathcal{M}$ $\hat{\nabla} = d + \frac{1}{\hbar}A$
  - $A \in \Omega^1(\mathrm{End}(\hbar\mathcal{H}^-/\mathcal{H}^-))$ gives product structure:

$$X *_x Y = Z \iff A_Z|_x = A_{X*Y}|_x = A_X A_Y|_x$$

- $(-,-)_{\mathcal{E}}$ on $\mathcal{E}/\hbar\mathcal{E} \cong T\mathcal{M}$ yields g

Summary So Far:

A-Side:

- Quantum cohomology is naturally a Frobenius mfld
- Frobenius mfld encodes physics of CFT family

B-side:

- Physics of LG model $\Leftrightarrow$ Singularity theory
- Singularity theory $\Rightarrow$ natural VSHS
- Quantum Cohomology has VSHS
- VSHS $\Rightarrow$ Frobenius manifold

Mirror Symmetry

(finally!)

Conjecture: Mirror Symmetry

For every σ-model (defined w/ Kähler manifold),
There is an LG-model (defined w/ quasihomogenous function)
Such that their VSHS's are isomorphic

Restrict to Calabi-Yau Manifolds:

- Kahler mfld X is Calabi-Yau if $c_1(TX)=0$ (or, $K_X=\Lambda^n T^* X$ is trivial)

Mirror symmetry takes special form for C·Y mflds

Quasihomogenous polynomial $F(x_1, \dots, x_n)$ has weights $\vec{\lambda}$: $\lambda_1, \dots, \lambda_n$:

- F naturally defined on weighted projective space $W\mathbb{C}P^n_{\vec{\lambda}}$
  - $F^{-1}(0)$ defines smooth submanifold of $W\mathbb{C}P^n_{\vec{\lambda}}$
    > $F^{-1}(0)$ is Calabi-Yau under simple constraint on degrees / weights

CY-LG correspondence: if $F^{-1}(0)$ CY, L.G model $\cong$ σ-model on $F^{-1}(0)$

"LG theory equivlent to strings propegating on vanishing set"

mirror symmetry equates σ-model on 1 CY-mfld to another

Mirror symmetry is T-duality

Complex Deformations (B-model)

- Almost complex structure $J : T^{0,1}M \to T^{1,0}M$

  $J \in T^{0,1*}M \otimes T^{1,0}M \cong \Omega^1(M, T^{1,0}M)$ TM-valued 1-form

- Deformations of J are Closed:

  $T\mathcal{M}_{complex} \cong H^1(M, T^{1,0}M)$ Kodira-Spencer class

- parametrize complex deformations of CY mflds

- Moduli space $\mathcal{M}_{LG}$ bigger: $T\mathcal{M}_{LG} \cong \bigoplus_{p,q}^{n} H^p(M, \Lambda^q T^{1,0}M)$

  $\mathcal{M}_{LG}$ is Extended/Thickened CY moduli space

"quantum version of $\mathcal{M}_{complex}$" "noncommunative cmplx deformations"

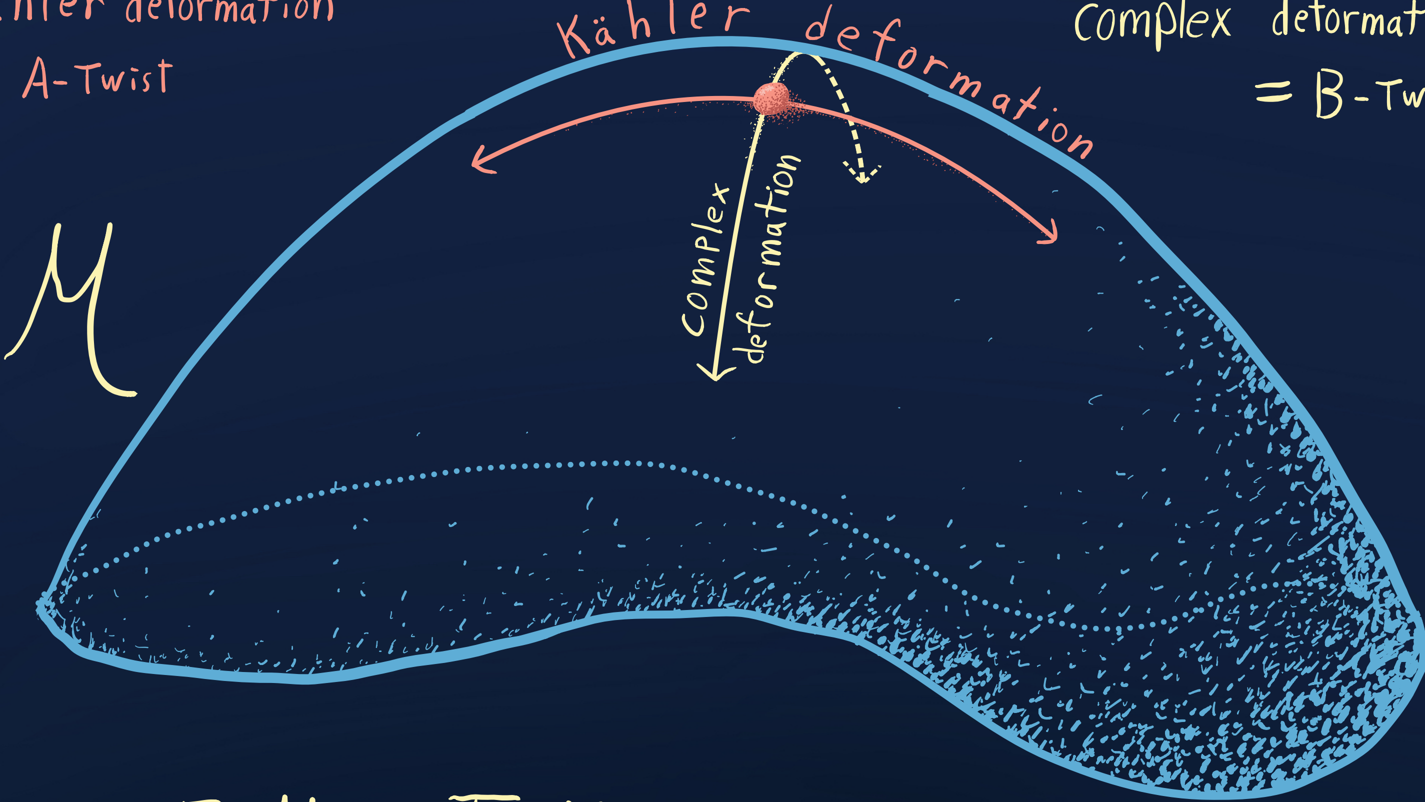
A-model: $T\mathcal{M}_{\text{Kähler}} = H^{1,1}(M)$, Quantum cohomo $= H^{\bullet}(M)$

No extention needed for CY 3-folds (I think)

Modulii Space of Kähler Manifolds

Kähler deformation
= A-Twist

complex deformation
= B-Twist



$$T_p\,\mathcal{M} \cong T_p\mathcal{M}_{\text{kähler}} \oplus T_p\,\mathcal{M}_{\text{complex}}$$

Mirror Symmetry for Calabi-Yau:

Mirror CY mflds exchange their (extended) modulii of Kähler deformations & (extended) modulii of Complex deformations

or...

Mirror CY mflds exchange their A & B twists

Preview: Tropical Geometry

tropical geometry $\Rightarrow$ curve counting $\Rightarrow$ A-side

?? tropical geometry $\Rightarrow$ B-side ??

Goal: Construct A & B side VSHS's thru tropics
isomorphism in tropics $\Rightarrow$ isomorphism of VSHS's

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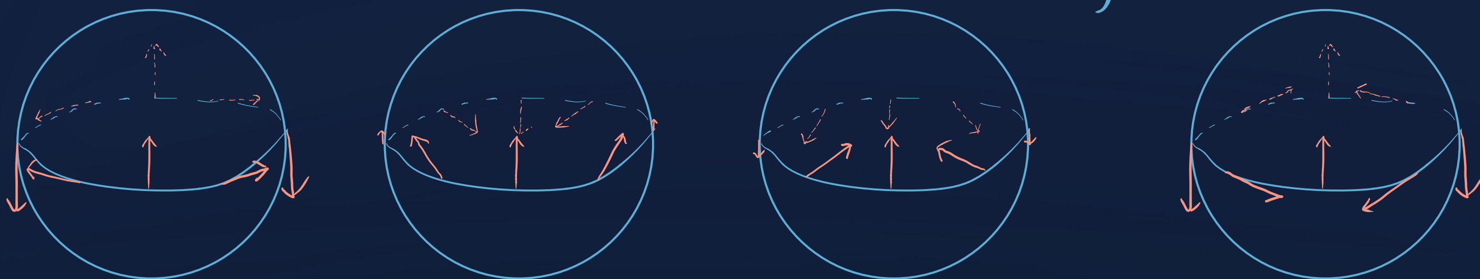
Appendix A: Geometric Picture of Fermions

Σ has line bundles $K = T^{(1,0)}(\Sigma)$, $K^{-1} = T^{(0,1)}(\Sigma)$

holomorphic 1-forms   antiholomorphic 1-forms

Spin bundles: $K^{1/2} \otimes K^{1/2} = K$ $K^{-1/2} \otimes K^{-1/2} = K^{-1}$

$\theta^{\pm} \in \Gamma(K^{1/2})$ $\bar{\theta}^{\pm} \in \Gamma(K^{-1/2})$



K^{1} +2 rotations $K^{1/2}$ +1 rotation $K^{-1/2}$ -1 rotation K^{-1} -2 rotations

New symmetries: $(\theta^{\pm}, \bar{\theta}^{\pm}) \xrightarrow{a,b} (e^{i\alpha}\theta^{\pm}, e^{(-1)^{a,b} i\alpha}\bar{\theta}^{\pm})$

"R-Symmetries"
mixing these w/ normal rotation gives A/B twists

Appendix B: Singularity Hodge Structure

$$[\omega/df] = \sum_{k\geq 0} \hbar^{\alpha} (\ln \hbar)^{k} A^{\omega}_{k\alpha}/k!$$

Hodge filtration: order of $[\omega/df]$ is smallest $\hbar^{\alpha}$

for each fiber $H^{n-1}(V_{\hbar}, \mathbb{C})$, take subspace $F^{p}_{\hbar}$ of elements w/ order $\leq n-p-1$

Weight filtration: let M be mononodromy on $H^{n-1}(V_{\hbar})$

Fact: $\exists n, k$ s.t. $(M^{n}-I)^{k}=0$: $M^{n}-I$ is nilpotent

Every nilpotent operator N defines a filtration:

$$N = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \qquad N\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_2 \\ \vdots \\ x_n \\ 0 \end{bmatrix} \qquad N^{k}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ \vdots \\ x_n \\ 0 \\ \vdots \\ 0 \end{bmatrix} \left.\vphantom{\begin{matrix}0\\0\\0\end{matrix}}\right\} k$$

* not the real Weight filtration, but gives an idea