An overview of 3D Mirror symmetry

content from AMS Notices article "3-dimensional mirror symmetry" by Webster & Yoo

this talk will go over a large-scale picture of 3D mirror symmetry, discussing the objects of study & their interrelationships.

I will tour the image above, wich I drew for Justin Hilburn.
I won't carefully define or describe any of the objects, leaving that for later talks.

Part 1: the physics framework

All different types of mirror symmetry are organized by their underlying quantum field theories. All the quantum field theories today stem from 1:

$$\underbrace{4\text{-}D}_{\text{Dimension of underlying space}} \quad \underbrace{N=4}_{\text{size of supersymmetry algebra}} \quad \text{Supersymmetric Yang-Mills theory.}$$

Such a theory is defined by a lie group G (gauge group) &
A Quaternionic vector space N (matter)

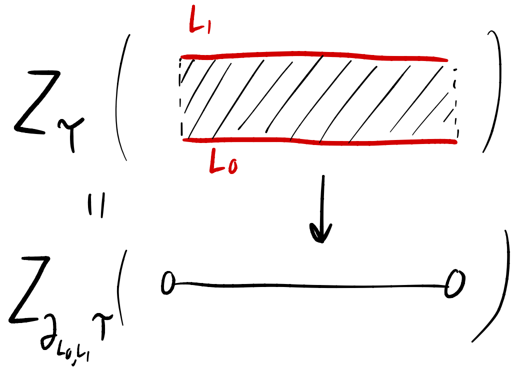
This theory assigns any 4-manifold with metric to a number. We can modify the theory via a "topological twist" to remove metric dependence, giving a TQFT

There are two possible twists the A & B twist. This leaves us with two TQFTs

B-twist Kapustin-Witten — 4D SUSY YM — A-twist Kapustin-Witten

We expect these to be fully extended Lurie-style TQFTs. We can obtain lower dimensional TQFTs by imposing bdry conditions

Let $\mathcal{T}$ be a TQFT in dimension n.



we can define $\mathcal{T}$ on manifolds w/ boundary if each bdry is labeled w/ a "bdry condition."

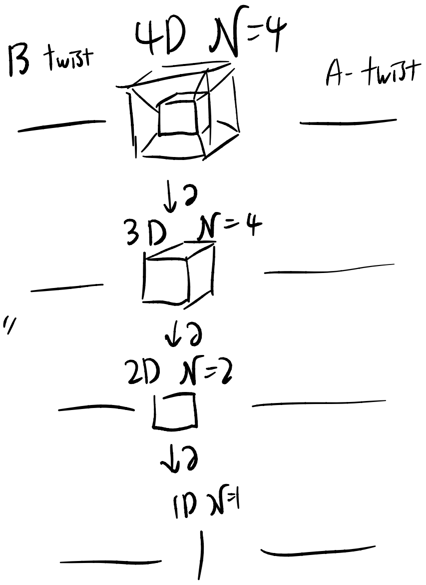
Applying $\mathcal{T}$ to $\mathbb{R}^{n-1} \times I$ w/ bdry labeled by L_0, L_1 defines a $n-1$ -dimensional "boundary theory" $\partial_{L_0,L_1}\mathcal{T}$

In the framework of the cobordism hypothesis, the tQFT $\mathcal{T}$ is defined by the $(n-1)$-category $Z_{\mathcal{T}}(Pt)$. This is the "category of boundary conditions"

- the objects of $Z_{\mathcal{T}}(Pt)$ are possible boundary labilings (e.g L_0 or L_1)
- the morphisms $\mathrm{Hom}(L_0, L_1)$ form an $(n-2)$-category $Z_{\partial_{L_0,L_1}\mathcal{T}}(Pt)$

Starting w/ Kapustin Witten TQFT, we repeatedly take boundary theories.

This is the "Mirror Symme-tree"



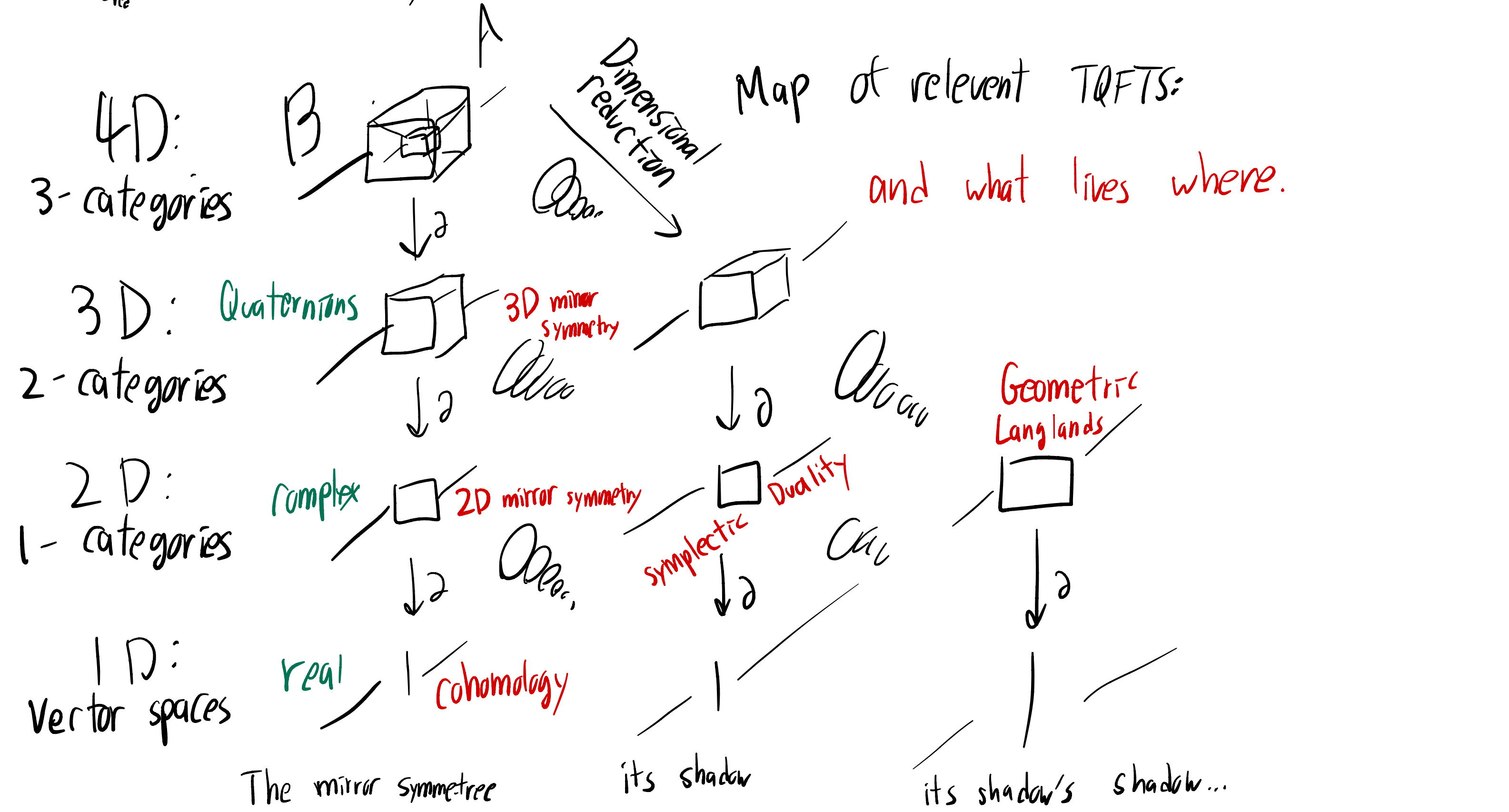
There is another way to reduce the dimension w/o choice of boundary conditions

Z_{τ^n}(cylinder) evaluate on a cylinder! This is called "Dimensional reduction"

$\downarrow$

$Z_{\tau^{n-1}_{red}}$(segment)

This extracts lower-categorical information out of a TQFT τ^n (Hoschild homology) it is a seperate axis from dimensional reduction.



Part 2: 2D mirror symmetry

For 2D mirror symmetry, we can describe the resulting TQFT as a σ-model: it is some counting of the space of maps from 2D space to target.

for a Kahler manifold M, we have TQFTs τ^{2A}_M, τ^{2B}_M $\quad \tau^{2A,B}_M(\Sigma) = \# \mathrm{Maps}_{A,B}(\Sigma \to M)$

The difference between the A, B twists lie in what "Maps" means.

Twist:	"complex side" B	"symplectic side" A
type of maps	locally constant	holomorphic
Boundary conditions	holomorphic submanifolds (generally, coherent sheaves)	Lagrangians
Chain complex of morphism	(derived) intersection pts	intersection pts
differential of morphisms	none	holomorphic strips L_0 L_1
Category of B.Cs.	$D^b coh(M)$	Fukaya category $\mathcal{F}uk(M)$

Mirror symmetry posits that any Kahler manifold M has a mirror pair $M^\vee$ such that $\mathcal{T}^{2A}_M = \mathcal{T}^{2B}_{M^\vee}$ & $\mathcal{T}^{2A}_{M^\vee} = \mathcal{T}^{2B}_M$ equivalence of underlying TQFTs

in particular, $\mathcal{F}uk(M) = D^b coh(M^\vee)$, the familiar statement.

1D theory: also exhibits holomorphic / locally constant dichotomy.

	B	A
target	M complex	M riemannian
Maps	constant	constant
Z(M)	$H_{\bar\partial}(M)$ Dolbeault cohomology "holomorphic"	$H_d(M)$ De Rahm cohomology. "locally constant"

Part 2: 3D Mirror symmetry

The Quantum field theories underlying 3D Mirror symmetry are sigma models w/ target M. Call these $\mathcal{T}_M^{3A}, \mathcal{T}_M^{3B}$

Now M is <u>hyperkahler</u> manifold: it has

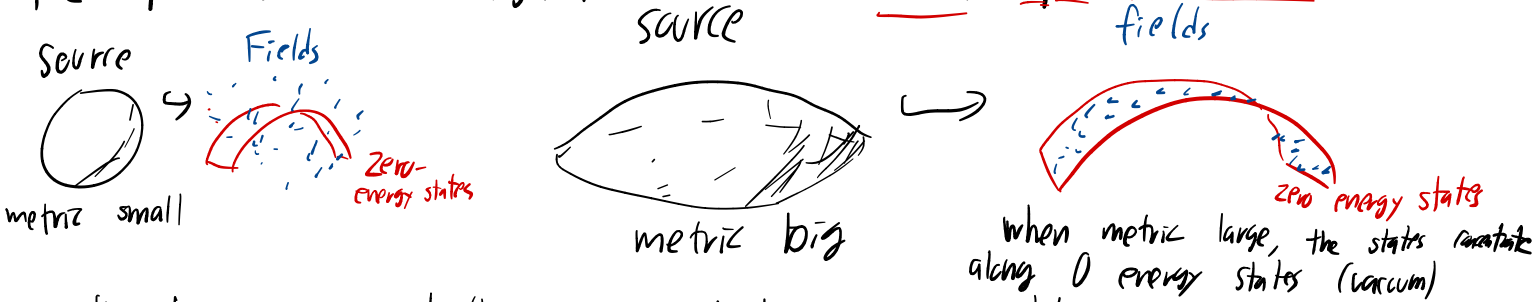
- Three complex structures I, J, K w/ $I^2 = J^2 = K^2 = IJK = -1$ (quaternions)
- Compatible kahler forms $\omega_I, \omega_J, \omega_K$

$\Rightarrow$ a holomorphic symplectic form $\Omega_I = \omega_J + i\omega_K$. (holo wrt I)

	B	A
target	—— M hyperkahler ——	
maps	constant	Feurter maps from $\mathbb{R}^3 \to M$ "quatornionic maps"
Boundary conditions	Holomorphic lagrangians (holo submflds, lagrangian wrt Ω_I)	Holo. lagrangians
$Z_{\mathcal{T}_M^3}(pt)$ 2-category	Rozansky-witten 2-category	Holomorphic Fukaya category or, "Feuter 2-category."

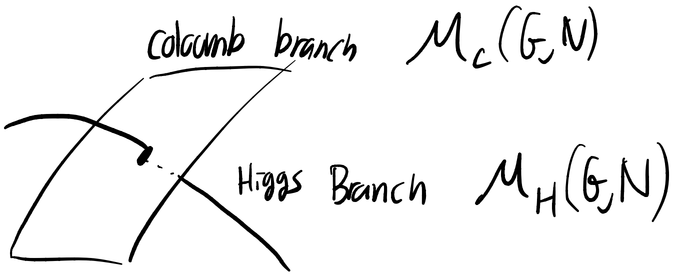
But what is the target manifold? remember, we started w/ just a gauge group G & a quaternionic representation $G \circlearrowright T^*N$ to get 3D TQFTs $\mathcal{T}^{A,B}(G,N)$

The theory $\mathcal{T}(G,N)$ has vacua: configurations of the fields with 0 energy. the space of such configurations is the moduli space of vacua



in the large metric limit, the QFT limits to a σ-model w/ target the moduli of vacua. A TQFT is independent of metric so this limit is exact.

The moduli of Vacua of $\mathcal{T}(G,N)$ is a hyperkahler manifold w/ 2 components.



we can compute things about the TQFT using either $\mathcal{M}_H$ or $\mathcal{M}_C$
But the mathematics behind each looks very different.

The Duality between $\mathcal{M}_C$ & $\mathcal{M}_H$ is 3D mirror symmetry

Conjecture: (3D Mirror symmetry, to a Physicist)
if $\mathcal{T}(G,N)$ is a 3D $\mathcal{N}=4$ QFT, then there exists a mirror pair $\mathcal{T}(G^!,N^!)$
such that
$$\mathcal{T}^A(G,N) = \mathcal{T}^B(G^!,N^!)$$
$$\mathcal{T}^B(G,N) = \mathcal{T}^A(G^!,N^!)$$
A & B twists swap

in particular
$$\mathcal{M}_C(G,N) = \mathcal{M}_H(G^!,N^!)$$
$$\mathcal{M}_H(G,N) = \mathcal{M}_C(G^!,N^!)$$
Higgs & Coloumb branches swap

As outlined above, we can define $Z_{\mathcal{T}^{A,B}(G,N)}(pt)$ as a 2-category associated to the holomorphic symplectic structure of the target $\mathcal{M}_H(G,N)$

$$Z_{\mathcal{T}^A(G,N)}(pt) = RW(\mathcal{M}_H(G,N))$$ Rozansky-witten 2-category (whatever that is)

$$\Vert$$

$$Z_{\mathcal{T}^B(G^!,N^!)}(pt) = Feut(\mathcal{M}_H(G^!,N^!)) = Feut(\mathcal{M}_C(G,N))$$ Feuter 2-category (whatever that is)

Conjecture (3D Mirror symmetry to a mathematician):

a holomorphic symplectic manifold $Y = \mathcal{M}_H(G,N)$ has a 3D-mirror $Y^! = \mathcal{M}_C(G,N)$ s.t.
$$RW(Y) = Feut(Y^!)$$
$$Feut(Y) = RW(Y^!)$$

To Do list:
- Define $\mathcal{M}_H(G,N)$ (easy)
- Define $\mathcal{M}_C(G,N)$ (Done in BFN I, II)
- Define $RW(Y)$ (Ben zvi-Nadler 2026 ICM ?)
  $Feut(Y)$ (Daan - Rezchikov ...)

Not Done Yet!!

let's get to know these spaces a little bit...

Example of Coloumb & Higgs branch:

let me describe the simplest case of abelian 3D Mirror symmetry which is analagous to toric Mirror symmetry, ...

$$G = U(1),\ N = \mathbb{C}^{n+1} \text{ with action } \theta.(z_0,\dots,z_n) = (e^{i\theta}z_0,\dots,e^{i\theta}z_n)$$
$$G^! = U(1)^n,\ N^! = \mathbb{C}^{n+1} \text{ w/ } (\theta_1,\dots,\theta_n)\cdot(z_0,\dots,z_n) = (e^{-i\sum\theta_i}z_0, e^{i\theta_1}z_1,\dots,e^{i\theta_n}z_n)$$

$\mathcal{M}_H(U(1), T^*\mathbb{C}^{n+1})$ is a Hyperkahler reduction $T^*\mathbb{C}^n /\!/\!/ U(1)$

identify $T^*\mathbb{C} \simeq \mathbb{H}$ quaternions so $T^*\mathbb{C}^{n+1} \simeq \mathbb{H}^{n+1}$ has a hyperkahler structure.

$U(1)$ is Hamiltonian in all 3 symplectic forms
$$\omega_I \quad \omega_J \quad \omega_K \qquad \text{w/ } d\mu_\bullet = i_X\omega_\bullet \quad \text{if } X \text{ generates } U(1) \text{ action}$$
$$\mu_I \quad \mu_J \quad \mu_K : T^*\mathbb{C}^{n+1} \to \mathbb{R}$$

organize into μ_I, $\mu_C = \mu_J + i\mu_K$ (moment map for holo symplectic structure $\Omega_I = \omega_J + i\omega_K$)

$$T^*\mathbb{C}^{n+1} /\!/\!/_{\xi} U(1) = \mu_C^{-1}(0) /\!/_{\xi} U(1) = \mu_C^{-1}(0) \cap \mu_I^{-1}(0) / U(1)$$

in Hyper kahler geometry, groups act 4 times.

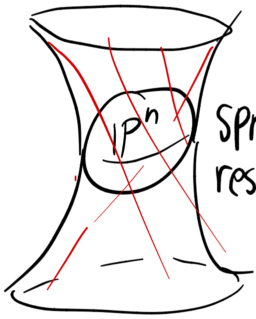
There is a "stability parameter" ξ controlling the level of the moment map

Physically, this is part of the data of $\mathcal{T}(G,N)$

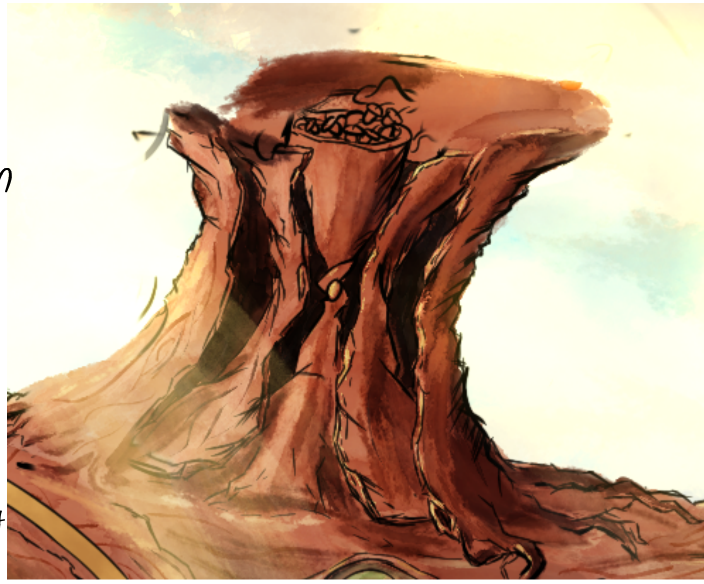
$\xi \neq 0$

$\mathcal{M}_H(U(1), \mathbb{C}^{n+1}) =$

$T^*\mathbb{C}^{n+1} /\!/\!/_{\xi}\, U(1) = T^*\mathbb{P}^n$



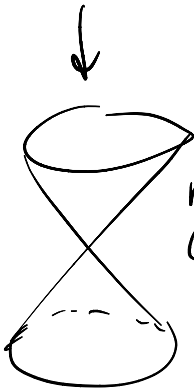
springer resolution



$\xi = 0$

$\mathcal{M}_H(U(1), \mathbb{C}^{n+1}) = T^*\mathbb{C}^{n+1} /\!/\!/_{0}\, U(1)$

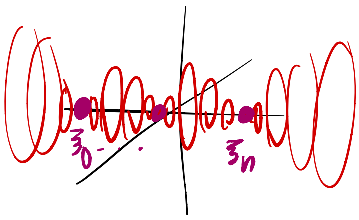
$= \left\{ \begin{array}{l} (n+1)\times(n+1) \text{ matricies} \\ \text{w/ rank} \leq 1 \end{array} \right\}$



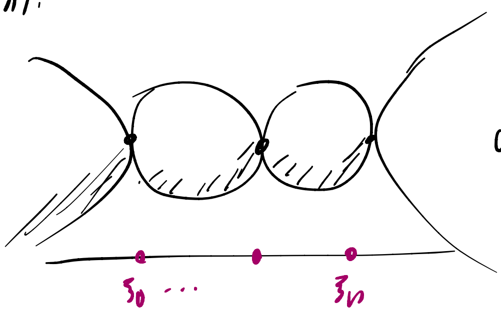
nilpotent cone

$\mathcal{M}_H(U(1)^n, \mathbb{C}^{n+1}) = T^*\mathbb{C}^{n+1} /\!/\!/_{(\xi_1,\dots,\xi_n)}\, U(1)^n$ hyperkahler reduction

Result is a 4-Dimensional space. Topologically, this is an S^1 bundle over $\mathbb{R}^3$ with $n+1$ circles contracted to a point.



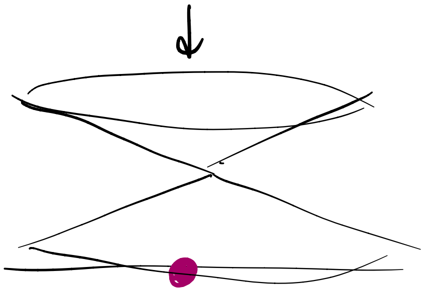
looks like



T^* of chain of n spheres

This lives over singular variety

$T^*\mathbb{C}^{n+1} /\!/\!/_{(0,\dots,0)}\, U(1)^n = \mathbb{C}^2/\mathbb{Z}_n$



$\mathbb{C}^2/\mathbb{Z}_n$

A_n singularity

For generic $\xi_1,\dots,\xi_n$, $T^*\mathbb{C}^{n+1} /\!/\!/ U(1)^n = \widetilde{\mathbb{C}^2/\mathbb{Z}_n}$ resolution of A_n singularity!

indeed, exceptional fiber is chain of n copies of $\mathbb{P}^1$ arranged in the dynkin diagram of A_n o—o— ... —o (Mackay correspondence)

Kronheimer constructed the Hyperkahler metric on $\widetilde{\mathbb{C}^2/\mathbb{Z}_n}$

3D mirror symmetry relates these spaces

	$\mathcal{M}_H$	$\mathcal{M}_C$
$\mathcal{T}(U(1), \mathbb{C}^{n+1})$	$T^*\mathbb{P}^n$	$\widetilde{\mathbb{C}^2/\mathbb{Z}_n}$
$\mathcal{T}(U(1)^n, \mathbb{C}^{n+1})$	$\widetilde{\mathbb{C}^2/\mathbb{Z}_n}$	$T^*\mathbb{P}^n$

$T^*\mathbb{P}^n \overset{!}{=} \widetilde{\mathbb{C}^2/\mathbb{Z}_n}$

note: dimension not preserved! (unlike 2D mirror symmetry)

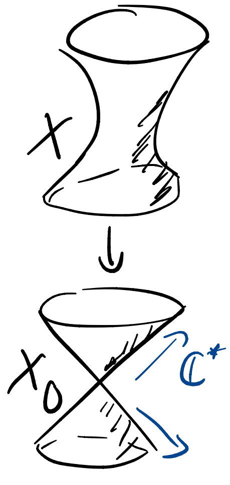
In general, Higgs branches are $\mathcal{M}_H(G,N) = T^*N /\!/\!/ G$

coloumb branches are trickier to get our hands on (more on that later)

Mathematically, these are captured by <u>symplectic resolutions</u>

Def: a <u>symplectic resolution</u> X consists of:

- A smooth variety X w/ holomorphic symplectic form
- a singular affine variety X_0 w/ resolution of singularities $X \to X_0$
- a scaling action of $\mathbb{C}^*$ on X_0 (<u>conical</u> singularity)



"The lie algebras of the 21st century" -Okounkov

3D mirror symmetry manifests mathematically as a duality between symplectic resolutions X & $X^!$, called <u>symplectic duality</u> (see last page for mathematical details)

What is a coloumb branch?

we would like to know how to build a 3D mirror to a Higgs branch, i.e construct a coloumb branch $\mathcal{M}_C(G,N)$. We can access this like an algebraic geometer, by defining the ring of functions $\mathbb{C}[\mathcal{M}_C(G,N)]$ (we assumed X_0 was affine in a symplectic resolution).

Recall that $\mathcal{M}_C$ paramertizes vacua of $\mathcal{T}$, which are configurations of fields. we can define a function on $\mathcal{M}_C$ by evaluating an observable for each vacua. we exploit the "state-operator correspondence"

the vector space $Z_\mathcal{T}(S^2)$ is the Hilbert space of states.

each element $v \in Z_\mathcal{T}(S^2)$ gives us an operator:

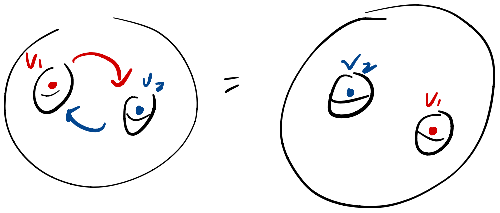
Take any 3-mfld M^3, & excise a small ball. this gives a cobordism $\partial M^3 \to S^2$

$Z(M^3) : Z(\partial M^3) \to Z(S^2)$ Pairing w/ $v \in Z(S^2)$ results in a number. This is "evaluating the operator v"

The TQFT structure gives a product on $Z(S^2)$, the "operator product"

$: Z(S^2) \otimes Z(S^2) \to Z(S^2)$

this is commutative:



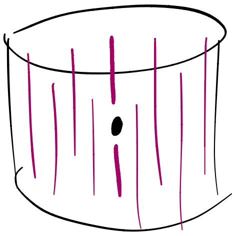
Each $v \in Z(S^2)$ gives an operator, and hence a function on $\mathcal{M}_C$.
in fact, this is all functions!

Higgs Branch is B side (easy) — $\mathcal{A}_B$ — algebra of observables — $\mathcal{A}_A$ — Coloumb branch is A side (hard)

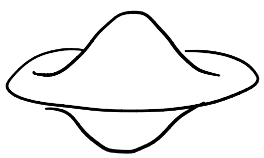
$$\mathcal{M}_H = \operatorname{Spec}[Z_{\mathcal{T}^B}(S^2)] \qquad \mathcal{M}_C = \operatorname{Spec}[Z_{\mathcal{T}^A}(S^2)]$$

when the quaternionic representation of $G_{\mathbb{C}}$ is T^*N for $G_{\mathbb{C}} \curvearrowright N$ a $\mathbb{C}$-rep this is easier to describe. We expect $Z(S^2)$ to be some geometric quantization of the space of admissable maps from S^2 into $N/G_{\mathbb{C}}$

we think about S^2 as the contraction of $\mathbb{R}^3 - \{0\}$ by vertical lines



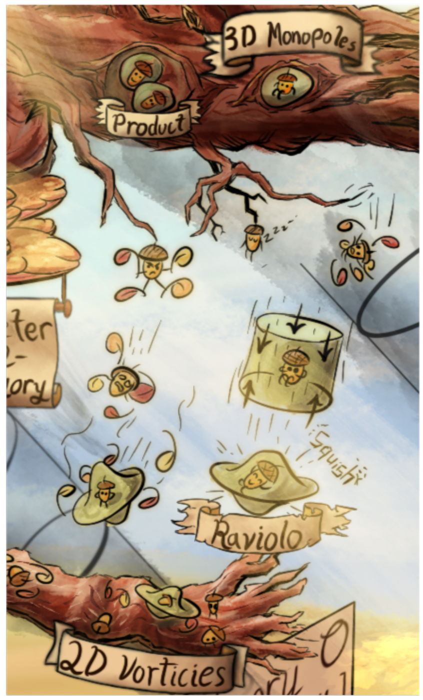
→ "Raviolo"



$D \sqcup D / \{D - \{0\}\}$

glue two discs on complement of 0.

"complex line w/ 2 origins"



$$\mathcal{A}_B = \mathbb{C}[\mathrm{Maps}_{\mathrm{const}}(\text{raviolo}, N/G_{\mathbb{C}})]$$

holo fns on space of locally constant maps

$$\mathcal{A}_A = \mathrm{Const}[\mathrm{Maps}_{\mathrm{Hol}}(\text{raviolo}, N/G_{\mathbb{C}})]$$

locally constant fns on space of holomorphic maps

$\mathcal{M}_H = \mathrm{Spec}\, A_B = \mathrm{Maps}_{const}(☁, N/G_{\mathbb{C}}) = \{$pairs of point $x \in N/G$ & covector @ $x\}$

(the covector remembers the orientation of the disc..)

(we're working in derived geometry)

$= T^*\overset{\text{stack}}{[N/G_{\mathbb{C}}]} = T^*N/\!/G_{\mathbb{C}}$ ← holomorphic symplectic quotient

$= T^*N/\!/\!/G$ Higgs branches are Hyperkahler Quotients

More succinctly, $A_B = \mathbb{C}[\mu_{\mathbb{C}}^{-1}(0)]^{G_{\mathbb{C}}}$ so $\mathcal{M}_H = \underbrace{\mu_{\mathbb{C}}^{-1}(0)/\!/G_{\mathbb{C}}}_{\text{affine GIT}} = \underbrace{\mu_{\mathbb{C}}^{-1}(0)/\!/G}_{\text{symplectic quotient}}$

The coloumb branch is more exciting. Think about the discs in the raviolo formally.

$D = \mathrm{spec}\, \mathbb{C}[[t]]$, $D^* = \mathrm{spec}\, \mathbb{C}((t))$, maps from $D \cup D / D^*$ into N/G are given by maps $n_\pm : D \to N$, glued together by a gauge transform $g_\pm : D^* \to G$, up to overall gauge transforms $h_+, h_- : D \to G$. Algebraically,

$$\mathrm{Maps}_{hol}(☁, G/N) = G[[t]] \backslash \underbrace{\{(n_+, n_-, g_\pm) \mid n_+, n_- \in N[[t]],\ g_\pm \in G((t)),\ g_\pm(t)\, n_-(t) = n_+(t)\}}_{Z_{G,N} \text{ (affine grassmannian kinda object)}} / \underbrace{G[[t]]}_{\text{looked at equivariantly}}$$

$\parallel$

$BFN(G,N)$

left action: $h_+ \cdot (n_+, n_-, g_\pm) = (h_+ n_+, n_-, h_+ g_\pm)$

right action: $h_- \cdot (n_+, n_-, g_\pm) = (n_+, h_- n_-, g_\pm h_-)$

We want to look at "locally constant" functions on this space. The right notion of this is Borel-Moore Homology. All together:

$$\mathcal{M}_C = \mathrm{Spec}\Big(H_{BM}\big(\underbrace{Z_{G,N}/G[[t]]}_{\text{stack quotient}} \big) \Big)$$

$$= \boxed{\mathrm{Spec}\big(H^{G[[t]]}_{BM}(Z_{G,N}) \big)}$$

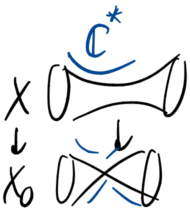
Extracting symplectic Duality from 3D mirror symmetry

The full 2-categorical information predicted by mirror symmetry is inaccessible right now. To extract provable theorems we need to decategorify.

This is where Dimensional reduction comes in: we can extract a 1-category shadow of 2D mirror symmetry.

This takes the form of "category $\mathcal{O}$" for a symplectic resolution.

Consider a symplectic resolution w/ compatible $\mathbb{C}^*$ action



Deformation quantize these manifolds: Replace $\mathbb{C}[X]$ w/ a noncommutative ring $A_\hbar(X)$ s.t $\lim_{\hbar \to 0} A_\hbar(X) = \mathbb{C}[X]$. The $\mathbb{C}^*$ action induces a grading on $A_\hbar$

Let $\mathcal{O}$ be the category of $A_\hbar$-modules w/ positive grading.

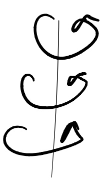
Geometrically, these are D-modules supported on stable strata of the $\mathbb{C}^*$ action.

If $X \downarrow X_0$ is a springer resolution of G, then $\mathcal{O}$ agrees w/ category $\mathcal{O}$ of reps of G.

This generalizes category $\mathcal{O}$ to arbitrary symplectic resolutions!

3D mirror symmetry suggests that category $\mathcal{O}$ of Y & category $\mathcal{O}$ of $Y^!$ are isomorphic, a phenomena generalizing Koszul duality. This is called symplectic duality.

Physically, this arises from Ω deformation: turn on S^1 action on $\mathbb{R}^3$.



operators fixed by this action lie on a line. so, they are no longer commutative. The algebra of operators is now a non-commutative algebra. This gives the deformation quantization of the coulomb branch :)