Hamiltonian G-spaces & Quantization

(Blue will mean verbal explinations, Black is written on the Board)

this is the first talk in a reading seminar on the relative langlands program, following the recent paper Relative Langlands Duality, by Ben-Zvi, Sakellaridis, and Venkatesh

Where we're Going: Relative langlands program

Langlands duality

Ordinary: $G \longleftrightarrow \check{G}$ Lie groups

relative $\begin{matrix} G \\ \circlearrowleft \\ M \end{matrix} \longleftrightarrow \begin{matrix} \check{G} \\ \circlearrowleft \\ \check{M} \end{matrix}$ G-spaces (specifically, hyperspherical varieties)

the relative langlands program extends langlands duality from groups to spaces with a group action. Both the group & the space are dualized. in the paper, they conjecture the spaces are what they define as hyperspherical varieties.

Today's Q: what (& why) are hyperspherical varieties? (BZSV sec. 3)

Section 3 describes several ways to construct symplectic spaces, defines hyperspherical varieties, & provides a uniform construction.

I had some trouble turning this into a talk, because it's not clear until later why we should care about hyperspherical varieties.

- Builds on study of $G \circlearrowleft L^2(X)$ for X spherical (Sakellaridis – Venkatesh)

BZSV builds on older work by SV, where it's easier to get motivation for spherical / Hyperspherical manifolds.

I found a corner of geometric rep. theory I hadn't seen before. it's a very classical symplectic geometry approach, extending the orbit method. And it all centers around Quantization.

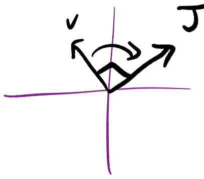
- We will motivate these spaces with Quantization

pt 1: Symplectic geometry & quantization

symplectic manifold (M, ω)

ω: - 2-form
- nondegenerate
- closed

Jv "=" $-i \cdot v$



$\omega(x,y) = \langle x, Jy \rangle$

$\langle , \rangle$ riemannian metric

$J^2 = -1$ "almost complex structure"

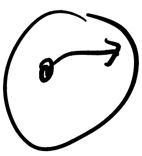
from Hamiltonian $H \in C^\infty(M)$, get hamiltonian vector field

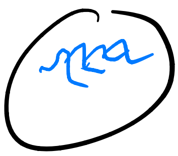
$X_H := J \nabla H$ first take the gradient, then rotate $90°$

Classically: particle in $\mathbb{R}^n$ has position & momentum
↳ lives in $T^*\mathbb{R}^n$, w/ canonical symplectic structure
↳ moves along vector field in $T^*\mathbb{R}^n$

Quantumly: particle is complex function on $\mathbb{R}^n$
↳ lives in hilbert space $L^2(\mathbb{R}^n)$
↳ moves according to linear Diffeq (wave equation)

nonlinear evolution of points $\xrightarrow{\text{Quantization}}$ linear evolution of functions





	Classical	Quantum
space of states	symplectic manifold (M, ω)	hilbert space $\mathcal{H}$
observables	$f \in C^\infty(M)$	$H: \mathcal{H} \to \mathcal{H}$ linear operator
lie algebra of observables	"poisson bracket" $\{f, g\} := X_f(g)$ infintesimal change in g along evolution by f	$[A,B] := AB - BA$ $= \frac{d}{dt}(e^{tA})^{-1} B (e^{tA})$ infintesimal change in B along evolution by A

Quantization (Dream): for (M, ω) symplectic manifold, construct hilbert space $\mathcal{H}$ & lie algebra homomorphism

$$(C^\infty(M), \{,\}) \rightsquigarrow (\text{operators } \mathcal{H} \to \mathcal{H}, [,])$$

This is provably impossible most of the time.

Geometric quantization: construct $\mathcal{H}$ as sections of line bundle L w/ curvature the symplectic form ω

A Beautiful idea, but it only half works, only applies in specific scenarios (e.g. Kahler mflds).

Archatypical examples: $M = T^* X \rightsquigarrow \mathcal{H} = L^2(X)$

X compact kahler $\rightsquigarrow \mathcal{H} = H^0(X, L)$ holomorphic sections

Motto: Quantization linearizes symplectic mflds

Part 2: G-actions

G compact reductive lie group

$G \circlearrowright M$ means $\forall g \in G$, $g: M \to M$ diffeo.

$G \circlearrowright M$ is <u>symplectic</u> if $\forall g$, $g^* \omega = \omega$. i.e, G preserves symplectic structure

ex: for <u>any</u> smooth action $G \circlearrowright X$, the induced action $G \circlearrowright T^*X$ is symplectic $\xrightarrow{\text{Quantization}}$ for any $G \circlearrowright X$, the induced action $G \circlearrowright L^2(X)$ is linear

Motto:

$$\underset{\text{symplectic}}{G \circlearrowright M} \xrightarrow{\text{Quant.}} \underset{\text{linear representation}}{G \circlearrowright \mathcal{H}}$$

$\Rightarrow$ symplectic approach to representation theory!!!

this philosophy is why we care so much abt symplectic geo in rep theory

This is a good start but we want to be able to "look inside" $G \circlearrowright M$. need internal structure. get it w/ slight strengthning

$G \circlearrowright M$ is <u>hamiltonian</u> if infitesimal action $D: \underline{\mathfrak{g}} \to \mathrm{Vect}(M)$ ($\mathfrak{g}$ = Lie(G)) is generated by hamiltonians.

↳ i.e, $\forall\, v \in \mathfrak{g}$, $D_v = X_{\mu(v)}$ for $\mu(v) \in C^\infty(M)$

$\mu(v)$ linear in $v \Rightarrow \mu|_{p \in M} \in \underline{\mathfrak{g}^*}$

↳ a hamiltonian G-action on M <u>defined</u> by its moment map $\mu: M \to \underline{\mathfrak{g}^*}$

fact: μ is <u>equivariant</u> map $\overset{G}{\underset{M}{\circlearrowright}} \overset{\mu}{\rightsquigarrow} \overset{G}{\underset{\mathfrak{g}^*}{\circlearrowright}} {}^{Ad^*}$ for rep. $Ad^*: \underline{\mathfrak{g}^*} \to \underline{\mathfrak{g}^*}$ "coadjoint representation"

i.e $\mu(g \cdot p) = Ad^*_g \mu(p)$

"coadjoint action" $G \circlearrowright \underline{\mathfrak{g}^*}$ is basic model of all hamiltonian $G \circlearrowright M$

restrict G action to maximal torus: $T \circlearrowright \mathcal{O}_+$

moment map is $\mu: \mathcal{O}_\alpha \to \mathfrak{t}^*$ orthogonally projecting to $\mathfrak{t}^*$ note SU(3) pic

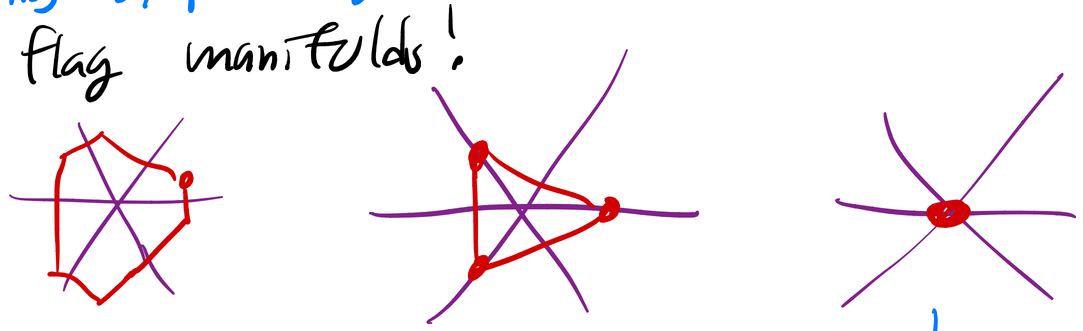
fixed points are exactly $\mathcal{O}_\alpha \cap \mathfrak{t}^*$ wave sphere around

Atiyah convexity thm $\Rightarrow$ $\mu: \mathcal{O}_\alpha \to \mathfrak{t}^*$ has image convex hull of weyl orbit!

atiyah was motivated by trying to re prove this result (of kirillov 70's) using symplectic geo

Can draw flag manifolds!

for SU(3):



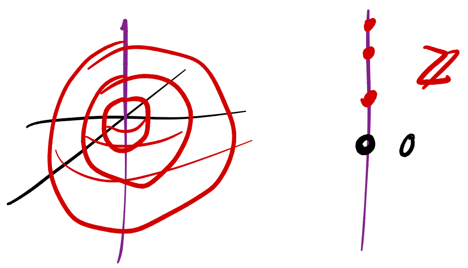
flag manifolds are classified by a pt in $\mathfrak{t}^{*+}$ ... what could their associated representations be?

<u>Quantizing $\mathcal{O}_\alpha$</u>: we need line bundle L whose curvature is the symplectic form ω_α. By essentially gauss-bonnet this needs ω_α to be <u>integral</u>: $\int_{\Sigma \text{ closed}} \omega_\alpha \in \mathbb{Z}$, or $\omega_\alpha \in H^2_{DR}(\mathcal{O}_\alpha, \mathbb{Z}) \subset H^2(\mathcal{O}_\alpha, \mathbb{R})$

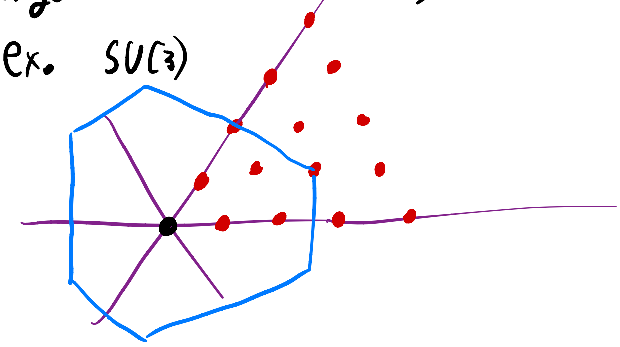
fact: ω_α integral iff α integral (belongs to the root lattice)

ex: so(3)

integer area spheres



ex. SU(3)



fact: associated line bundle L_α defined by character $\alpha: T \to \mathbb{C}$

Quantum hilbert space $G \circlearrowright H^0(G/T, L_\alpha)$

by Borel-Weil thm, $G \circlearrowright H^0(G/T, L_\alpha) = E_\alpha$, irriducible rep. w/ heighest wt α!

coadjoint orbit $\mathcal{O}_\alpha$ w/ α integral $\xrightarrow{\text{Quantize}}$ E_α irridicible rep w/ heighest wt α

motto: coadjoint orbits are irriducible

Symplectic reduction

moment map $\mu: M \to \mathfrak{g}^*$ sends M to a collection of coadjoint orbits

decompose M according to these orbits:

- start w/ $\mathcal{O}_0 = \{0\}$. $\mu^{-1}(0) \subset M$ is G-invariant, so divide out by G

  define the symplectic reduction $M /\!/ G = \mu^{-1}(0)/G$

Thm (Marsden-Weinstein) $M /\!/_0 G$ has a symplectic structure

motivating Example: $\mathbb{P}^1$ as symplectic reduction

$M = T^*\mathbb{R}^2 = \mathbb{C}^2$, $\omega = dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2$

$U(1)$ action $e^{i\theta}: (z_1, z_2) \longrightarrow (e^{i\theta} z_1, e^{+i\theta} z_2)$ has moment map
$$\mu: \mathbb{C}^2 \longrightarrow \mathfrak{u}(1)^* \simeq \mathbb{R}$$
$$(z_1, z_2) \longrightarrow \|z_1\|^2 + \|z_2\|^2$$

physically, this describes a 2D harmonic osscilator

$U(1)$ action preserves μ, so might as well restrict to $\mu^{-1}(1) = S^3 \subset \mathbb{C}^2$

define $\mathbb{C}^2 /\!/ U(1)$ to be the "space of orbits with fixed energy"

$$\mathbb{C}^2 /\!/ U(1) := \mu^{-1}(1)/U(1) = S^3/U(1) = \mathbb{P}^1 \simeq S^2 \text{ symplectic!}$$

Remark: for $G \circlearrowright X$ any G-action w/ X/G smooth,
$$T^*X /\!/ G = T^*(X/G)$$

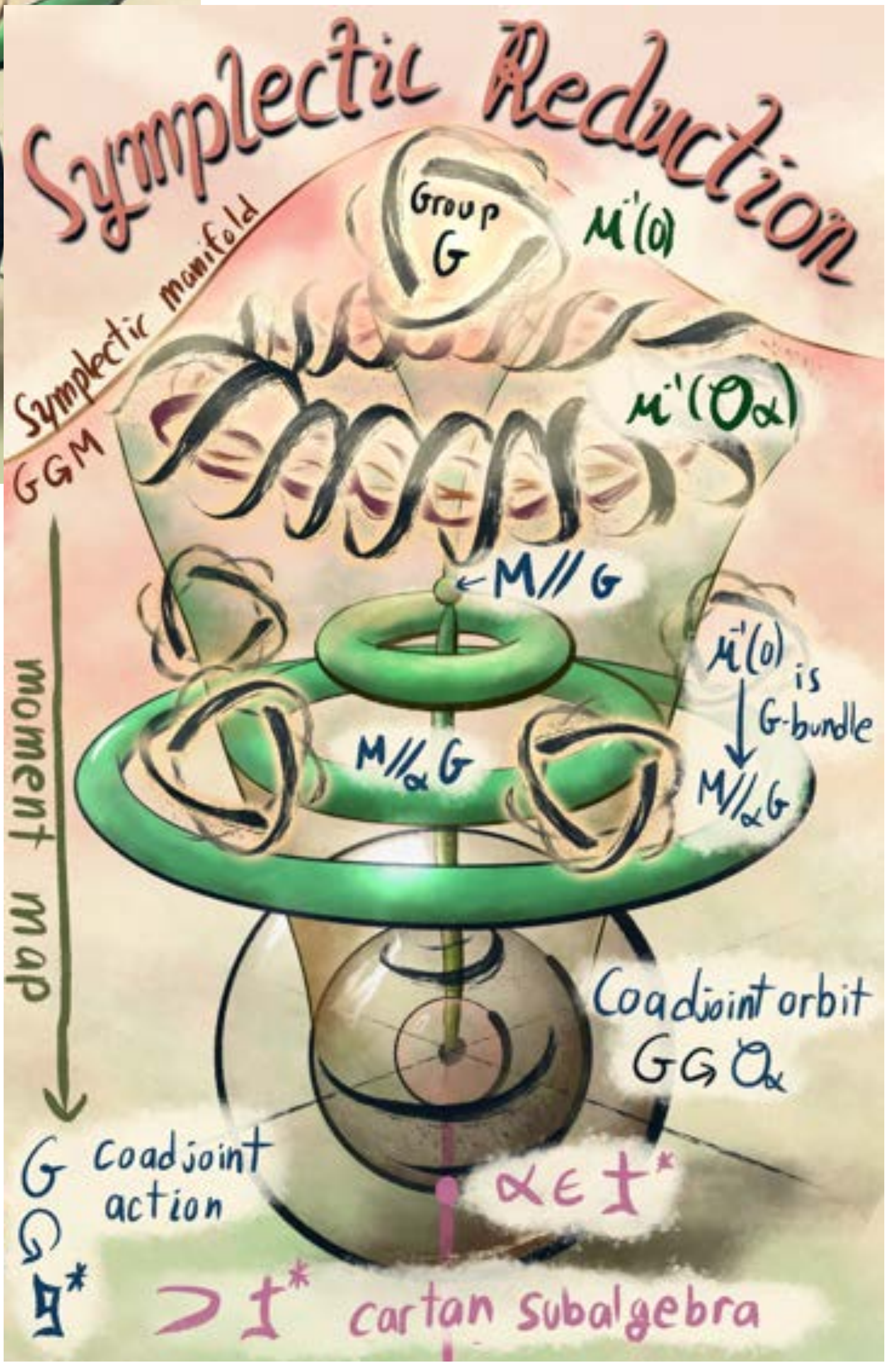
Remark: $\dim M /\!/ G = \dim M - 2\dim G$

motto: in symplectic geometry, $\boxed{\text{Groups act twice!}}$

position & momentum come in pairs, so G acts in pairs

Thm (Guillemin-Shlomo Sternberg): for M compact, Kahler
$$\mathcal{H}(M /\!/_0 G) = \mathcal{H}(M)^G \quad G\text{-invariant vectors in } \mathcal{H}(M)$$

"Quantization commutes w/ reduction"


Symplectic Reduction
Group G
$\mu^{-1}(0)$
Symplectic manifold
$G \circlearrowright M$
$\mu^{-1}(\mathcal{O}_\alpha)$
$\leftarrow M /\!/ G$
$\mu^{-1}(0)$ is G-bundle
$M /\!/_\alpha G$
$M /\!/_\alpha G$
moment map
Coadjoint orbit
$G \circlearrowright \mathcal{O}_\alpha$
$\alpha \in \mathfrak{t}^*$
G coadjoint action
$G \circlearrowright \mathfrak{g}^*$
$\supset \mathfrak{t}^*$ cartan subalgebra

Multiplicities of coadjoint orbits

Reduction along other coadjoint orbits: $M/\!/_\alpha G = \mu^{-1}(\mathcal{O}_\alpha)/G$

note $\dim(M/\!/_\alpha G) = \dim M - 2\dim G + \dim \mathcal{O}_\alpha$

M splits into G-bundles over symplectic mflds, parametrized by coadjoint orbits

$$\begin{array}{c} G \hookrightarrow \mu^{-1}(\mathcal{O}_\alpha) \\ \downarrow \\ M/\!/_\alpha G \end{array}$$

ex: $\mathbb{C}^2 = \bigsqcup_{\alpha\in[0,\infty)} S^3_\alpha = \bigsqcup_{\alpha\in[0,\infty)}$ $\begin{array}{c} S^1 \hookrightarrow S^3_\alpha \\ \downarrow \\ S^2_\alpha \end{array}$ S^2_α, S^3_α denotes spheres of radius α

Motto: moment map splits M into irriducible components $\mathcal{O}$

Thm: (Guilleman-Sternberg) The heighest weight representation $G \circlearrowright E_\alpha$ occours in $G \circlearrowright \mathcal{H}(M)$ iff $\mathcal{O}_\alpha$ occurs in $\mu(M)$

Consider spaces where decomposition is simple:

<u>Def</u> a hamiltonian space $G \circlearrowright M$ is <u>multiplicity-free</u> if $\dim(M/\!/_\alpha G) = 0$ $\forall \alpha$

if $G \circlearrowright M$ compact connected, multiplicity-free $\Rightarrow$ $M/\!/_\alpha G$ is a point

this only uses properness of μ, follows from thm of Kirwan $\mu^{-1}(\mathcal{O}_\alpha)$ connected, using a bit of morse theory

$\Rightarrow$ $G \circlearrowright M$ entirely charecterized by moment map image

Convexity thms $\Rightarrow$ $G \circlearrowright M$ structure is combinatorial!

motto: $\boxed{\text{multiplicity-free manifolds have maximal symmetry}}$

<u>multiplicity-free examples</u>

- coadjoint orbits $\mathcal{O}$ (moment map $\mu: \mathcal{O} \hookrightarrow \mathfrak{g}^*$ is identity $\Leftrightarrow$ $\mathcal{O}$ are irriducible)
- for $G \circlearrowright G/K$ transitive group action, induced action $G \circlearrowright T^*(G/K)$ is <u>multiplicity-free</u> for simple necessary & suffcient condition on G/K
- Toric manifolds: $T = U(1)^n \circlearrowright M$ $\dim M = 2n$
  symplectic manifolds with effective half-dimnl torus action

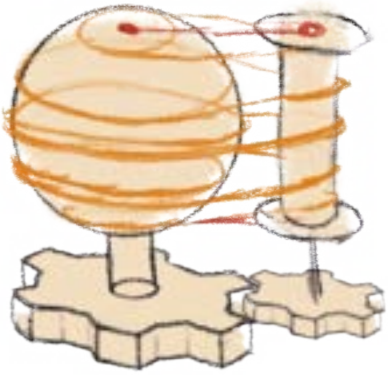
Atiyah: moment map $\mu: M \to \mathfrak{t}^* = \mathrm{Lie}(T)^* \simeq \mathbb{R}^n$ image is a convex polytope

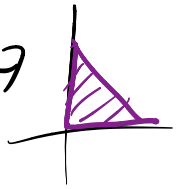
$Ad^*: T \circlearrowright \mathfrak{t}^*$ is trivial $\Rightarrow$ $\mathcal{O}_\alpha = \{\alpha\}$: coadjoint orbits are points

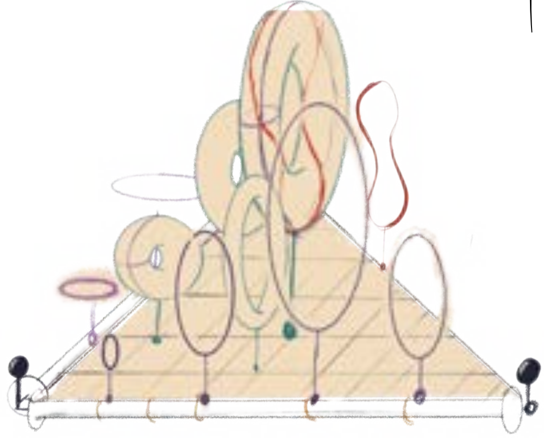
$\mu^{-1}(\alpha)$ is a torus $\Rightarrow$ $M/\!/_\alpha T = \{pt\}$

e.g. $U(1) \circlearrowright S^2$

$S^2 \subset \mathbb{R}^3 \longrightarrow \underline{t}^*$



$U(1)^2 \circlearrowright \mathbb{P}^2$ $\quad \mu : \mathbb{P}^2 \longrightarrow$ 



non-example: $U(1) \circlearrowright \mathbb{C}^2$ $\quad e^{i\theta} \cdot (z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2)$

$\mathbb{C}^2 /\!/ U(1) = S^2 \neq pt.$ for torus actions, <u>need</u> dimensionality constraint

Thm the following are equivalent:

- $G \circlearrowright M$ multiplicity free

  G-invariant functions are determined by their values on $M/\!/_\alpha G$ for all α, which is defined by their value on α

- all G-invariant functions on M lift from $\underline{g}^*$

  i.e, $f \in C^\infty(M)^G \Rightarrow f = h \circ \mu \quad h \in C^\infty(\underline{g}^*)^G = C^\infty(\underline{t}^*)^W$ — weyl-invariant functions on dual cartan subalgebra

  $\underline{t}^*$ is poisson commutative, so this gives

- ✱ all G-invariant functions poisson-commute: $f, f' \in C^\infty(M)^G \Rightarrow \{f, f'\} = 0$

  $\{f, f'\} = \{h \circ \mu, h' \circ \mu\} \overset{\mu \text{ is Poisson}}{=} \{h, h'\} = 0$

Quantum version: all G-invariant $A, B : \mathcal{H} \to \mathcal{H}$ satisfy $[A, B] = 0$

split $\mathcal{H}$ into irreps V_i. A G-invariant $\Rightarrow \begin{cases} A \text{ preserves irrep type} \\ A : V_i \to V_i \text{ is scalar} \end{cases}$ (schur's lemma)

G-invariant operators commute $\Rightarrow$ each irrep type has multiplicity ≤ 1

a G-invariant operator respects decomposition into G-irreps

$$\begin{array}{ccccc} \mathcal{H} = & V_1^{n_1} & \oplus & V_2^{n_2} & \oplus \cdots \\ {\scriptstyle \text{G-invariant } A} \downarrow & \downarrow & & \downarrow & \\ \mathcal{H} = & V_1^{n_1} & \oplus & V_2^{n_2} & \oplus \cdots \end{array}$$

$A : V_1^{n} \to V_1^{n}$

$$\begin{array}{c|c} V_1 & \begin{bmatrix} a_{11} I & & \\ & \ddots & \\ & & a_{nn} I \end{bmatrix} \\ \vdots & \\ V_1 & \\ \hline & \underbrace{\quad}_{V_1} \cdots \underbrace{\quad}_{V_1} \end{array}$$

by schurs lemma, A acts block-scalarly, interchanging factors of V_i

$\boxed{\text{all G-invariant } A, B \text{ commute} \iff \text{each } V_i \text{ appears once}}$

Thm (GS) $G \circlearrowright T^* X$ multiplicity free $\Rightarrow$ each irrep in $G \circlearrowright L^2(X) = \mathcal{H}(T^*X)$ has multiplicity ≤ 1

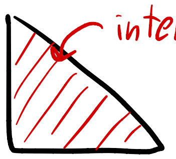
fav multiplicity-free spaces:

T^*X multiplicity free $\longrightarrow$ $G \curvearrowright X$ transitive, or $X = G/K$

note $G \curvearrowright X$ extends to complexified action $G^{\mathbb{C}} \curvearrowright X$

Def: $G^{\mathbb{C}} \curvearrowright X$ is a spherical variety if it has an open dense orbit of the Borel subgroup $B \subset G^{\mathbb{C}}$

ex: toric manifolds



interior is B-orbit of $T^{\mathbb{C}} = \mathbb{C}^{*n}$

intuition: B-orbit moves you all along moment polytope in $\mathfrak{t}^{*+}$. this takes you thru the whole manifold if $M/\!/_{\xi} G = pt$

Thm: T^*X multiplicity free $\Leftrightarrow$ X spherical

T^*X are archtype "of hyperspherical" varieties

Def 3.5.1 $G \curvearrowright M$ is hyperspherical if:

(1) M affine
(2) M is multiplicity-free
(3) the moment map of M intersects the nilpotent cone
(4) the G-stabilizer of a generic pt in M is connected
(5) M carries a $\mathbb{C}^*_{gr}$ action (the "grading" action) which is "neutral"

true for whitt after inductions

implies it is a whitacre induction

Notes: - if $M = T^*X$, $\mathbb{C}^*_{gr}$ scales fibers (needed structure for later)

(3) implies $\mu(M) \cap \mathfrak{t}^*$ contains 0, & $\mu^{-1}(0)$ is a G-orbit

w/ $\mathbb{C}^*_{gr}$, implies $\mu(M) \cap \mathfrak{t}^* = \mathfrak{t}^*$: every coadjoint orbit appears

(5) "neutrality" implies we can reconstruct $G \curvearrowright M$ w/ grading action from local data of a point $x \in \mu^{-1}(0)$

Building Multiplicity-free spaces

convert constructions from representation theory to symplectic.

induced representations

for $H < G$, & rep $\rho: H \circlearrowright V$, we build the induced rep: $\mathrm{ind}_H^G(\rho)$

G/H is a symmetric G-space, $G = SO(3)$, $H = U(1)$, $G/H \simeq S^2$

$H \hookrightarrow G \downarrow G/H$ principle H-bundle

using ρ, build associated bundle $E_\rho = G/_H \times_H V = G \times V / H$

$\Rightarrow$ obtain G-rep. on sections of E_ρ $\quad \mathrm{ind}_H^G(\rho): G \circlearrowright L^2(E_\rho)$

G acts by either interchanging fibers, or rotating fibers by H

if G is finite, $L^2(E_\rho)$ is a finite dimensional rep.

Symplectic analouge: replace $H \circlearrowright V$ w/ hamiltonian H-space $H \circlearrowright S$

Def the Hamiltonian induction is $\quad h\text{-}\mathrm{ind}_H^G S = (S \times T^*G) /\!/_0 \, H$ $\quad$ need symplectic result

$E_\rho = V \times G / H$

in symplectic geo, groups act x2! $\quad \dim h\text{-}\mathrm{ind}_H^G(S) = \dim S + 2\dim G - \dim H$

ex: $h\text{-}\mathrm{ind}_H^G (T^*Y) = (T^*Y \times T^*G) /\!/ H = T^*((Y \times G)/H)$

in rep theory, we build reps as sub-reps induced from simpler reps!

$G > H = 1$

representation theory	symplectic
trivial rep $\mathbb{C}$	$S = \{pt\}$
bundle $G/_1 \times_H \mathbb{C} = \begin{matrix} \mathbb{C} \times G \\ \downarrow \\ G \end{matrix}$ trivial	$h\text{-}\mathrm{ind}_1^G(\{pt\}) = \{pt\} \times T^*G /\!/_1 = T^*G \simeq G \times \mathfrak{g}^*$
rep $\mathrm{ind}_1^G(\mathbb{C}) = L^2(G)$	moment map $\mu: (g, \dot{g}) \to \dot{g}$
irrep decomp $L^2(G) = \bigoplus_{\text{irreducible}} V_i^{\otimes \dim V_i}$ (peter-weyl thm)	$\mathrm{im}\, \mu = \mathfrak{g}^* \Rightarrow$ every coadjoint orbit appears

every irreducible component occurs in reps. induced from trivial!

Whittaker induction

G reductive, B a borel subgroup, U its unipotent radical
$\begin{pmatrix} * & * \\ * & * \end{pmatrix}$ $\begin{pmatrix} * & * \\ & * \end{pmatrix}$ $\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$

I only know this from wikipedia lol

a **Whittaker model** of a representation $\rho: G \to V$ is a realization of ρ as a sub rep of $\mathrm{ind}_U^G(\chi)$ for $\chi: U \to \mathbb{C}$ all irreps $U \to \mathbb{C}$ are characters

these are common! irreps w/ a whittaker model are "generic"
irreps w/o ——— " ——— are "degenerate"

these are interesting, #-theoritic criteria, that encompas most reps we care about

symplectically: χ defines hamiltonian action $U \circlearrowright \mathbb{C}$
following symplectic induction, we consider

$(\mathbb{C} \times T^*G) /\!/ U = \begin{array}{c} \Psi \\ \downarrow \\ T^*(U\backslash G) \end{array}$ $\mathbb{C}$ - principle bundle over $T^*(U\backslash G)$

Basic Whittaker space is $\Psi /\!/ \mathbb{C}$: Twisted cotangent bundle $T^*(U\backslash G)$
(same topology, new symplectic form)

whitacre induction sends $S \circlearrowleft H$ to G-space using homo. $H \times SL_2 \to G$
the morphism $SL_2 \to G$ picks out a unipotent $U \subset G$ (associated to positive eigenspace)

Whittaker induction is h-$\mathrm{ind}_{HU}^G(S)$ or something...

note that I'm kinda lying here
this interpolates between ordinary induction & the twisted bundles of whittaker spaces.

Thm (3.6.) of relative langlands program):
All hyperspherical varieties are built by whittaker induction

Dictionary between symplectic geo & rep theory

symplectic manifolds $\overset{\text{quantize}}{\rightsquigarrow}$ representations

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$G \circlearrowright M$ hamiltonian G-space — $G \circlearrowright \mathcal{H}(M)$ G-rep on hilbert space

$M/\!/_0 G$ symplectic reduction —[GS 84]→ $\mathcal{H}(M)^G$ G-fixed vectors

Guillemin-sternberg 1982: geometric quantization and multiplicities of group representations

coadjoint orbit $\mathcal{O}$ of G — irrep of G

$\mathcal{O}_\alpha \subset \mu(M)$ — highest weight rep E_α occurs in $\mathcal{H}$

for $H \circlearrowright S$, $H < G$,
$h\text{-ind}_H^G(S) = (S \times T^*G)/\!/H$ symplectic induction — $\text{ind}_H^G(\mathcal{H}(S))$ induced representation

Kazdan - Konstant- Sternberg 1978: Hamiltonian group actions and dynamical systems of calogero type

$G > H \circlearrowright M$ treating G-space as an $H<G$ space — $\text{Res}_H(\mathcal{H}(M))$ restriction of rep. to H

$h\text{-ind}_H^G(S)/\!/_\alpha G \simeq$???

$\varphi: G \to \mathcal{H}$ $\quad \psi: H \to \mathcal{H}$

$\langle \varphi, \text{ind}_H^G(\psi)\rangle = \langle \text{Res}_H(\varphi), \psi\rangle$ frobenius reciprocity

Guillemin-sternberg 1983: The frobenius reciprocity theorem from a symplectic point of view

whitakker induction — whitakker model

Extra: Spherical Varieties

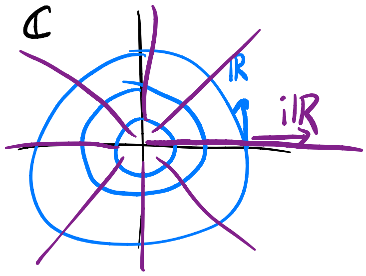
every hamiltonian $G \circlearrowright M$ extends to $G^{\mathbb{C}} \circlearrowright M$

infinitesimal action $\mathfrak{g}^{\mathbb{C}} \to \mathrm{Vect}(X)$ splits $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g}$

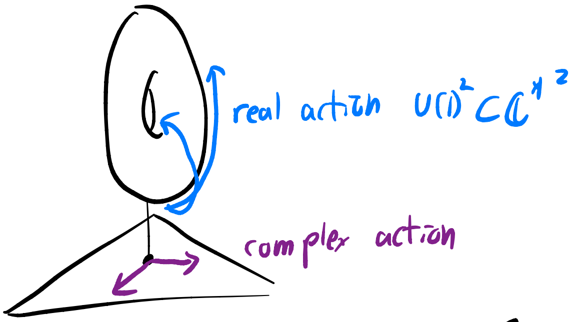
$\mathfrak{g} \to$ hamiltonian vector field $J \nabla H$

$i\mathfrak{g} \to$ gradient vector field ∇H

ex: $U(1) \circlearrowright \mathbb{C}$ extends to $\mathbb{C}^* \circlearrowright \mathbb{C}$



ex: $\mathbb{C}^{*2} \circlearrowright \mathbb{P}^2$



remark (Kempf-Ness thm) $\mathbb{C}[M /\!/_0 G] = \mathbb{C}[M]^{G^{\mathbb{C}}}$

roughly, $M /\!/_0 G = \mu^{-1}(0)/G = M/G^{\mathbb{C}}$ "GIT quotient"

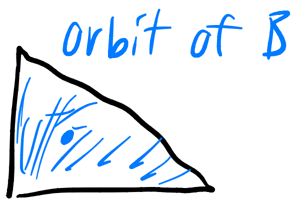
so, setting value of μ is like choosing complex part of $G^{\mathbb{C}}$ action

the complex version of a maximal torus is a borel subgroup B

$G^{\mathbb{C}}/B = G/T$ $\begin{bmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{bmatrix}$ $\begin{bmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{bmatrix}$

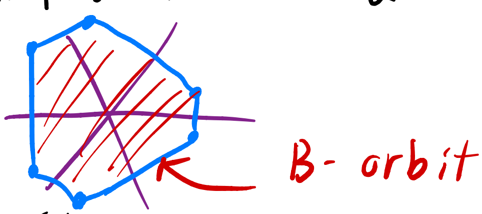
Def a $G^{\mathbb{C}}$ variety $G^{\mathbb{C}} \circlearrowright M$ is *spherical* iff M has a dense B-orbit

ex: toric varieties have $T = U(1)^n$, $B = T^{\mathbb{C}} = \mathbb{C}^{*n}$



flag manifolds: moment map of T on $\mathcal{O}_\lambda$

$G = SU(3)$:



$G^{\mathbb{C}} \circlearrowright M$ spherical then hamiltonian $G \circlearrowright M$ multiplicity-free

Somehow, multiplicity-free manifolds are locally modeled by spherical varieties.