Hyperbolic String Art

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Abstract

Stretching straight lines across a circle according to mathematical rules produces emergent patterns known as *string art*. We re-contextualize string art, envisioning the circle as the circle at infinity of the hyperbolic plane. The strings stretch across the Beltrami-Klein model of the hyperbolic plane, each line a hyperbolic geodesic. We examine the string art coming from Möbius transforms, by studying the envelope of the strings, the curve tangent to every string. We describe the envelopes of a Möbius transform in terms of the underlying hyperbolic symmetry: elliptic Möbius transforms give hyperbolic circle envelopes, parabolic transforms give horocycles, and hyperbolic transforms give hypercircles. To visualize these envelopes, we use the Poincaré disc model, rendering each string as a circular arc orthogonal to the boundary. This draws all envelopes described above as Euclidean circles. We conclude with a purely aesthetic application, showing a hyperbolic string art fractal.

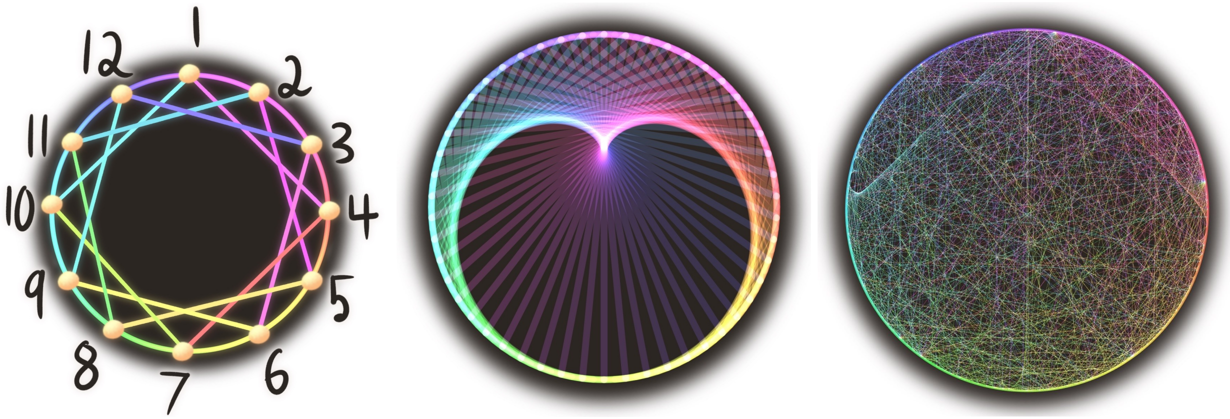
Introduction

String art (or *curve stitching*) is an art form composed of thin strings stretched between points. Placing the strings according to mathematical rules results in aesthetic designs that illustrate the mathematics of the generating rule. For example, in a classic classroom activity, students place 12 equally spaced pins around a circle, labeled 1 to 12. Each student stretches strings from the pin labeled i to that labeled $i + 3$ modulo 12. (See Figure 1a. For more detailed instructions, consult [3, p.25].) The strings accumulate around a smaller central circle, manifesting the rotational symmetry of modular addition. The smaller circle is a curve built from straight lines. It is tangent to each string and "envelopes" the family of strings, so is called the *envelope*. Modular multiplication is similarly illustrated by stretching strings from i to $2i$. The strings once again accumulate around their envelope, the heart-shaped cardioid (Figure 1b, see [3, p.44] for detailed instructions.)

Other common functions (squaring, exponential, etc.) produce noisy results without an envelope (see Figure 1c). These functions are not natural with respect to the modular structure, so their string art lacks obvious mathematical or aesthetic patterns. So, we change perspective, replacing the integers labeling the pegs with their angle around the circle. The strings are defined by maps from the circle to itself. Modular addition (Figure 1a) is a rigid rotation, while modular multiplication by two (Figure 1b) is the angle doubling map. To make string art, we choose a set of input points, and stretch a line from each point to its image under the defining map. This viewpoint suggests several other natural functions, like sine and cosine, which produce aesthetic envelopes (See Figure 2). To emphasize the envelope, we draw the strings so thin to be individually invisible.

In this paper, we explore a further leap in perspective, exploring the geometry of the disc that the strings stretch across. We introduce *hyperbolic string art*, which identifies the disc with the hyperbolic plane. Strings are straight lines across the disc, representing geodesics (hyperbolic lines [1]) in the Beltrami-Klein model of the hyperbolic plane. The bounding circle is now hyperbolic infinity (or "the circle at infinity," or the "ideal points"), the set of possible directions a geodesic can point. Hyperbolic geometry suggests several natural functions that map the circle at infinity to itself. In particular, symmetries of the hyperbolic plane extend as Möbius transforms to the circle at infinity. We will explore the string art coming from these

[1] In this paper, we will refer to lines in hyperbolic space as geodesics, and lines in Euclidean space as lines



(a) 12 *pins,* $i \longrightarrow i + 3$ *modulo* 12 **(b)** 50 *pins,* $i \longrightarrow 2i$ *modulo* 50 **(c)** 200 *pins,* $i \longrightarrow i^2$ *modulo* 200

Figure 1: *String art illustrating modular arithmetic. Colors chosen from the starting angle of each string*

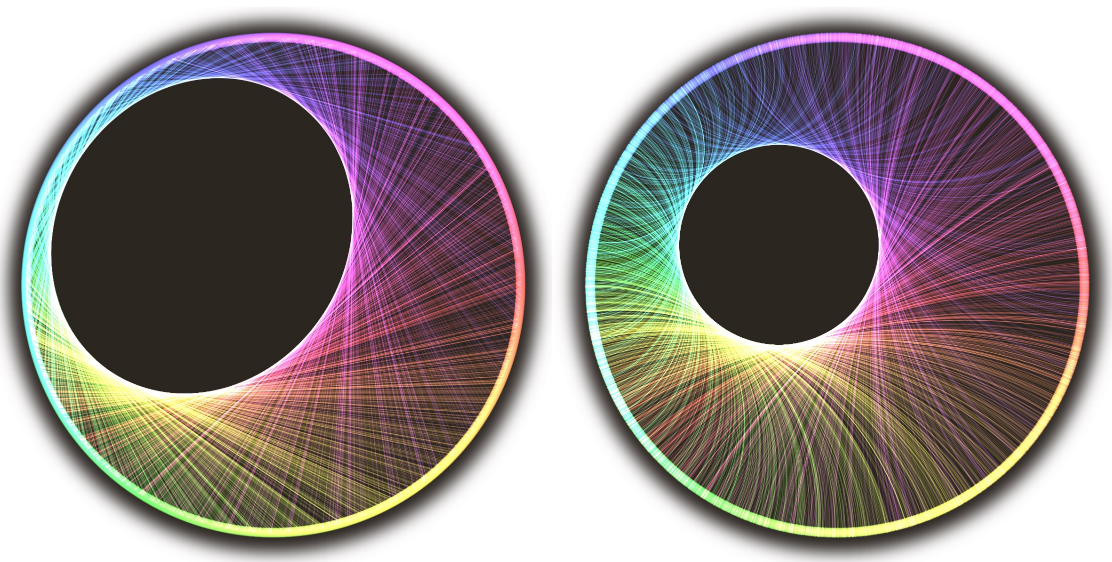
Figure 2: *String art for the function* $\theta \longrightarrow 2\left(\theta + \frac{1}{2}\sin(\theta)\right)$*, drawn with 1000 equally spaced strings.*

maps, understanding their envelopes and relating them to the underlying symmetries of the hyperbolic plane. Through this, string art artistically visualizes properties and themes of hyperbolic geometry. We also show purely artistic applications, such as fractal string art.

This work follows a long history of string art interconnecting math and aesthetics. The activity was invented by Mary Everest Boole to make geometry concepts accessible to children. It has since branched into many variations. These include 2D string art with non-circular boundaries (see [3] for many examples), 3D sculpture (see [3, chap. 7]), and computational image generation from string art (For example, [1]). The choice to render strings as circles perpendicular to the bounding circle was used for aesthetic reasons in [4]. Robert Bosch used hyperbolic string art to render a portrait of Henri Poincaré [1]. This work follows Boole's tradition, and uses hyperbolic string art to elucidate the underlying of hyperbolic geometry.

String Art in the Hyperbolic Plane

Let us first recap the necessary aspects of hyperbolic geometry. (See, for instance, chapter 2 of [2].) The hyperbolic plane is a two-dimensional plane with constant negative curvature. Like the Euclidean plane, there are geodesics (which act as lines), angles, distances, and an axiomatic approach to geometry analogous to Euclid's. However, the negative curvature forces geodesics to curve away from one another, possibly never



(a) *Beltrami-Klien model* **(b)** *Poincaré model*

Figure 3: *String art of a Möbius transform, shown in two different models of hyperbolic space.*

intersecting. This is a failure of Euclid's fifth postulate, as there are infinitely many geodesics through a given point, which don't intersect with a given geodesic (hereafter, we say the geodesics are "parallel"). Despite this, we can still draw the hyperbolic plane on Euclidean paper using *models*. These are akin to flat maps of the spherical earth, and like maps, every model has its own uses. Let us start with the *Beltrami-Klein model*, where the hyperbolic plane is drawn as the interior of a unit disc, and geodesics are drawn as straight lines. A fixed hyperbolic distance looks smaller the closer it is to the boundary of the disc, vanishing at the boundary. In fact, the bounding circle is infinitely far from any point inside the disc, hence its moniker "the circle at infinity."

Note that every line passing through the interior of the unit disc intersects the boundary circle twice, and is uniquely defined by its intersection points. Translating to hyperbolic geometry, each geodesic is uniquely defined by its two asymptotic directions, each a point at hyperbolic infinity. This manifests a creed of hyperbolic geometry: *an object in the hyperbolic plane is uniquely determined by its behavior at hyperbolic infinity.* This stands in contrast to Euclidean space, which also has a circle of possible asymptotic directions for lines, but forward and backward directions of a line are necessarily opposite, and there are parallel lines with the same asymptotic forwards and backwards directions. The core insight of hyperbolic string art is to treat the boundary of the disc as hyperbolic infinity and to treat strings as hyperbolic geodesics.

The hyperbolic plane carries similar symmetries (or isometries) as the Euclidean plane, including rotation around a point and translation along a direction. Unlike Euclidean space, hyperbolic translations and rotations mix together, as translation along a loop results in a net rotation. This mixing is captured in the symmetry group of the hyperbolic plane $SL(2,\mathbb{R})$, which consists of 2×2 real matrices with determinant one. Hyperbolic symmetries send geodesics to other geodesics, hence they naturally act on the space of asymptotic directions. They restrict to functions mapping the circle at infinity to itself, known as *Möbius transforms*. In fact, hyperbolic symmetries are uniquely defined by their action at infinity, manifesting the above hyperbolic creed.

Let us form string art from a hyperbolic symmetry. First, we choose a symmetry, such as rotation around a point. This acts on the circle at infinity via a Möbius transform. Then we draw the string art for this map, shown in Figure 3a. The envelope forms an oval shape, surrounding the pivot point of the rotation.

However, the Beltrami-Klein model obfuscates hyperbolic shapes, because it distorts angles. To understand the envelope's shape, we switch to the *Poincaré disc model*, which preserves hyperbolic angles. The Poincaré disc model is based on the unit disc, but geodesics are drawn as circular arcs intersecting the boundary circle at right angles. Using these circle arcs to render the strings from Figure 3a results in Figure 3b. In the Poincaré disc model, the envelope appears to be a perfect circle, and the center of the envelope circle is the original pivot point of the hyperbolic rotation. This is not a coincidence. In the next section, we will explain why the envelope is a hyperbolic circle, and discuss the envelopes for all other types of Möbius transforms.

Möbius Transform Taxonomy Through String Art

Orientation-preserving symmetries of the hyperbolic plane fall into three classes: rotations, parabolic transforms, and translations. Their associated Möbius transforms are called elliptic, parabolic, and hyperbolic respectively. These are illustrated in Figure 4: The orange arrows circumnavigating the border circles show how the Möbius transform acts on each point of the circle at infinity. An elliptic Möbius transform has no fixed points at infinity and comes from a hyperbolic rotation about the point marked in purple. Moving to column 2, a parabolic Möbius transform has one fixed point at infinity, shown by the orange dot. The corresponding symmetry of the hyperbolic plane does not have a description familiar to Euclidean beings. Shown in column 3, a hyperbolic Möbius transform pulls points at infinity away from a repelling fixed point, and towards an attracting fixed point. This is the boundary action of hyperbolic translation along a geodesic which starts at the repelling point and ends at the attracting point called the *axis* of translation (marked in purple).

Algebraically, this trichotomy arises from classifying the conjugacy classes of the associated elements of $SL(2, \mathbb{R})$. Each conjugacy class is uniquely defined by its trace. When the trace is > 2 (resp. $= 2, < 2$), the associated Möbius transform is called elliptical (resp. parabolic, hyperbolic). Geometrically, a conjugacy class represents symmetries of the hyperbolic plane up to change in perspective. All hyperbolic rotations by θ degrees are conjugate to one another by a translation: translate one pivot point to another, then rotate by θ, then translate back. The angle θ defines a conjugacy class and is uniquely determined by the trace. Parabolic transforms all have trace 2, so are conjugate to one another. Finally, hyperbolic translations are conjugate by rotations, and the only invariant under conjugacy is the (signed) length of translation. Each class of Möbius transforms can be understood from a single family of basic transforms, such as rotation around a fixed point or translation along a fixed line. We will use this technique to study the string art of Mobius transforms. These are shown in Figure 4, with their envelopes shown in green. The descriptions of each envelope are defined and justified below.

Elliptic Möbius Transforms

As mentioned above, elliptic Möbius transforms all arise from hyperbolic rotations. We start with the simplest rotation, pivoting around the center of the Poincaré disc. This acts on the boundary circle by rigid rotation. All the strings are thus circular arcs of the same radius. The envelope is a circle with radius equal to the smallest distance between one string and the center. To see this, consider the distance function to the center of the Poincaré disc. At the minimum distance along one string, the string must be tangent to the level set of the distance function, which is a circle (this is a geometric realization of the method of Lagrange multipliers). By rotational symmetry, every string has the same minimal distance, so all strings are tangent to the same level set. Hence, the envelope is a circle. Note that this argument also justifies the circular envelope in string art for modular addition, shown in Figure 1a. The Poincaré projection preserves angles, so hyperbolic circles are drawn as Euclidean circles. Thus, the envelope is the image of a hyperbolic circle.

Now we extend the simple case of rigid rotation R to all elliptic Möbius transforms. Let us translate

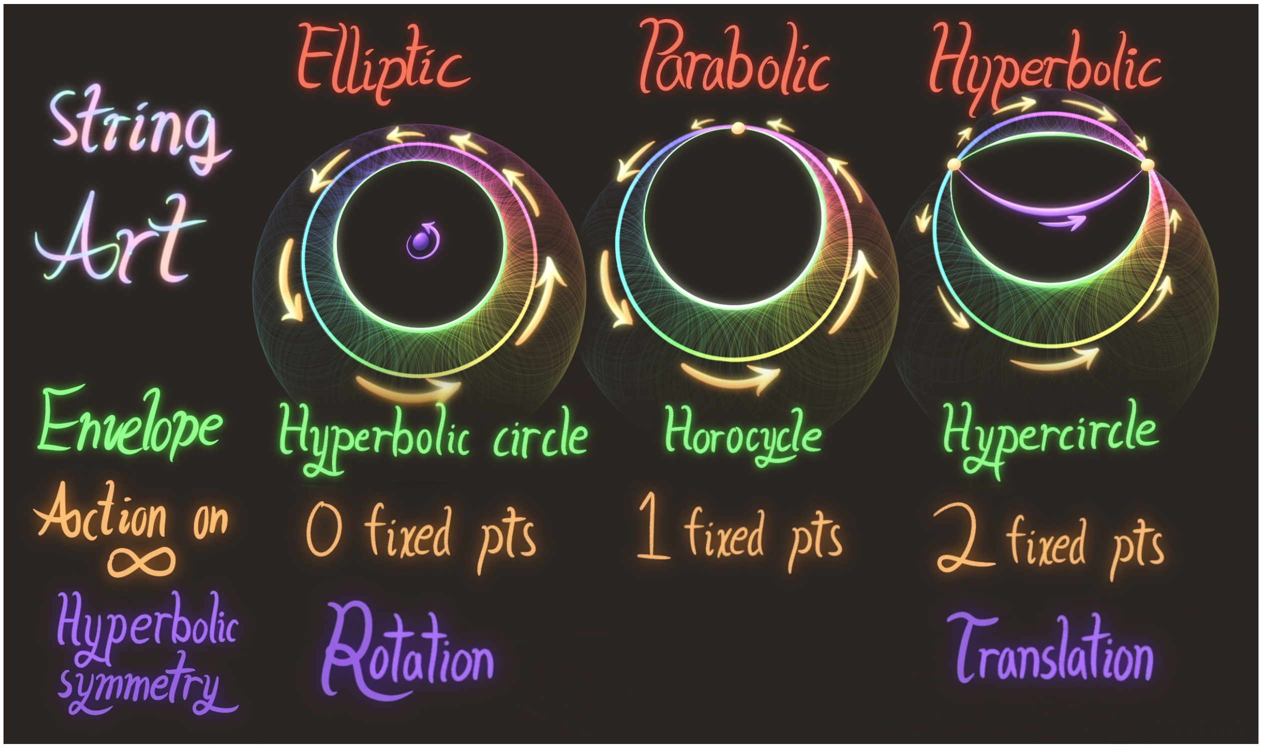


Figure 4: *Description of string art for each type of Möbius transform.*

the string art of rigid rotation by a hyperbolic translation T_p sending the center of the Poincaré disc to a point p. Since T_p is a hyperbolic symmetry, it sends geodesics to geodesics. After translating, the geodesics form string art from a new function f, which maps the start of each translated geodesic to its endpoint. Symbolically, if d is a point at hyperbolic infinity, the function f maps $T_p(d)$ to $T_p(R(d))$. Labeling $T_p(d)$ as d', we see

$$f : d' \to T_p R T_p^{-1}(d')$$

So, the image is the string art for the Möbius transform $T_p R T_p^{-1}$, which is a hyperbolic rotation around the point p. Every elliptic Möbius transform is conjugate to R in this way.

The image under the translation T_p of the envelope for rigid rotation R gives the envelope for rotation around p, $T_p R T_p^{-1}$. However, the envelope for R is a hyperbolic circle. The translation T_p is an isometry, so it sends hyperbolic circles to hyperbolic circles. Thus, *the envelope of any elliptic Möbius transform is a hyperbolic circle.* This is drawn in the Poincaré disc as a Euclidean circle, as seen in column 1 of Figure 4.

Parabolic Möbius Transforms

We establish the string art for parabolic transforms following the blueprint of the above argument. First, we choose a specific model of hyperbolic space and specific parabolic transform with a very simple description. In particular, it will be realized by a Euclidean isometry of the model. Then, we find the envelope and describe it intrinsically with hyperbolic geometry. Finally, we describe the string art of all other parabolic Mobius transforms by conjugating the simple case. We notice that our intrinsic description of the envelope is preserved by isometries, and conclude that all envelopes of parabolic transforms share the same intrinsic description.

We will describe our base parabolic transform in the Poincaré half-plane model, shown in the right half of Figure 5. This consists of the complex numbers with positive imaginary part, endowed with a constant

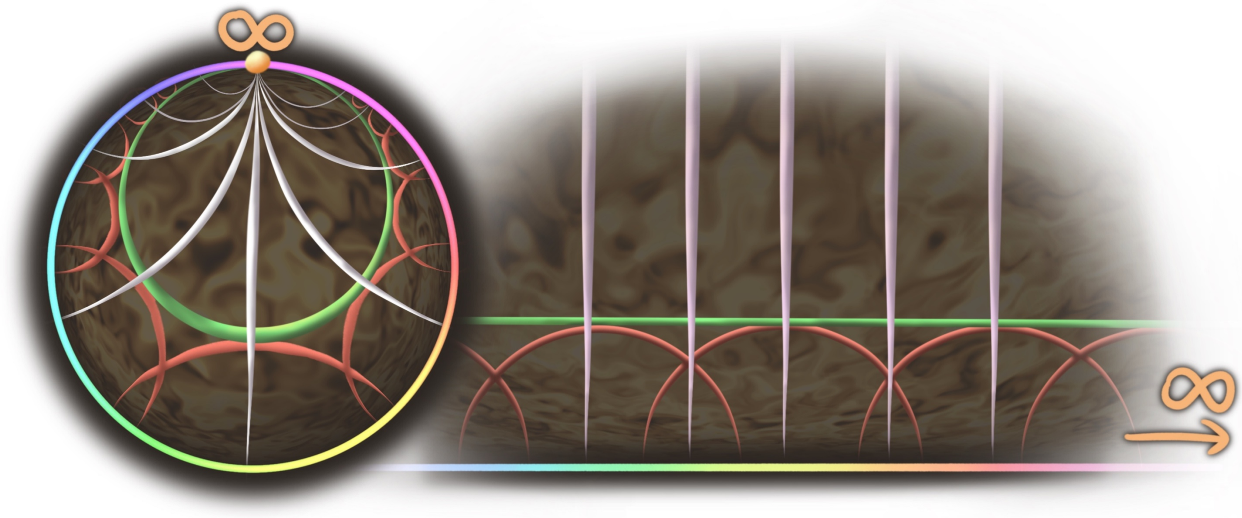


Figure 5: *String art for a parabolic Möbius transform in the Poincaré disc (left) and half-plane(right) models. The rainbow boundary is hyperbolic infinity, the red curves are geodesic strings, the green curve is the envelope of the strings, and the white shows normal geodesics to the envelope.*

negative curvature metric which scales down distances near the real line (visualized in Figure 5 by the scale of the background pattern). This metric is invariant under translation by a real number. Hyperbolic infinity consists of the real line (drawn in rainbow), along with the point at infinity in the complex plane. Geodesics consist of semicircles intersecting the real line orthogonally. Symmetries of the hyperbolic plane extend to the real line as Möbius transforms, explicitly written as fractional linear transforms:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}) \qquad A : x \longrightarrow \frac{ax + b}{cx + d}$$

All parabolic Möbius transforms are conjugate to those of the form $x \longrightarrow x + 1$, which are induced by a translation of the complex plane.

The strings for this action are semicircles stretching from x to $x + 1$ on the real line (some examples are drawn in red in Figure 5). The envelope is a horizontal line at height $1/2$ (drawn in green.) Indeed, every string has maximal height $1/2$, so the level set of the height function at $1/2$ is tangent to every string. Returning to the hyperbolic plane, this horizontal line is a *horocycle*, A curve whose perpendicular geodesics all converge to the same point at hyperbolic infinity. Indeed, the perpendicular geodesics of a horizontal line are vertical lines (infinitely large semicircles, drawn in white), which converge at infinity in the half-plane model. Convergence is clear after converting back to the Poincaré disc model, where the horocycle is drawn as a circle tangent to the top of the boundary circle (see the left half of Figure 5).

All parabolic Möbius transforms are conjugate to that described above. Following the argument for elliptic transforms, the string art of every parabolic transform is related by hyperbolic symmetries. These symmetries preserve horocycles, so *the envelope of a parabolic Möbius transform is a horocycle*. In general, horocycles are drawn in the Poincaré disc model as circles tangent to the boundary circle, as seen in column two of Figure 4.

Hyperbolic Möbius Transforms

For hyperbolic Möbius transforms, we follow the same strategy as above. We describe the base hyperbolic translation in the *band model*, shown on the right side of Figure 6. This consists of the strip of complex numbers z with $\mathrm{Im}\, z \in (-1, 1)$, endowed with a constant negative curvature metric that scales distances

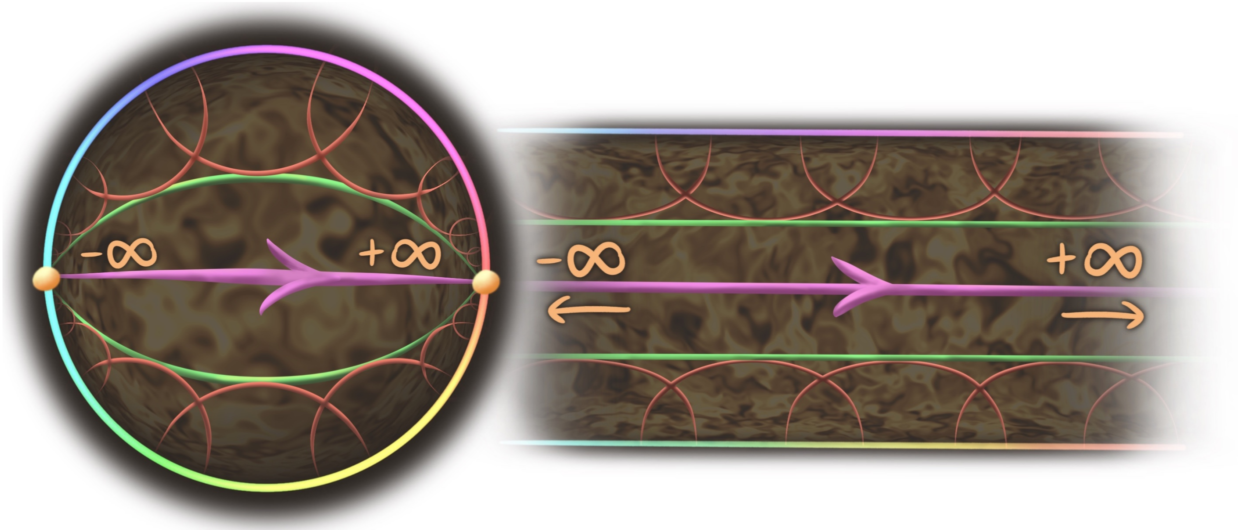


Figure 6: *String art for a hyperbolic Möbius transform in the Poincaré disc model (left) and the band model (right). The rainbow boundary is hyperbolic infinity, the red curves are geodesic strings, the green curve is the envelope of the strings, and the purple geodesic is the axis of the hyperbolic translation.*

to zero at the top and bottom line (visualized again by the scaling of the background pattern). The metric is translation invariant under $z \to z + b$ for real b. Hyperbolic infinity consists of top and bottom lines $\mathrm{Im}\, z = \pm 1$, along with the two points at infinity of the real line, denoted $\pm\infty$. A geodesic stretching between two points on the same boundary line is not a semicircle, but is a convex shape that does not cross the real axis (some examples are drawn in red). In particular, each geodesic has a unique point closest to the middle of the band.

Every hyperbolic translation is conjugate to a Euclidean translation $z \to z + b$ in the band model. The fixed points of this action are $\pm\infty$, and the geodesic connecting the two fixed points is the real axis, drawn in purple in Figure 6.[2] The string art consists of geodesics connecting z with $z + b$ with $\mathrm{Im}(z) = \pm 1$. Every string is the same shape, so has the same minimal distance to the real axis. Hence, the envelope consists of two parallel lines equidistant to the real axis (drawn in green). This envelope is the locus of points of fixed hyperbolic distance to the geodesic connecting $\pm\infty$, a shape called a *hypercircle*.

Converting back to the Poincaré disc model (Figure 6, left), this hyperbolic Möbius transform has fixed points at the left and right extremes of the unit disc. The geodesic connecting these two points is again the real axis. The hypercircle is drawn as two circular arcs connecting the fixed points at infinity of the translation, mirror to one another about the real axis, as depicted in green. In general, hypercircles about a given geodesic in the Poincaré disc are drawn as a pair of circular arcs sharing the endpoints of that geodesic. See the string art for the hyperbolic Möbius transform in Figure 4.

Once again, we realize the string art of all hyperbolic Möbius transforms by applying a hyperbolic symmetry to the string art above. Since hyperbolic symmetries preserve the hyperbolic distance to geodesics, the image of the envelope is still a hypercircle. Hence, *the envelope of a hyperbolic Möbius transform is a hypercircle*. The only conjugacy invariant of hyperbolic translations is the translation length, which controls the angle of the envelope at infinity. For example, small translations will yield strings that stay far from the translation axis, so the envelope will have a wide angle.

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[2] Compare to the parabolic transform, which was realized as a Euclidean translation with a single fixed point at Euclidean infinity. The band model realizes a hyperbolic transform as a Euclidean translation, with two fixed points at Euclidean infinity.

Generalities About Möbius Transforms

In this section, we argued that an elliptic (resp. parabolic, hyperbolic) Möbius transform yields string art with an envelope forming a hyperbolic circle (resp. horocycle, hypercircle). This is summarized in Figure 4. The arguments above can be formalized into a proof without much trouble. In the Poincaré disc model, all three envelope types share a uniform description:

- Elliptic transforms have circular envelopes that remain within the boundary circle.
- Parabolic transforms have circular envelopes tangent to the boundary circle.
- Hyperbolic transforms have circular envelopes intersecting the boundary circle.

The points of intersection with the boundary circle are the fixed points of the transform. The genesis of the two-component envelope for hyperbolic transforms is clarified by extending the strings to full circles (as in Figure 4). Then, the envelope of any transform has two circular components, one being the inversion of the other about the boundary circle. For elliptic and parabolic transforms, this second envelope lies outside the disc, but for hyperbolic transforms both envelopes intersect the disc.

Aesthetic Applications

On top of illustrating mathematics, we can pick string art functions purely for their artistic merit. Figure 7 shows string art inspired by the Mandelbrot set, a fractal built from the repeated dynamics of the map $z \to z^2 + c$. This squares a complex number and then applies a parabolic Möbius transform (shifting by c). Restricting from the Riemann sphere to the unit circle, $z \to z^2$ multiplies angles by 2, and shifting by c is a Möbius transform. A hyperbolic analog is the map $\theta \to 2\ \text{Möbius}(\theta)$. Figure 7 shows the string art for this function, iterated various numbers of times.

Summary and Future Work

Hyperbolic string art recontextualizes string art in the framework of hyperbolic geometry. In this paper, we explored only symmetries, the simplest functions. These produced envelopes with natural hyperbolic descriptions, reflecting the symmetry of the hyperbolic plane. String art visualizes Möbius transforms and hyperbolic geometry in a novel, artistic way. However, hyperbolic geometry cares about more than mere symmetries.

Maps from infinity to itself capture information about hyperbolic tessellations, as a deformed hyperbolic lattice characteristically deforms infinity. Circle maps thus parameterize all hyperbolic lattices, and hence all hyperbolic surfaces [2]. We hope for string art to visualize the boundaries of this space, elucidating the appearance of geodesic laminations. The space of circle maps is also a core object in, ironically enough, string theory: Perhaps one day, we could use string art to visualize string theory.

Acknowledgements

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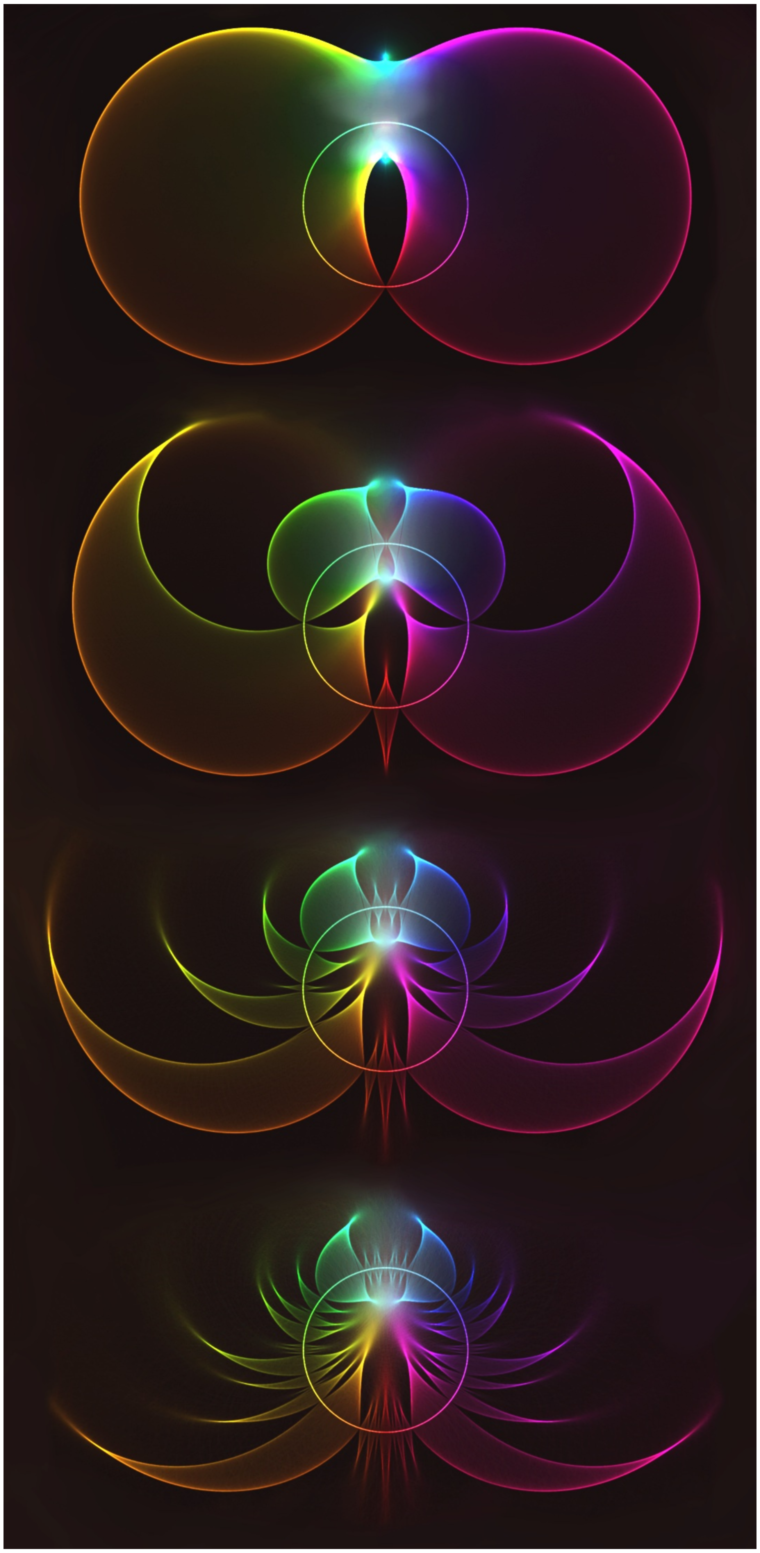

Figure 7: *String art for iterations of the map $\theta \to 2$ Möbius(θ). From the top down, we depict 1, 3, 5, and 7 iterations.*

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