Bunda On the Difference Geometry of Surfaces of Constant Negative Curvature

Von

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(With 8 Figures)

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I. Introduction

In differential geometry, a "Chebyshev net" is understood to be a system of surface coordinates u, v with the special property that u and v measure the lengths of the coordinate lines (in the Gaussian notation, this is determined by E = G = 1). Thus, if one draws the v-lines through a series of equidistant points on the u-axis, and similarly draws the u-lines through a series of points at equal intervals s along the v-axis, a discrete net of generally curvilinear rhombuses with a common side length s is obtained, roughly resembling a fishing net It is obvious that such a net can be laid (at least in certain areas) on any surface, whereby the course of the u- and v-axes can still be arbitrarily determined. This idea of "clothing" a surface with a net or fabric with a fixed mesh size and variable cutting angles had provided inspiration for P. L. Chebyshev and was subsequently taken up again by A. Voss.2

1 P. L. Chebyshev, Sur la coupe des vêtements. A. F. A. S. Paris 1878; Œuvres II, 708. Apart from this short fragment, unfortunately no publication on the subject exists, since the posthumous notes on the matter did not bear an imprimatur

2 A. Voss, On a new principle for mapping curved surfaces. Math. Ann. 10 (1882), 1-26. Furthermore: On equidistant curve systems on curved surfaces. Model catalog of the DMV, edited by W. Dyck (1892).

If one disregards the requirement of a support surface and stretches such a rhombic grid freely, the question arises as to the equilibrium positions it assumes under the influence of external boundary forces, whereby self-weight and elongation may initially be disregarded. That this presents a potentially very interesting static problem is demonstrated by the observation made in 1940 by J. Radon and almost simultaneously by H. Thomas that the rotationally symmetric equilibrium form observed in fish traps, which assumes a tube-like closed rhombic grid stretched between two coaxial circular rings, represents, in the limiting case of infinitesimal compression, a surface of rotation with constant negative curvature and its oscillation lines (see Fig. 1).

Such a model, presented by J. Radon in a lecture on Chebyshev nets given in Vienna on May 8, 1950, tempted the author to investigate it using the methods of difference geometry, thus postponing the limit as far as possible. The key to this was the observation that the four thread segments emanating from each node lie in a plane (for reasons of symmetry). It was natural to consider, more generally, rectilinear rhombic grids with planar nodes (without particular symmetry), and it turned out that these can always be regarded as difference-geometric models of surfaces of constant negative curvature (pseudospheric surfaces) together with their oscillation lines. This agrees with the well-known differential-geometric fact that pseudospheric surfaces are the only ones whose oscillation lines form a Chebyshev net By using the spherical mapping of the grid, which yields rhombic nets formed from great-circle arcs on the sphere, —

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the most important properties of pseudospherical surfaces can be derived in a highly intuitive and entirely elementary way. The rhombic nets with planar nodes can be obtained by attaching

3 J. Radon, On Chebyshev nets on surfaces of revolution and a problem of the calculus of variations. Mitt. Math. Ges. Hamburg 8 (1940), 147-151.

4 H. Thomas, On the question of the equilibrium of Chebyshev nets made of knotted and taut threads. Math. Z. 47 (1942), 66-77.

suitable boundary forces are always obtained as equilibrium positions. Furthermore, the consideration of the associated force diagrams establishes an interesting connection with the movable quadrilateral polyhedra of H. Wiener; these polyhedra, recently rediscovered by R. Sauer and H. Graf, are difference-geometric models of the surfaces considered by A. Voss with a conjugate network of geodesic lines. Due to the reciprocity between truss and force diagram, these models can also be realized as equilibrium positions of networks, specifically those made of unknotted threads; such networks have occasionally been investigated by H. Thomas

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rhombic grids—or rather, rhombic lattices—allow for — The another remarkable realization at their planar nodes, namely by means of lamellae of equal length and twisted by the same amount. If these lamellae (made, for example, of metal), one set of which is to be right-handed and the other left-handed, are hinged together in groups of four at the nodes, a movable lattice is created that, in every state, represents a surface of constant negative curvature and with which any such surface can be approximated using difference geometry. Each lattice mesh forms a skew hinge rhombus, a (special) Bennettian "isogram." If twisted lamellae of another type (of arbitrary length and specific torsion) are attached at all lattice nodes, their ends can be connected by a second lattice of the same type as the first. In this process, the extremely illustrative difference-geometric model of the famous Bäcklund-

[5] H. Wiener, Proceedings of the Society of Natural Scientists and Physicians, 75th Meeting, Kassel 1903, 29–30. Cf. also Wiener-Treutlein, Collection of Mathematical Models (2nd ed., Teubner, 1912), 38 ff.

[6] R. Sauer–H. Graf, On surface bending in analogy to the buckling of open facet surfaces. Math. Ann. 105 (1931), 499–535. A. Voss,

[7] On those surfaces on which two sets of geodesic lines form a conjugated system. Proceedings of the Academy of Sciences, Munich 18 (1888), 95–102

8 H. Thomas, On surfaces on which special types of geodetic lines can be spread. Math. Z. 44 (1939), 233-265. G. T. Bennett, A new mechanism. Engineering

[9] 76 (1903), 777-778

Transformation to be seen which allows further transformations to be derived from a given pseudo-spherical surface.

After completing his investigations, the author became aware of a recently published treatise by R. Sauer, 10 which is devoted to the same set of questions and in which the pseudo-spherical surfaces are represented somewhat more generally by (skew) parallelogram lattices with planar nodes. Naturally, this results in extensive overlaps with the present work Sauer does not seem to have considered the realization by means of a hinge system made of twisted lamellae, which is also possible in the case of a parallelogram lattice. However, the author was recently able to ascertain that G. T. Bennett already possessed a perfectly clear idea of such a lattice model of pseudospherical surfaces, as is unequivocally evident from the concluding remarks of a little-known work. If the author nevertheless undertakes to give a summary presentation of the outlined area of inquiry here, he believes he can justify this, quite apart from some additions, primarily with the importance of the subject matter itself: The fact that it is possible to make a not entirely easily accessible field, such as the differential geometry of pseudospherical surfaces, extremely elementary and to bring it as close as possible to intuition through the application of differential geometric methods is of considerable importance. Moreover, the possibility of constructively and graphically treating the lattices opens up a new field of activity for descriptive geometry, which has always received special attention and development in Austria.

II. Rectilinear Rhombic Lattices with Plane Nodes

A doubly extended system of discretely arranged points in space, Pir, along with the connecting

10 R. Sauer, Parallelogram lattices as models of pseudospherical surfaces. Math. Z. 52 (1950), 611-622.

11 G. T. Bennett, The skew isogram mechanism. Proc. London Math. Soc. 13 (1914), 151-173.

The line segments Pik Pi+1, and Pit Pi, 2+1 are understood. The (real, assumed) points shall be called the nodes, the mentioned line segments the bars (1st and 2nd kind, respectively), and the generally skew quadrilaterals Pik Pi+1, k Pi + 1, k+1 Pi, k+1 the meshes of the grid. We now make the following requirements:

A) All grid bars shall be of the same length s ("Rhombic or diamond grid with mesh size s").

B) Four bars emanating from each node shall lie in one plane ("Grid with planar nodes").

C) The meshes shall not overlap or fold together ("Simple grid")

The plane spanned jointly by the four rods of a node Pik is called the nodal plane, the perpendicular erected at Pik is the lattice normal. Due to restriction C), all nodal planes—and thus all lattice normals—can be oriented uniformly by declaring the direction of rotation for Pik, for example, -π+1, kπ, k+1π-1, kπ, k-1, as positive.

The two nodal planes assigned to the ends of a rod enclose a certain angle, which, according to the orientation, is determined to multiples of 2 and receives a sign when considering the screw that transforms one nodal plane into the other. The value thus determined between - and += is called the twist angle, and the quotient (sin c)/s is called the "twist" belonging to the rod in question

Let us consider a single mesh of a lattice that satisfies the given conditions—whose existence is yet to be proven: We can view it as a rhombus that was originally flat and folded along a diagonal, thus recognizing the two existing planes of symmetry, each containing one diagonal and bisecting the other. From the double symmetry, it immediately follows that the twists are opposite for adjacent bars, but the same for opposite bars. Moving on to the neighboring meshes, we gradually determine that only the two twist angles and occur in the entire lattice

Theorem 1: If all bars in a quadrilateral lattice with planar vertices have the same length, then the angle of adjacent vertex planes is constant in magnitude. Along bars of the same type, the same twisting occurs; along bars of the other type, the opposite twisting occurs.

If, in an arbitrary spatial polygon, we call the connecting planes of successive sides "vertex planes" and define "twisting" here as the sine of the angle of successive vertex planes divided by the side length, then we can pronounce the

Theorem 2: In a simple rhombic lattice with planar vertices, the continuous polygonal chains composed entirely of identical bars have constant twisting of the same magnitude. Their vertex planes are identical to the vertex planes of the lattice

For the purpose of the still outstanding proof of existence for the lattices considered here, we arbitrarily define a space polygon Pok (k=0, +1, +2,...) of the type mentioned in Theorem 2, i.e., with a fixed side length s and a fixed angle of rotation - ; furthermore, another rod Poo P10 of the same length s, lying in the vertex plane of Poo. The two rods Poo Po and Poo Por now span a uniquely determined skew rhombus with angles of rotation +, whose fourth vertex P lies in the vertex plane of Po1: P11 is obtained by rotating the point Poo about the diagonal P10 P01 until it reaches the vertex plane of Po1 for the second time 12 The new side Po1 P11 now determines, in the same way with the rod P01 P09, an equally perpendicular rhombus whose fourth vertex P_{12} lies in the vertex plane of Pos, etc. The sequence of new points P_1 obtained by constantly repeating this rhombus construction, which can also be carried out in the opposite direction, now forms a second equilateral polygon of constant perpendicularity. This therefore leads, after assuming a further rod P10 P20 s lying in the vertex plane of Pro, analogously to a

12 The descriptive-geometric implementation thus requires only elementary constructions.

third similar polygon Pak, etc. The sequence of selected additional bars obviously forms an equilateral spatial polygon Pio (i = 0 ± 1, 2,...) with the fixed skew (sin c)/s, which could be specified as a whole from the outset and would then, together with the first polygon Pok, uniquely determine the rhombic lattice.

Theorem 3: Two equilateral polygons with the same side length and fixed, oppositely equal skew angles, which have a vertex and the corresponding plane in common, uniquely determine a rhombic lattice with planar vertices to which they belong.

By means of a very similar process, applied to a lattice polygon whose sides alternately belong to different sets (and which will therefore exhibit a zigzag pattern), a second possibility of determining the rhombic lattice by such an "alternating polygon" arises

Theorem 4: An equilateral, zigzag-shaped polygon whose sides alternately exhibit the angles of rotation + and , uniquely determines a rhombic lattice with planar nodes to which it belongs.

As an example, Fig. 1 shows the construction of the Radon-Thomas fish trap model, i.e., the rotationally symmetric rhombic lattice with planar nodes. The starting point is the row of nodes, which forms a regular polygon (n=12) assumed to be in the plan plane; the corresponding node planes are normal to the base plane for reasons of symmetry and touch the circumcircle. After specifying the mesh size s, the next row of nodes, and thus an alternating lattice polygon, can be immediately determined. The essential auxiliary lines necessary for the step-by-step construction of the lattice are included in the figure. inneb Jim

Considering that the meshes of a grid exhibit different shapes, it is to be expected that a skew rhombus will allow constraint-preserving deformations. Now, if we denote the angles occurring at opposite corners A and C, and at the junction D, in rhombus ABCD by a and ß respectively (see Fig. 2), then

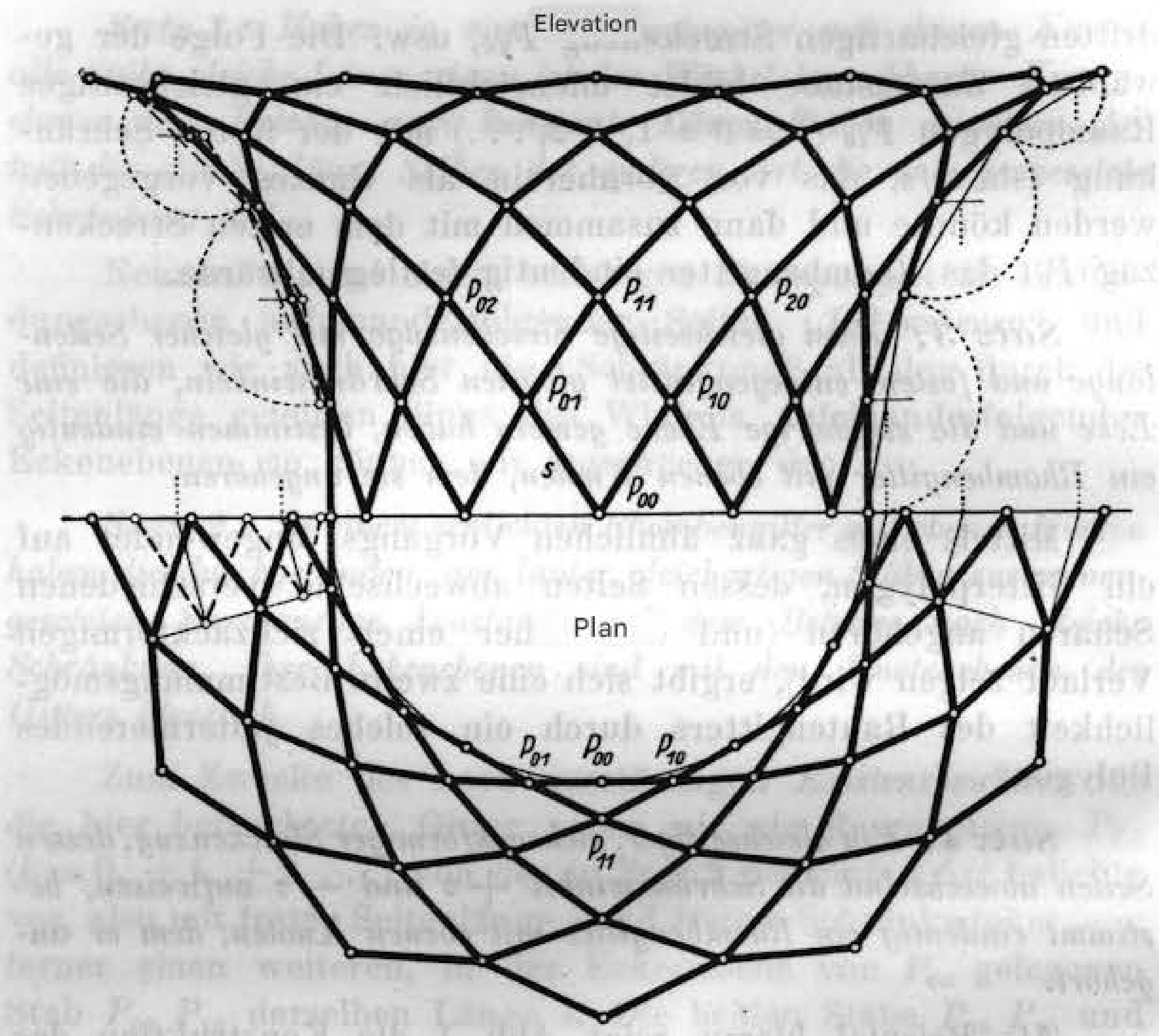

Fig. 1: Partial representation of the rotationally symmetric rhombic lattice with plane nodes. (Radon-Thomas lattice model of a surface of revolution with constant negative curvature).

By solving the isosceles triangular A(BCD) for the angle of rotation, we obtain the condition

$$\cos \sigma = \operatorname{tg} \frac{\alpha}{2} \operatorname{tg} \frac{\beta}{2}. \tag{1}$$

Since the shape of the rhombus is completely determined by a and 3, for a given side length s there are still 11 meshes with the same twist: A skew rhombus with rigid sides allows a continuous deformation of one degree of freedom, in which the twist is preserved. (See Sections VI and VII.) Let the "co-circles" of the rhombus ABCD be those two circles which each contain a pair of opposite vertices, touch the nodal planes there, and lie in the corresponding plane of symmetry

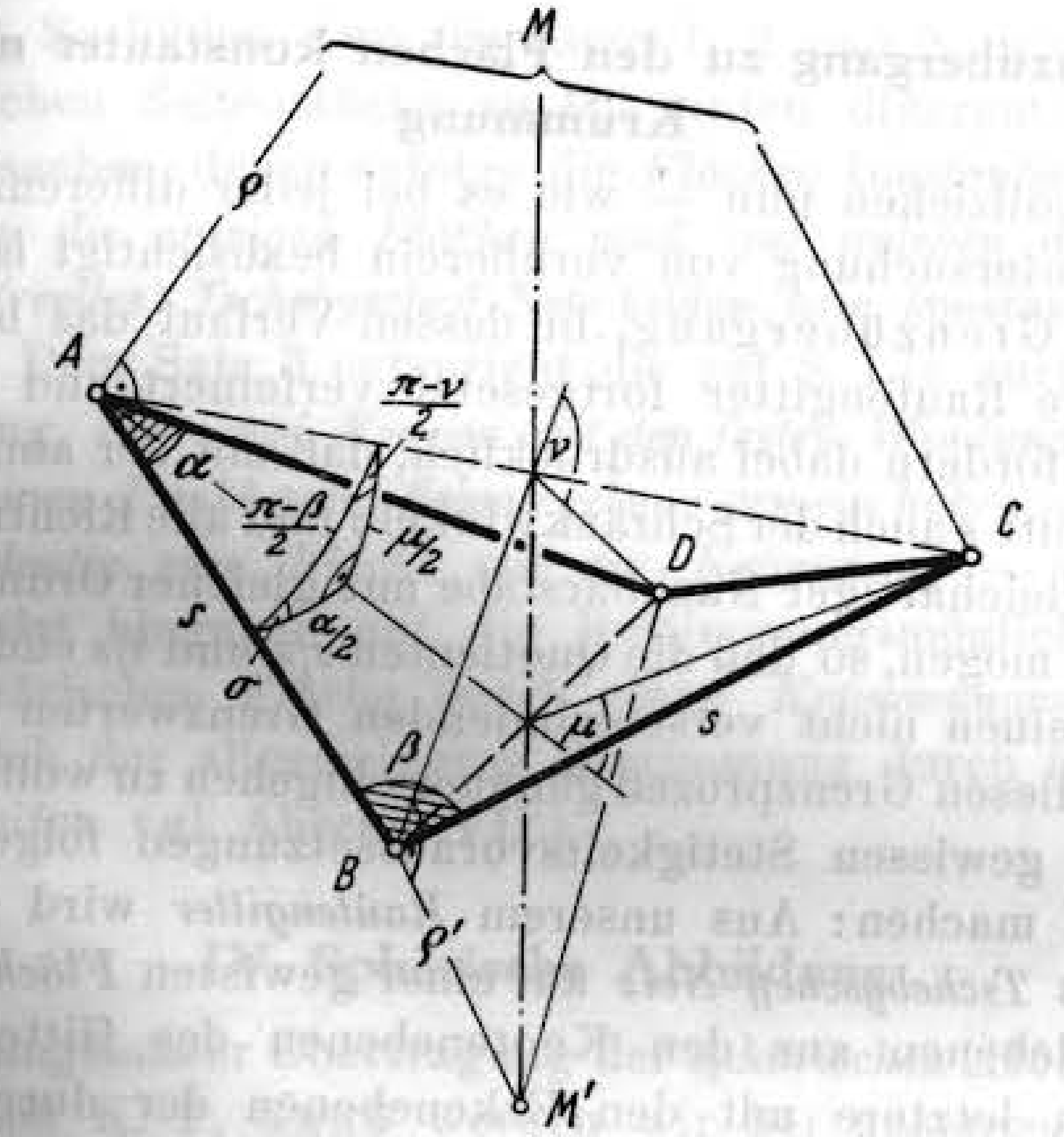

Fig. 2: Rhombic single-mesh scheme

lie. Their centers M, M' are found at the intersection of the lattice normals of A and C, and B and D, respectively (see Fig. 2); for their radii, the values

$$\rho = s \cos \frac{\alpha}{2} \operatorname{ctg} \frac{\mu}{2} = s \operatorname{ctg} \sigma \operatorname{ctg} \frac{\alpha}{2},$$
$$-\rho' = s \cos \frac{\beta}{2} \operatorname{ctg} \frac{\nu}{2} = s \operatorname{ctg} \sigma \operatorname{ctg} \frac{\beta}{2}. \tag{2}$$

Forming the product, we obtain, with reference to (1), the fundamental relationship

$$\rho \rho' = - \frac{s^2 \cos \sigma}{\sin^2 \sigma}, \tag{3}$$

which states that for constraint-preserving deformations of a skew rhombus, the product of the circumradii remains constant. For our lattices, the following therefore holds:

Theorem 5: In a rhombic lattice with planar nodes, the product of the circumradii has the same negative value for all meshes

III. Limit Transition to Surfaces of Constant Negative Curvature

We now carry out, as is intended from the outset in every difference-geometric investigation, a suitable limit transition in the course of which the considered planar rhombic grid is continuously refined and densified. We expressly require that with the decreasing mesh size s, the angle of twist and all directional differences 0 of similar neighboring bars of the same order may also approach zero, so that the quotients 0/s and 0/s tend towards finite and generally non-vanishing limits. Without going into this limit process in more detail, we can make the following observations under certain continuity assumptions: Our rhombic grid becomes a continuous Chebyshev grid on a certain surface Þ, whose tangent planes arise from the nodal planes of the grid Since the latter coincide with the corner planes of the continuous bar lines, the oscillation planes of the net curves are identical to the tangent planes of the supporting surface; the net curves thus represent the oscillation lines (principal tangent or asymptote lines) of 4. These curves also have constant turns = lim /s due to the fixed twisting of the grid polygons; the (generally variable) curvature x = Φ lim 0/s could be used for the design definition. Furthermore, the curvature lines of the surface obviously emerge from the diagonal lines of the grid, the circles of curvature of which appear as the boundary positions of the circumcircles. Based on formula (3), the principal radii of curvature have a constant product, and is characterized by constant surface curvature

$$\frac{\sin^2 \sigma}{s^2 \cos \sigma} = - - \lim()^* = - :^2$$

Klim (4). $\sin^2 \sigma$

Rhombic lattices with planar nodes are therefore suitable for illustrating pseudospherical surfaces. Their properties are represented in a clear and transparent way by certain elementary relationships in the lattice models.

mirrored. Thus, Theorems 1, 2, and 5 form the differential-geometric counterparts to known differential-geometric facts, according to which surfaces of constant (negative) curvature are the only surfaces on which the Schmeg lines form a (real) Chebyshev net, or possess constant turns. 13 Theorem 3 corresponds to the finding, attributed to S. Lie, that two curves with fixed turns and -, which share a point and a Schmeg line, uniquely define a surface with curvature τ^a as Schmeg lines. Theorem 4, on the other hand, points to the possibility of determining a pseudospherical surface by a strip of curvature. (Regarding the more general determination by an arbitrary strip, see Section VII.)

IV. Spherical Mapping

By analogy to the spherical mapping of a surface according to K. F. Gauss, we will assign to each node P in our rhombic grid that point P on the unit sphere whose tangent plane is parallel to the node plane of P, taking into account the orientation: The radius belonging to P should be parallel to the grid normal in P, which points towards Section II.

When moving along a grid bar, the node plane rotates by the angle and the image point on the sphere describes a great-circle arc of length : We thus arrive at a spherical rhombic grid with the constant mesh size σ.

Theorem 6: The spherical image of a rhombic grid with planar nodes is a rhombic grid of the image sphere constructed from great-circle arcs. 2 Bassatorio Giletsgynowy Poole mom

Fig. 3 shows the spherical rhombic grid obtained by spherical mapping of the "fish trap" shown in Fig. 1,

13 A detailed presentation of the theory of pseudospherical surfaces is given, for example, by L. Bianchi in his "Lectures on Differential Geometry" (trans. by M. Lukat, Leipzig 1899). See also Enz. math. Wiss. III D 5, Section VII.

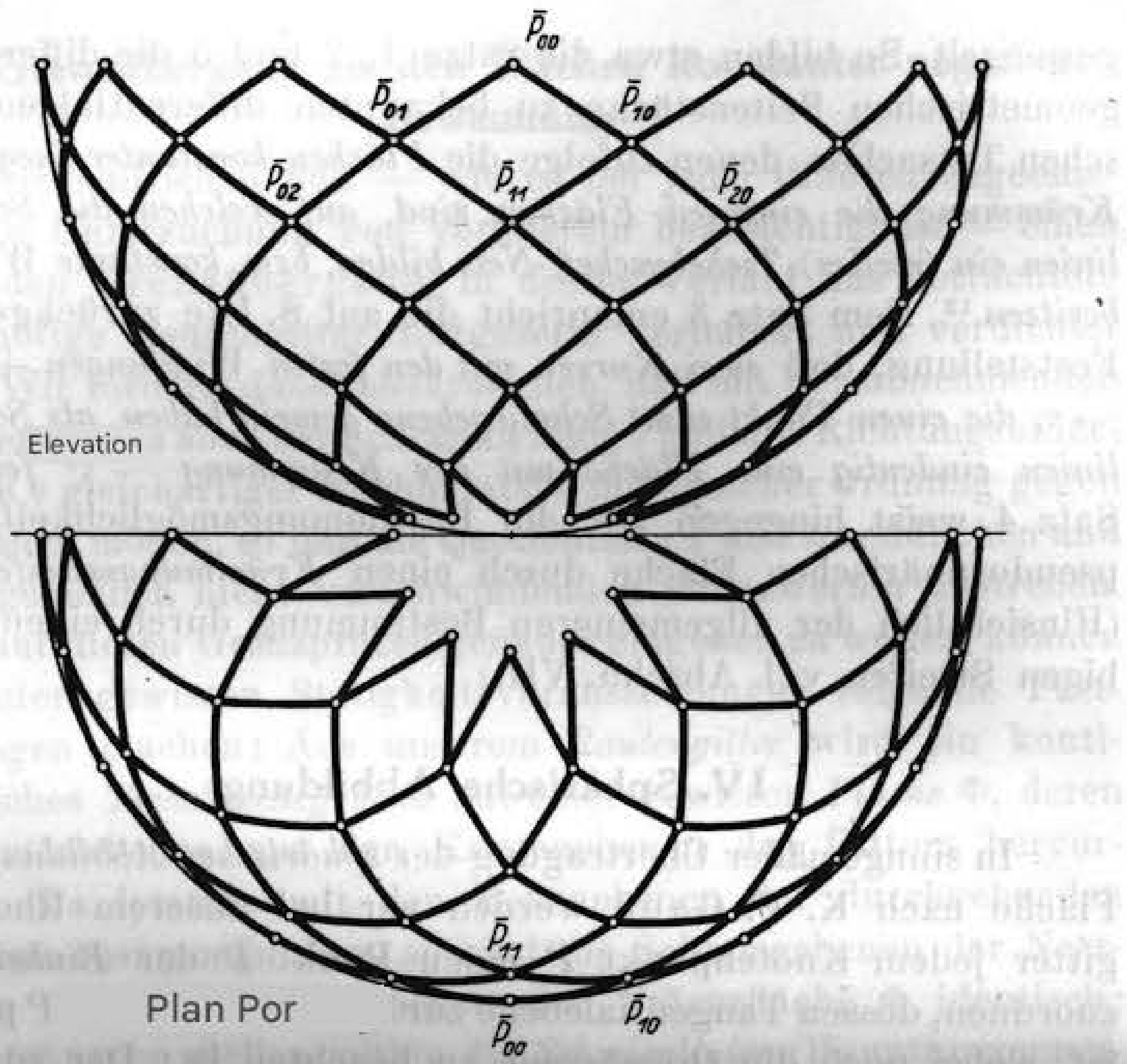

Fig. 3: Partial representation of a rotationally symmetric rhombic grid on the sphere. (Spherical image of the Radon-Thomas grid model in Fig. 1.)

which is of course also rotationally symmetric. The constructive-drawing process is quite elementary

The converse belonging to Theorem 6 is also easy to understand: Every rhombic grid of the sphere corresponds to a perfectly defined rectilinear and planar rhombic grid. Each great-circle arc indicates the direction of its associated grid member by its axis. The spherical grid can be conceived as being defined by two continuous arcs belonging to different sets (with a common node) or by an alternating arc; the completion to form a rhombic grid is clear. The selected arcs correspond to certain equilateral spatial polygons of constant twisting, which, according to Theorem 3 or 4, uniquely define a rectilinear rhombic grid with planar nodes

its spherical image must then coincide with the proposed rhombic grid of the sphere

The limiting process indicated in Section III ultimately leads to the well-known fact that, when a surface of constant curvature is spherically mapped, the Chebyshev net of oscillation lines corresponds to another Chebyshev net, and conversely, every real Chebyshev net on the sphere — defined, for example, by two different net curves, uniquely determines the shape of a certain pseudospherical surface.13 The limiting form of the "fish trap" and its spherical image would, incidentally, already require elliptic integrals for their mathematical description, which puts the methodological significance of elementary difference-geometric treatment into proper perspective

The constancy of curvature of the interfaces can also be easily seen with the help of the spherical mapping: The partial triangle ABD of a lattice rhombus ABCD corresponds on the sphere to an equally isosceles image triangle ABD, whose vertical angle is supplementary to the vertical angle α of the lattice triangle. The areas, taking into account the opposite direction of rotation,

(5) $\sin \alpha$ and $f = -\sin(\pi - \alpha) + O(\sigma)$,

such that

$$\lim f/f - \lim (5/8)^3 = -\tau^2.$$

That the quotient f/f actually converges to the Gaussian curvature K of the interface would require further consideration due to the intertwining of the two boundary processes f→0 and lattice → surface

Here, the difference-geometric kernel of a theorem by J. N. Hazzidakis will be presented, according to which the area of a quadrilateral bounded by four oscilloscope lines of a surface with curvature 1 is equal to the excess of the angle sum over 2. 14 We first consider for a rhombic

14 J. N. Hazzidakis, On some properties of surfaces with constant curvature. Crelles J. 88 (1880), 68-73.

lattice with planar nodes Pik the region 0 ≤ i ≤ m, 0 ≤ k ≤ n and its spherical image. The pairwise equal angles aik, Bik of a single mesh Pik Pi-1, k Pi-1, k-1 Pi, k-1 appear on the sphere as exterior angles of the image rhombus and determine its (negative) area

$$\bar{f}_{ik} = 2\alpha_{ik} + 2\beta_{ik} - 2\pi.$$

When summing over the specified region, apart from the subtrahend 2mn, the sum of all lattice angles appears; since every four angles meeting at one of the (m-1) (n - 1) interior nodes contribute 2 π, we provisionally obtain the expression for the total area of the spherical image

(6) F = Fik = Σ boundary angle $- 2(m+n-1)\pi$.

We now denote as "strip of the first kind" gk the sequence of adjacent lattice meshes with the nodes Pik and Pi, k-1 (0<i≤m), with ḡ the corresponding spherical image. If we extend the above summation not over the entire area, but only over the strip pair gtk+9k+1, then the sum of the lattice angles is also equal to twice the sum of the angles occurring in the middle row of nodes Pik, where all interior nodes again contribute 2. Thus, for the spherical image, we obtain

$$(7)\quad \bar{g}_k + \bar{g}_{k+1} = 2(\beta_{1k} + \alpha_{1,k+1} + \alpha_{mk} + \beta_{m,k+1}) - 4\pi,$$

and an analogous expression for pairs of adjacent "stripes of the second kind" hi. The content of the spherical image of such a pair of stripes thus depends essentially only on the angles occurring in the middle of the narrow sides, and conversely, we are thus able to express the marginal angle sum in formula (6) in terms of the gu and hi:

$$\bar{F} = (\alpha_{11} + \beta_{m1} + \beta_{1n} + \alpha_{mn}) + \frac{\bar{g}_1 + \bar{g}_2}{2} + \cdots + \frac{\bar{g}_{n-1} + \bar{g}_n}{2} + 2\pi(m-1) + \frac{\bar{h}_1 + \bar{h}_2}{2} + \cdots + \frac{\bar{h}_{m-1} + \bar{h}_m}{2} + 2\pi(n-1) - 2\pi(m+n-1).$$

However, since the sum of all gk, as well as the sum of all hi, has the value F', we thus obtain the relationship

1 (8) -F= corner angle-2-2 (91+In+h+hm).

The last term originating from the boundary strips now converges to zero as the lattice is progressively refined, and F finally agrees with the area of the oscillating parallelogram due to the curvature -1, thus obtaining Hazzidakis' theorem. 15

Finally, two model representations of spherical rhombic grids should be mentioned. It is obvious to stretch a net of constant mesh size, knotted from threads, over a smooth sphere, such as a ball; the thread pieces then lie along great circles, and depending on the applied boundary forces, all possible shapes can be achieved (at least in certain areas). The rotationally symmetrical forms shown in Fig. 3, for example, occur practically in balloon nets. Another possibility, inspired by the well-known Japanese paper flowers, is to staple together a number of congruent circular ring sectors made of cardboard, radially pre-scored at equal intervals and folded in a zigzag pattern, along the folds in such a way that numerous spherical rhombic shapes are formed. The model is largely movable and, in every state (except for the flat boundary positions), shows a spherical rhombic grid on both the outer and inner surfaces. 16

V. Equilibrium Models and Force Diagrams

Even the planar rhombic lattices themselves can be represented in a remarkable way by knotted thread nets-

15 This theorem, as is well known, provided D. Hilbert with the means to prove that there can be no singularity-free surfaces of constant negative curvature (Foundations of Geometry, 2nd ed. 1903, Appendix V). In the present context, one would have to try to show that there is no completely simple rectilinear rhombic lattice with planar nodes.

16 The paper flowers mentioned exhibit a surprising richness of form. This is because they are made of very thin material, which also allows twisting. If they were made of perfectly rigid material, they would at best permit spherical deformations

place, namely through certain self-supporting equilibrium positions. This can be verified by developing a force diagram for a given lattice using the methods of graphical statics.

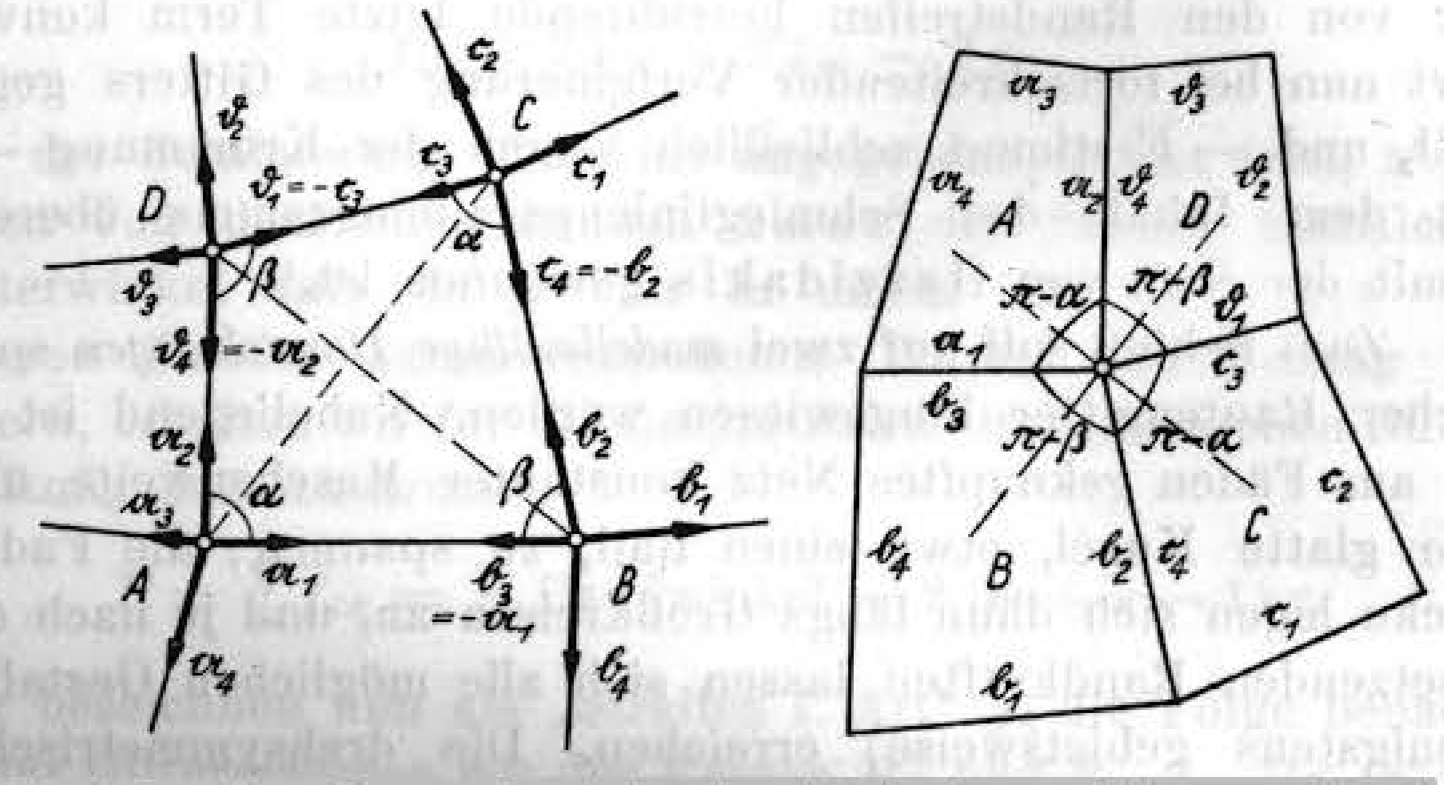

Fig. 4: Section of an equinodular rhombic lattice and the associated force diagram

The four tension forces acting at a node act in the direction of the bars and maintain equilibrium when the vectors representing them can be arranged to form a closed quadrilateral. Corresponding to the fact that any two neighboring nodes have two opposing partial forces located in the connecting bar (for example, $a_1 = b$, in Fig. 4), the associated force quadrilaterals in the diagram share a common side. If the magnitudes of the — tension forces along two continuous sets of bars of different types (or for an alternating set of bars) are arbitrarily specified, the force diagram can be developed quite simply and unambiguously by drawing parallel lines. Conversely, from its existence, it can then be concluded that the bar system is in equilibrium after the application of the boundary forces shown in the diagram. The equilibrium is certainly stable if only tensile stresses occur, which can always be achieved in simple grids by a suitable choice of boundary forces. It therefore holds that

Adet Theorem 7: A simple rectilinear rhombic grid with planar nodes can always be realized as the equilibrium position of a freely stretched net of inextensible and knotted threads; some freedom remains in the choice of boundary forces.

For example, the rotationally symmetric grid from Fig. 1 can be realized very simply by stretching a tube-like closed, uniformly meshed thread net between two circular rings, as Radon and Thomas did.17 It should also be emphasized that rhombic nets are capable of other free equilibrium positions in which they do not have planar nodes; however, they then naturally do not represent pseudospherical surfaces.

The quadrilateral polyhedra that have appeared here as force diagrams may claim independent interest As can be seen in Fig. 4, four (planar) force quadrilaterals originating from the nodes of a grid mesh meet at each corner, with angles that are supplementary to or coincide with the mesh angles 2, 3, depending on whether the bar forces have the same sign (stability under purely tensile stresses) or opposite signs (unstable equilibrium18). In any case, the corners are formed by axially symmetric quadrilaterals with identical pairs of opposite sides (π-α and π-β or a and ß), whose opposite angles are naturally also identical in pairs (-- σ and π+ σ or π±5). The axis of symmetry is the common bisector of the angles spanned by the two pairs of opposite edges. The additional symmetries present are irrelevant here. It should only be mentioned that the lines of intersection of opposite faces run parallel to the mesh diagonals and therefore, like them, enclose a right angle.

17 H. Thomas shows beautiful photographs of such net models in the same place. In the same way, by means of suitable non-equal mesh nets, oscilloscope models of arbitrarily negatively curved surfaces of revolution could be produced. 18 This case would be the case for R. Sauer 10, who, however, does not address questions of equilibrium

In the course of the boundary transition according to Section III, the present quadrilateral polyhedron, with increasing refinement, converges towards a certain curved surface P, on — which, due to the flatness of the original quadrilaterals, the surface curves originating from the edge lines form a conjugate net. A strip formed from adjacent quadrilaterals converges towards a torse circumscribed along a net line, whose generators arise from the transverse edges and thus touch the net curves of the other set. Because the vertex planes of the passing edge lines contain the quadrilateral axes, the oscillation planes of the net curves in the boundary contain the surface normals: The net curves are therefore geodesic lines of P. Surfaces with a net of conjugate geodesics are named after A. Voss, who, although not the first, studied them in detail.19

Theorem 8: If a rectilinear rhombic lattice with planar vertices is considered an equilibrium system, then every corresponding force diagram represents a polyhedron of planar quadrilaterals that has axially symmetric vertices and uniformly equal edge angles. Such a quadrilateral polyhedron can be regarded as a difference-geometric model of a (special) Voss surface.

The special feature of the Voss surface appearing here is based on the aforementioned orthogonality of the rhombic diagonals and can be characterized as follows: The conjugate system of geodesics has a diagonal network conjugate to an orthogonal system

The force diagram belonging to the Radon-Thomas fish trap, under the obvious (but not necessarily required) assumption of a rotationally and mirror-symmetric stress distribution, possesses an easily discernible helical structure (Fig. 5). The force quadrilaterals assigned to the nodes of a meridian then have a common diagonal line intersecting the helical axis at a right angle, from which it is immediately apparent that a helical surface appears as the limiting form. This is the most familiar example of a Voss surface.

19 Before A. Voss (op. cit. 7), these surfaces were already considered by K. Peterson (Über Kurven und Flächen. Moscow and Leipzig 1868).

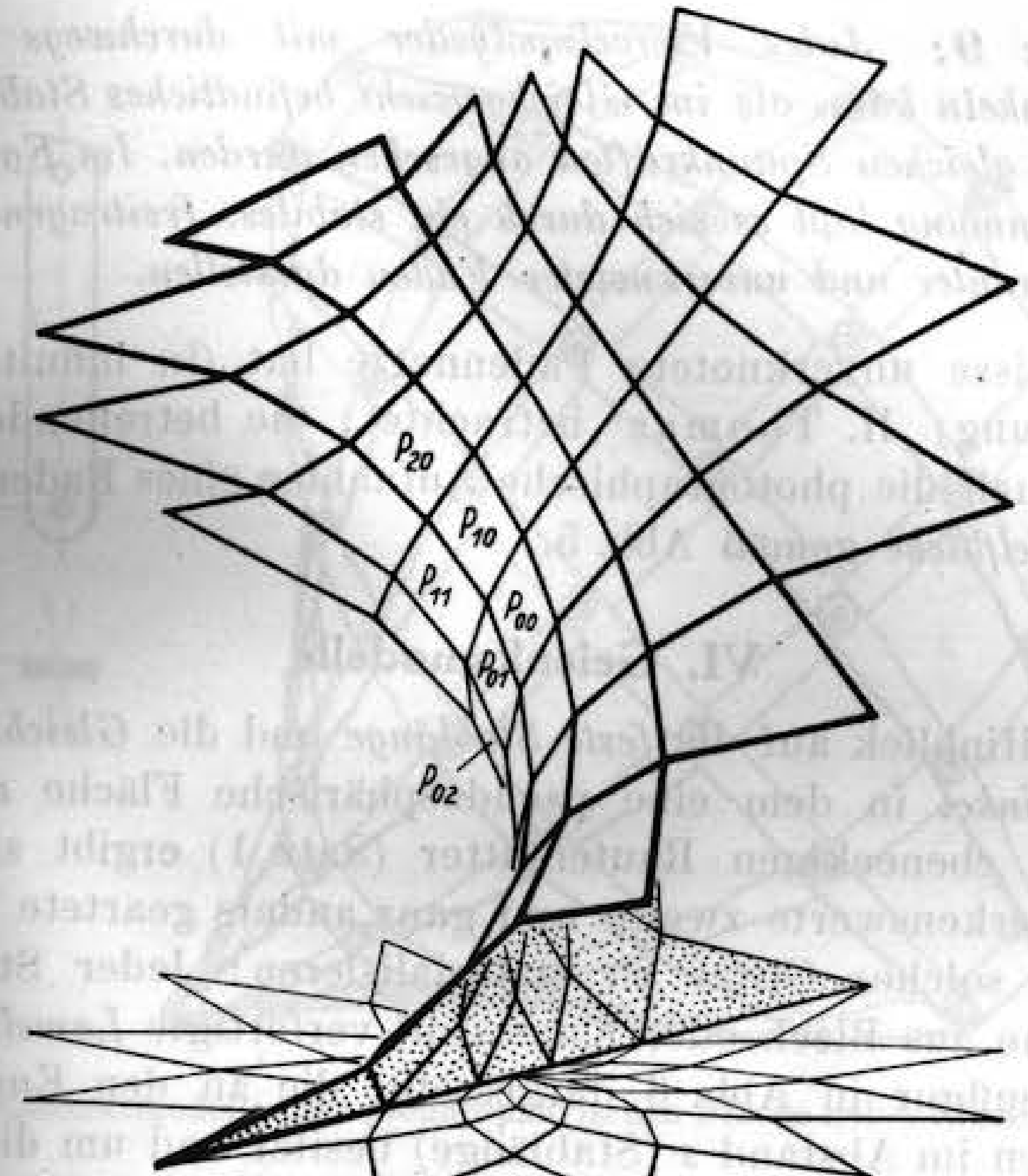

Fig. 5: Force diagram for the Radon-Thomas fish trap according to Fig. 1. (Difference geometric model of a helical surface with a conjugate network of geodesic lines.)

The relationship between the original grid and the corresponding force polyhedron is characterized exclusively by the parallel position of corresponding line segments, and is therefore entirely reciprocal and reversible; indeed, one speaks of "reciprocal force diagrams". Consequently, the edge system of the polyhedron can also be considered an equilibrium system and, in the stable case, — can be represented by a self-supporting network of threads. The threads exhibit consistently constant and equal stresses because the corresponding force diagram now consists entirely of rhombuses, and they obviously do not need to be knotted.

Since each of the quadrilateral polyhedra identified in Theorem 8 corresponds to a parallel-edged rhombic grid with planar nodes, the following holds:

Theorem 9: Every quadrilateral polyhedron with uniformly equal edge angles can be considered a truss structure in equilibrium with uniformly equal tension forces. In the case of negative curvature, it can be represented by a stable, self-supporting network of uniformly tensioned and unknotted threads.

Certain unknotted thread networks have been considered (in infinitesimal refinement) by H. Thomass; the work in question also includes a photograph of a thread model of the helical surface according to Fig. 5.

the surface

VI. Hinged Models

With regard to the fixed rod length and the equality of all twist angles in the planar rhombic lattice representing a pseudospherical surface (Theorem 1), a remarkable second and quite different possibility arises for materializing such a lattice: Each rod is represented by a lamella made of sheet metal or plastic, as shown in the inset figure in Fig. 6, which has two holes at its ends spaced s apart (rod length) and is twisted about its longitudinal axis by the amount (twist angle). The rods of the first type are approximately right-handed, the rods of the second type left-handed. If these lamellae are then hinged together in groups of four at a node, preferably by means of rivets, a largely movable mechanism is created which, in every state, represents a planar rhombic lattice and thus a surface of constant curvature - (5/8)2 including its oscillation lines

The hinge model shown in Fig. 6, which has 6.6 = 36 meshes and consists of 49 steel lamellae, each 1 mm thick and twisted by +20° or 20°, with a 5 cm hole spacing, thus approximating surfaces with a constant curvature of approximately (1/14 cm), proved to be extraordinarily instructive and useful for demonstrating the appearance and many properties of pseudospherical surfaces.20 The figure

20 The author is indebted to Mr. K. Wanka, Engineer, for the production of the model.

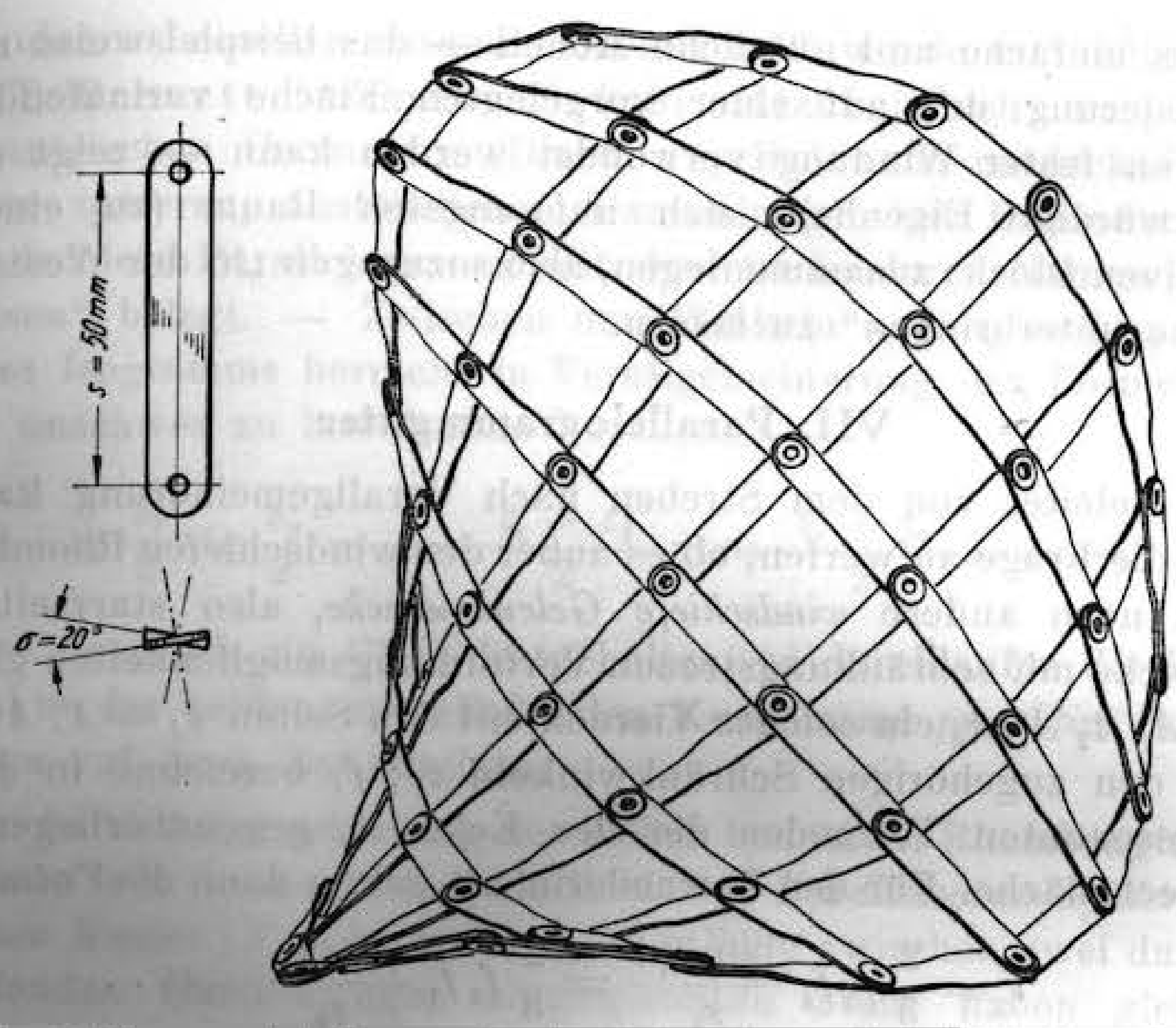

Fig. 6: Hinge model of a pseudospherical surface with individual lamellae

illustrates the surface of constant curvature that can be formed by a (right-angled) intersecting pair of straight lines, the investigation of which was suggested by L. Bianchi at the time, 13 but apparently has not yet been carried out. If the two aligned lamellae were each made from a single piece, the variability of the model would be limited to one degree of freedom (variation of the angle of intersection) and it would exclusively represent the aforementioned special surfaces. The hinged version of the "fish trap" according to Fig. 1 would also be attractive, as it would result in a mechanism of two degrees of freedom capable of exclusively representing pseudo-spherical surfaces of revolution.

On this occasion, explicit reference should be made to the difference-geometric model of the space curves of constant torsion (which is included several times in the described hinged model), consisting of the hinged arrangement of a larger number of similar twisted lamellae

This simple and useful model, which can be — used, for example, to realize the curves of fixed turns running on a given surface, shows the curious property of being able to be folded up in the — smallest space (into a massive block), i.e., to fit "in a vest pocket", so to speak.

VII. Parallelogrammgitter

Guided by the pursuit of generalization,

One might ask whether, besides skew rhombuses, there are other skew quadrilaterals, i.e., rigid-sided quadrilaterals with constraint-preserving deformation possibilities. Let A1 A2 A3 A4 be such a quadrilateral with sides si AiAi+1 and the corresponding twist angles i; fi denote the triangular face opposite vertex A; in the spanned tetrahedron. The formulas V = 2 3 $1 2 faf sing remab then apply to the area of the tetrahedron. With constraint-preserving deformation, the area ratios f_i: fs and fa fa thus remain

(9) fixed.The $= f_3 f_4 \frac{|\sin \sigma_1|}{}$ = the to $\frac{|}{s_2} = \dots$ common

base A, A_1 of f_1 and fs can therefore only vanish simultaneously, which implies relationships of the form $8_1=S2 S3 and S1 S2 S3 sy. A more detailed examination of all possible sign combinations reveals the equality of opposite sides S, S8, S9 S, as a necessary condition; that this condition is also sufficient follows from the associated equality of areas f_1 = fs, f_1. According to (9), the angles of rotation , = and = are subject to the condition

σ_1

the | sin ,| sing | = $1:$2. (10) Besides rhombus, only the skew parallelogram is also capable of being moved with respect to twisting.21 This curious spatial mechanism (which is most easily reconstructed from ver-

TOHO 21 G. Darboux, Sur un problème relatif à la théorie des courbes gauches. Mem. ac. sci. 50 (1908), 1-31, тov пoly hlasna почет

(which can be assembled from wound lamellae) was first described by G. T. Bennett, but soon after also discovered by French geometers (Darboux, Bricard).22 In a later, more detailed presentation,11 Bennett gave the skew parallelogram the more fitting name "isogram". - Between the angles a_1 = a and a of an isogram, the following easily provable relationship $= \alpha_4$ prevails, in generalization of formula (1).

$$(11) \quad \operatorname{tg} \frac{\alpha_1}{2} \operatorname{tg} \frac{\alpha_2}{2} = \cos \frac{\sigma_1 - \sigma_2}{2} : \cos \frac{\sigma_1 + \sigma_2}{2} = \text{const.}$$

Therefore, if one link is held in place, the endpoints of the two subsequent crank arms traverse projective series of points on their orbital circles.

In generalization of the rhombic lattice used so far, we can therefore consider a (skew) parallelogram lattice with planar nodes Pik). All bars clamped between two continuous adjacent lines have the same length (12) SiPi-1,k Pik (k = 0, ± 1, ± 2, ...),

$$s_k'' = P_{i,k-1} P_{ik} \quad (i = 0, \pm 1, \pm 2, \dots)$$

and are equally twisted. To prevent the meshes from overlapping, the twist angles are assumed to be acute from the outset and positive (i) for the bars of the first kind and negative (-) for the bars of the second kind; moreover, they must satisfy condition (10): bona sin σ sin O < σε < π/2,

$$(13) \quad \frac{}{s_i'} = \frac{\sigma_k''}{s_k''} = \tau = \text{const}; \quad 0 < \sigma_k'' < \pi/2.$$

The theory of the general isogram lattice can be developed in a completely analogous way to that of the rhombic lattice, but naturally it no longer possesses quite the same degree of intuitiveness. Extending Theorems 3 and 4, such a lattice is defined by

22 From the point of view of "symmetrical scrapings", the mechanism was considered by J. Krames: On the geometry of the Bennett mechanism. Sitzgsber. Ak. d. Wiss. Wien 146 (1937), 159-173.

two bar pulls of different types or an alternating bar pull are clearly defined

From the outset, the intention is again to arrive at a Chebyshev net on a curved surface by means of a limiting process that continuously reduces the sides and angles of curvature while adhering to the Bennett relation (13). Because of the flatness of the lattice nodes, this is naturally the net of oscillation lines from 4, and these obviously have fixed torsions and τ. Thus, from this more general starting point, one again arrives only at pseudospherical surfaces. The constancy of the surface curvature can again be made plausible by means of the spherical mapping, which assigns to the lattice a parallolegrammatic net of great-circle arcs of lengths i and σë on the unit sphere (see Section IV); a partial triangle f of an isogram with angle a corresponds to a spherical triangle f with angle -z, and, in generalization of (5)

$$(14)\qquad K=\lim\frac{\bar f}{f}=-\lim\frac{\sigma_i'\sigma_k''\sin(\pi-\alpha)}{s_i's_k''\sin\alpha}=-\tau^2.$$

The isogram lattice can also be viewed as the equilibrium position of a knotted thread net (see Section V). The corresponding force diagram is again a quadrilateral polyhedron with axially symmetric vertices and represents a Voss surface, this time, however, of a much more general nature; obviously, any quadrilateral polyhedron with axially symmetric vertices can be related by parallel edges to an even-nodular quadrilateral lattice, whose meshes must then also be axially symmetric, i.e., are all isograms. Such a quadrilateral polyhedron can, in the case of negative curvature, again be represented by a stable system of unknotted threads (see Theorem 9); this time, however, only each individual thread has constant tension, while the tension can vary from thread to thread. moy

The construction of a hinged model made of twisted lamellae is obvious, and a corresponding reference can already be found in G. T. Bennett;1" however, it offers virtually no advantages over the simpler rhombic system.

In summary, we note:

Theorem 10: In a parallelogram lattice with planar nodes, the quotient of rod length and sine of the twist angle has a constant value. A simple lattice of this type can serve as a difference-geometric model of a pseudospherical surface and its oscillations. It can be constructed hinged from twisted lamellae or represented as the equilibrium position of a knotted thread net; any corresponding force diagram forms a planar quadrilateral polyhedron with axisymmetric vertices, which can be regarded as a model of a Voss surface with its network of conjugate geodesics. Ferodo dittoisimm

After all this, it might seem that considering the general parallelogram lattice compared to the more manageable rhombic lattice would be entirely irrelevant for the theory of pseudospherical surfaces. Indeed, in most cases the rhombic model will suffice. However, as an application of the isogram model, let us discuss here the determination of a surface of prescribed fixed curvature by an arbitrarily given strip. First, we construct a $-\tau^2$ difference-geometric model of the given strip by selecting a (sufficiently dense) sequence of points P; the strip curve along with the associated strip planes; which we consider bounded by the lines of intersection ei and ei+, with the neighboring planes 1-1 and i+1; normally, two consecutive points Pi-1 and Pi will lie on opposite sides of the generating line ei and be approximately equidistant from it To achieve precisely equal distances, we slightly modify the model by shifting each point Pi, along with its subsequent part, in the direction of the perpendicular to ei until its distance is equal to Pi-1er. Since these are second-order shifts, their overall influence decreases without limit with increasing refinement. The new positions of the elements (Pi, ,) can now be represented as a diagonal series of node elements (Pii, ii) of a planar parallelogram lattice with a prescribed twisting con-

constant s/sint (=1/t) can be considered because a suitable isogram can be found for each pair of consecutive elements. Let us consider, for example, the points Poo, P11 with the associated planes 200, €11 and their lines of intersection; if 11 denotes the angle of inclination of the diagonal d11 = POO P11 to 11 (and €00), then the cylinder of revolution with axis d_{11} and radius t sin v11 cuts out the missing isogram vertices Po1 and P10, as can easily be verified; Po1 is the point whose connecting rod to Poo has positive twist. Continuing in this manner, one arrives at an alternating line of rods that uniquely defines a parallelogram lattice with planar nodes and with twist =1/t The only remaining question requiring examination is whether the rotary cylinder used always yields real additional nodes. Since these nodes Po and P10—to remain with the isogram under consideration—lie on an ellipse extending 200° around Poo with semi-axes t and t sin 11, where t belongs to the normal projection of d_{11}, it is easy to see that with a sufficiently dense sequence of points, real intersection points are always to be obtained if the angle between the diagonal d_{11} and a does not fall below a certain limit, i.e., if the given strip does not contain a spherical element. If it is therefore not a spherical strip at all, then a surface containing it with a prescribed curvature measure certainly exists. In the case of a spherical strip, it would have to exhibit constant torsion; then, as is well known, a second spherical strip of opposite curvature could be added arbitrarily.

The role played by a parallelogram lattice of medium generality used by R. Sauer10, which contains only two bar lengths sis' and ss" and accordingly only two twist angles = ' and - σ" = σ", will be reported in the next section. Here, only a theorem on curves of constant torsion, which results from considering two adjacent bar sets of such a lattice, should be mentioned. The ladder-like structure in question may again be imagined as being made from twisted lamellae and movable

think. The two "spars" are equilateral polygons of constant and equal twist, one of which can be arbitrarily specified; the second is then still doubly indeterminate, but after assuming a "rung" (in terms of length, direction, and twist sign) is uniquely determined and can be constructed using elementary principles. If the length and twist of the rungs are fixed, but their sequence is progressively condensed, the spar polygons converge to two curves of fixed and equal twist. Accordingly, for every curve of constant torsion, a companion curve of equal torsion can be found in ∞ª ways, such that corresponding points have a fixed distance s and corresponding arcs are of equal length; corresponding oscillation planes contain the line connecting the associated points and form a fixed angle = arcsin 3τ. 23 The connecting lines generate a radiating surface of constant

angular momentum p=tg 6, whose line of restriction is the rung segments bisects; the two spar curves are oscillations of the surface.

We will encounter such Bäcklund pairs of curves of fixed winding in Section IX. Here, let us mention only one simple relevant model: If the ends of a number of identical rods are tied at equal intervals into two strings, a kind of "rope ladder" is created which, neglecting its own weight, forms an isogram strip of the type considered earlier in every taut equilibrium position. The Bäcklund pair resulting from this through infinitesimal refinement is, however, quite special.24

VIII. Rigid Buckling of Force Plans and Angle-Preserving Lattice Transformations

Quadrilateral polyhedra with axisymmetric vertices, such as those that appeared as force plans in Sections V and VII, have a remarkable property, which was discovered by H. Wiener in 1903.5

23 Analytically, the determination of such a companion curve requires the integration of a Riccati differential equation

24 The two curves of constant torsion are congruent and periodic. Their tangents are parallel to the generators of those two cones that form a 5

and later again demonstrated by R. Sauer and H. Graf: They can be continuously and necessarily folded along the edges while preserving the lateral faces. Between the lateral angles a, β and the edge angles of an axially symmetric quadrilateral, the following relationship exists:

(15) ψ α-tgtg = cos : cos +, 2

so that during a folding, the tangent product of the half edge angles must also be preserved with axial symmetry.25 Therefore, if new edge angles φ', ' are determined for all quadrilaterals of the polyhedron using a common parameter by means of

$$\text{(16)} \qquad \operatorname{tg}\frac{\varphi'}{2} = \frac{1}{\lambda}\operatorname{tg}\frac{\varphi}{2}, \quad \operatorname{tg}\frac{\psi'}{2} = \lambda\operatorname{tg}\frac{\psi}{2}$$

If the parameter is set to 0, equation (15) remains satisfied; the new polyhedron can be assembled from the old quadrilaterals and therefore represents a buckled form of the original shape. The continuous increase of the buckling parameter from 1 to the final value describes the corresponding continuous buckling process. For = 0 or ∞, two completely flat limiting forms result, which, however, are generally not practically realizable

If we now consider in particular the force polyhedron associated with an equatorial rhombic lattice, whose edge angles according to Section V have the values = π - σ, ψ = π + 6, the question immediately arises whether a parallel-referenced lattice can be assigned to each kinked shape. This is generally not possible for any quadrilateral polyhedron, as the consideration of a cap made of nine abutting quadrilaterals shows. However, it is possible in the case of vertices that are entirely axially symmetric: Each such vertex corresponds to a skew parallelogram, which, up to parallel translations and scale

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project a geodesic line drawn from the foci on an egg-shaped ellipsoid of revolution. The rungs are parallel to the tangents of the geodesic.

Formula (15) is obtained by applying one of the so-called Napier's formulas of spherical trigonometry to one of the triangular faces into which the quadrilateral is divided by a diagonal plane

changes are determined; by mutually coordinating the sides, the isograms can be seamlessly arranged and then, due to the flatness of the polyhedral surfaces, form a parallelogram lattice with planar nodes. Since the original mesh angles a, 3 have remained unchanged, an angle-preserving lattice transformation is present. The twist angles change in this process, and their new values σ', - σ result from (16)

(17) 1 tg to = tg, to. tg 2

Because of the difference in the magnitudes, it is no longer a rhombic lattice, but one of Sauer's parallelogram lattices, which are constructed from rods of two different lengths s', s".10 This generalization of the rhombic lattice is necessarily imperative here

Taking advantage of the freedom of the scale factor, the rod lengths can be normalized such that the original twist is preserved. This then yields sin σ' s

σ

sin σ" s" = sin S

$$\text{(18)} \qquad s' = \frac{\lambda s}{l'}, \quad s'' = \frac{s}{\lambda l'} \qquad \text{where } S \qquad l' = \cos^2\frac{\sigma}{2} + \lambda^2\sin^2\frac{\sigma}{2}$$

= cos ± sin σ ² 2.

Theorem 11: Every rigid buckling of a quadrilateral polyhedron (force diagram) associated with a rhombic lattice with planar vertices by parallel edges induces an angle- and twist-preserving lattice transformation that yields a planar parallelogram lattice with two different rod lengths

Through infinitesimal refinement (s→ 0, 0) one arrives at a transformation of the pseudospherical surfaces with the following properties: ∞ surfaces of the same curvature can be assigned to each surface of constant negative curvature, whereby in addition to the turns of the oscillation lines, their

Angles of intersection are preserved. The arc lengths and radii of curvature are multiplied by a constant factor in the first set, and by 1/2 in the second set.26 This is the long-known Lie transformation.27 The buckling of the quadrilateral polyhedra corresponds to the — bending of the Voss surfaces while preserving a conjugate network of geodesic lines. 6, 7, 18.

A buckling model of the helical surface polyhedron belonging to the "fish trap" according to Fig. 5 was published at the time by Wiener's Model Publishing House. The parallelogram lattices belonging to its buckling forms represent certain helical surfaces of constant curvature.

IX. Skew Parallelepiped. Bäcklund Transformation

We also owe to G. T. Bennett, the discoverer of the "skew-joint parallelogram," the knowledge of an interesting twelve-membered spatial mechanism of two degrees of freedom, which can be called a "skew parallelepiped."11 It is best constructed again from twisted lamellae of the same twist: Three types of four lamellae are needed, whose lengths a, b, c and twist angles α, β, γ correspond to the Bennett condition

$$a : b : c = |\sin\alpha| : |\sin\beta| : |\sin\gamma| \qquad (19)$$

must suffice, but are otherwise not subject to any restrictions. If a "basic isogram" P1 P2 P3 P_1 with sides a, b is formed and four lamellae of length c are attached hingedly at the corners (where the direction of one of them can be chosen arbitrarily), the endpoints can be connected by a second isogram Q1 Q2 Q3 Q4 in the manner of the basic isogram; in each node, a common cylindrical hinge serves for all three lamellae. According to the variability of the

26 For this purpose, on the one hand, l', l'1 is to be used, and on the other hand, 8/0p, when denotes the difference in direction of successive bars.

27 S. Lie: On the theory of surfaces of constant curvature. Arch. Math. Naturw. 5 (1881).

basic isogram and the change in direction of the side bars, a two-parameter movable mechanism is present

The resulting spatial figure, due to the three groups of identical rods and their arrangement, resembles a parallelepiped (Fig. 7). However, instead of flat faces, six skew parallelograms appear. Three rods meeting at each corner lie in a plane, so that the tetrahedra P_1 Q2 P3 QA and QPQP are mutually inscribed and circumscribed, forming a Möbius pair. All 12 edges of the parallelepiped therefore belong to a thread (linear beam complex).

For the proof of existence of the described spatial figure, which is yet to be provided here, the last-mentioned property may be guiding: The edges of the basic isogram P_1 P2 P3 P4 lie in ∞1 threads that fill the bushing spanned by the reciprocal diagonals P_1 P3, P2P. We pick out one of the threads; its axis o must 𝔖 intersect the axis of symmetry of the isogram, which is formed by the common perpendicular of the diagonal pair, at a right angle. For reasons of symmetry, the pairs of points P_1Ps and PP_1 are distributed on two rotating cylinders 2 and 2, with axis o (see Fig. 7).

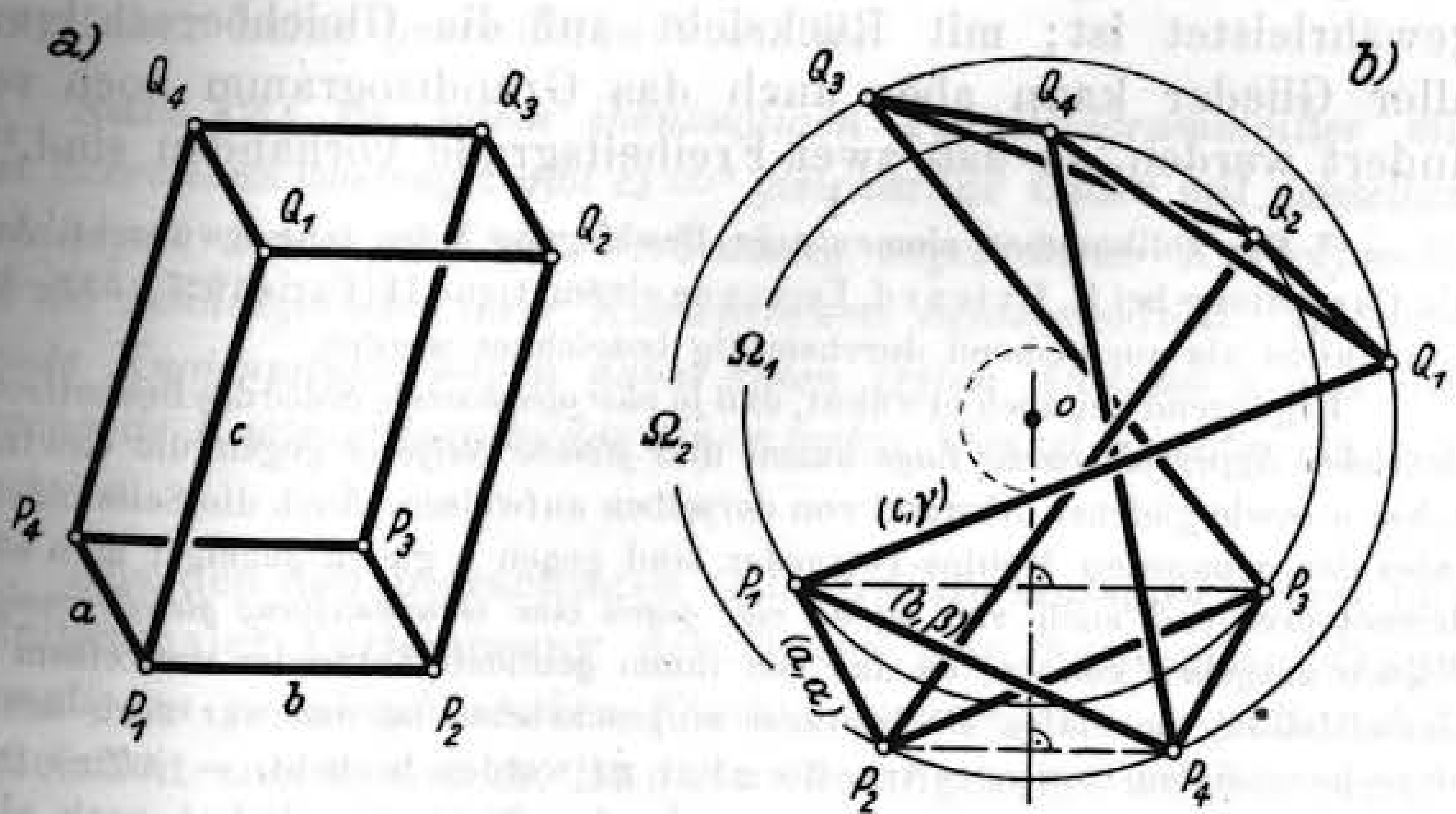

Fig. 7: Bennett's skew-angled parallelepiped. a) Schematic representation; b) Perpendicular projection onto a plane normal to the thread axis

We now rotate the rod P_1c, fixed at P_1, in the nodal plane about P_1 until Q_1 lies on 2 (four possibilities); it always remains in S. Now we perform a series of thread-preserving operations:

1) Screw P_1Ps about 0; Q_1 ends up at Q; PQ.
2) Reflection $P_2 Q_1$ across a normal of 0; P_1 ends up at Q; $P_2 Q_2$, $Q_1 Q_2 \in G$.
3) Reflection P_1Qs across a normal of 0; Ps ends up at Q_4; $P_4 Q_4$, $Q_3 Q_4 \in B$

From Fig. 7, it is immediately apparent that the line segments QQ and Q4 Q1 also arise from P P and P_1 P_1 by axial reflections across normals of o, so that the following line segment equations exist: P_1PP P4 Q1 Q2 Q3 Q4 = a, P2 P3 = PPQs = Q4 Q1 = b, PiQic. = Since, furthermore, all the aforementioned line segments are supported by threaded rays, three rods emanating from each corner lie in one plane (the zero plane of the corner), and all six isograms must therefore exhibit the same twist. Finally, if we allow the aforementioned threaded bundle to pass through, the rod $P_1 Q_1$ continuously changes its direction, thus ensuring the mobility of the structure while maintaining the fixed basic isogram; however, with regard to the equality of all elements, the basic isogram can also be changed, so that two degrees of freedom are present.28

28 A completely elementary proof would be very desirable. The presentation in R. Bricard, Leçons de cinématique II (Paris 1927), 332-336, cannot be described as sufficiently transparent either

It should also be mentioned that any four identical rods of the Bennett mechanism have a hyperboloid position and exhibit the same inclination to the thread axis o and the same distance from it. The side faces of each of the aforementioned Möbius tetrahedra are also inclined equally to o; accordingly, one could also have started with any four planes inclined equally to a basic position, since the tetrahedron formed by them is always inscribed in a rotating cylinder normal to the basic position and only needs to be transformed by means of a coaxial thread. Affine dilation of the Bennett mechanism in the direction o again yields an identical mechanism; in particular, the planar twelve-rod mechanism appearing as a projection in Fig. 7, which contains six antiparallelograms, is also deformable in two parameters.

Let us now start from an even-node rhombic lattice Pik, which we imagine to be assembled according to Section VI from twisted lamellae of length s. We further attach another lamella of the same twist hinged at each node, whose fixed length a and fixed twist angle a thus satisfy the condition

$$a : |\sin \alpha| = s : |\sin \sigma| \quad (20)$$

suffice. After all, all the lamella ends can then be connected by a similar rhombic grid {Qik}, with corresponding meshes forming a "skew parallelepiped" with the four connecting elements. Since the rod length a (<s: sin) can be chosen arbitrarily and the direction of, for example, the rod Poo Loo within the node plane of Poo can be arbitrarily prescribed, we thus have the possibility of deriving ∞ further rhombic grids from any given one. If the original grid {Pik} is fixed, as is the length a of the connecting rods, they still allow rotations around the node points Pik, whereby, according to a remark in Section VII, the endpoints Qik traverse projective point series on their circular paths and the grid {Qik} is constantly changing. The generalization to parallelogram grids is obvious, but offers no particular advantages —

Theorem 12: For every two-node parallelogram lattice with the degree of twisting, there exist similar lattices with the same bar lengths such that the line connecting associated nodes coincides with the line of intersection of their node planes. Corresponding nodes have a fixed distance a and corresponding node planes form a fixed angle α, where sin a = ± τα.

From the lattice transformations considered here, with infinitesimal refinement of the lattices, those famous transformations of pseudospherical surfaces emerge which were first shown in full generality by A. Bäcklund.29 Their properties are immediately recognizable: A given

29 A. Bäcklund: Om ytor med konstant negativ krökning. Lunds Univ. Arsskr. 19 (1883).

Surfaces with constant curvature $K-\tau^2$ can be assigned surfaces of the same curvature such that the lines connecting corresponding points lie in the associated tangent planes; the points have a fixed distance a and the tangent planes form a fixed angle a = ± arcsin ta. For each Bäcklund pair, the oscillation lines correspond to each other, with equal arc lengths and in the manner described at the end of Section VII. Likewise, there is a correspondence between the lines of curvature, which arise from the diagonals of the rhombic grids. The lines connecting corresponding points fulfill a so-called "pseudospheric ray congruence," which is characterized by a constant focal point distance and a constant focal plane angle and possesses the surface pair as a focal pattern The Bäcklund transforms of a surface corresponding to a certain distance a intersect the circles with radius a, which can be described in the tangent planes of the original surface around the points of tangency, at the fixed angle α and generate projective series of points on the same surface. The special case $\alpha = \pi/2$, $\alpha = 1/t$ deserves particular mention: This distinguished Bäcklund transform, usually called the "complementary transformation," had already been discovered earlier by A. Ribaucour and L. Bianchi, the former in connection with the aforementioned circle system.[30]

Finally, let us briefly discuss the remarkable commutation law of Bäcklund transformations, which, according to L. Bianchi, states: If a Bäcklund transformation Ba is applied to a pseudo-spherical surface and a second such transformation B is applied to the result, the same final result can be achieved by two identical transformations B and Ba applied in reverse order. (The indices indicate the intervals.) - The same commutation already applies to the analogous lattice transformations, which are described in

30 G. Darboux: Leçons sur la théorie générale des surfaces, Vol. 3 (Paris 1894), 420 ff.

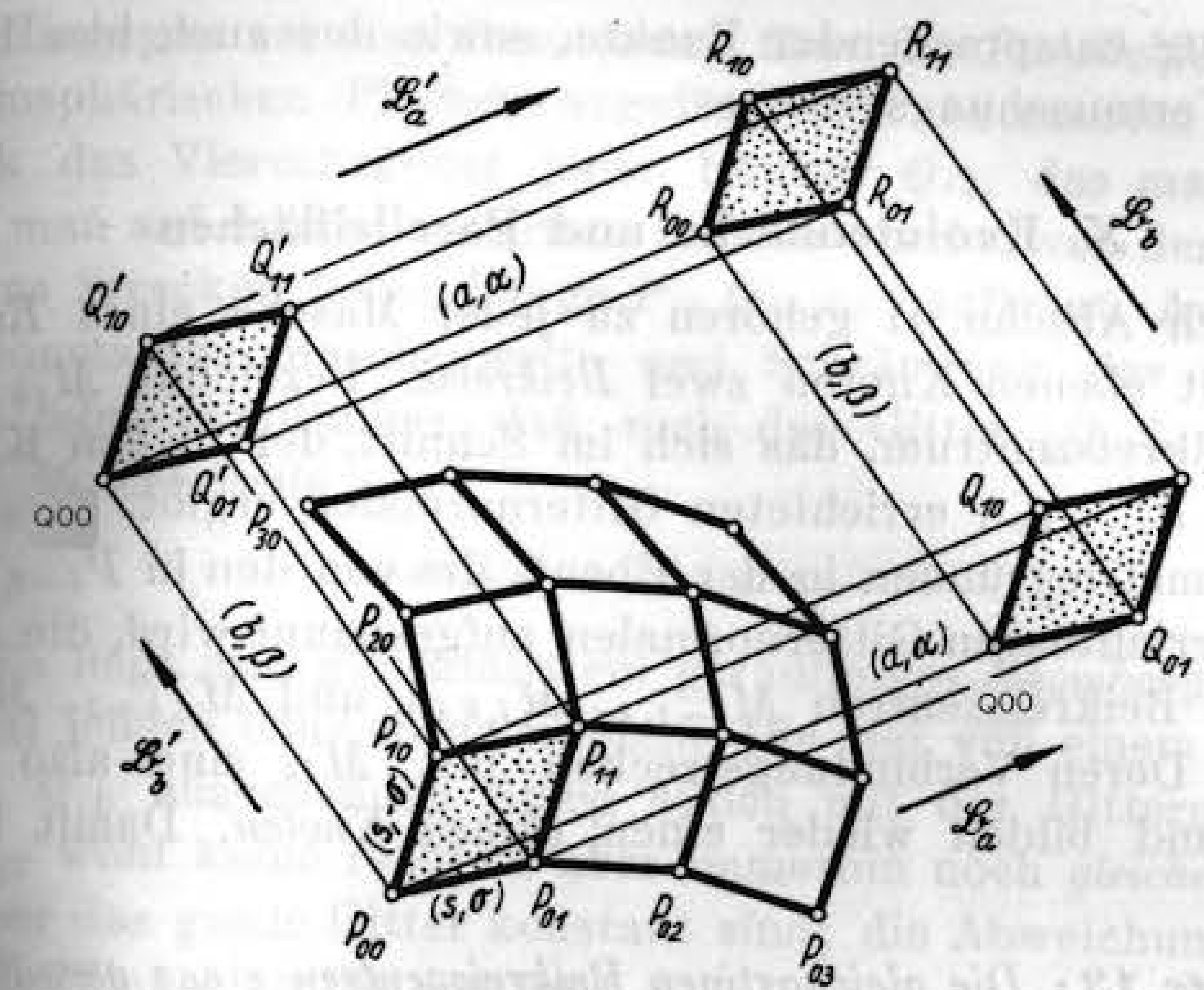

Fig. 8: Schematic of Bianch's commutation theorem smo of the Bäcklund transformations

may be designated by the same symbols. Let the original lattice {Pik} be transformed by a first transformation Ba into the lattice (Q), and this be transformed by B into {Rik}; if we complete the pair of lamellae Poo Qoo Roo to form an isogram whose missing corner is called Qoo, then the rods Poo Qoo and Qoo Roo determine two transformations B and B'a, which lead from Pik} via a lattice {Qik} to the same final lattice {Rik}. The agreement of the final result, starting at Poo and constantly progressing to neighboring nodes, is repeatedly ensured by the existence of skew parallelepipeds (see Fig. 8). Thus, Bianchi's theorem appears to be placed on a completely elementary foundation

Finally, it should be noted that the spherical figure, appearing as a spherical image of a skew parallelepiped and by no means trivial, consists of 8 points and 12 connecting great-circle arcs, four of which are congruent. This figure forms the basis of the transformations of spherical Chebyshev nets with constant , corresponding to the Bäcklund transformations

Distance of corresponding points, as well as the commutation law that also exists here.

X. Evolute Surface and Parallel Surfaces

According to Section II, each mesh of a rhombic lattice with planar nodes has two circumcircles. Let Mik denote, for example, the circumcircle center that results from the intersection of the lattice normals constructed at nodes Pik and Pi-1, 4-1. For reasons of symmetry, Mik lies in the plane spanned by the lattice normals that can be constructed at Pi-1, 4 and Pi, k-1, which in turn carry the circumcircle centers Mi-1, k, Mi, k+1 and Mi, k-1, Mi+1, k. Their connecting lines with Mik are therefore coplanar and again form a planar node. Thus, we have the

Theorem 13: The similar centers of circumcircles of a rectilinear rhombic lattice with planar nodes form a quadrilateral lattice with planar nodes if the centers originating from adjacent meshes are connected by line segments.

From this elementary observation, through the usual limiting process that transforms the rhombic lattice {Pik} into the osculating line network of a pseudospherical surface and thereby transforms the lattice {Mik} into the osculating line network of the central or evolute surface, flows the theorem that the osculating lines correspond to a surface of constant curvature on the evolute surface, in turn to its osculating lines. According to L. Bianchi (op. cit. 243), this correspondence property is even suitable for characterizing surfaces of fixed curvature. enoa154

Other properties of the normal congruence of a surface also become apparent in the normal system of the grid. Tracing the diagonal lines, for example, leads to the lines of curvature with their developable, mutually orthogonally intersecting normal surfaces, whose ridge lines represent geodesics on the lateral surfaces of the evolute surface, etc. However, since these are not phenomena specific to surfaces of constant curvature, these matters will not be discussed in more detail.

However, a remark on the parallel surfaces of the pseudospherical surfaces should be added. For this purpose, we consider the quadrilateral grid of those points Qik that is obtained by plotting line segments of fixed length Pik on the (oriented) grid normals from the nodes Pilk. Due to the constant mesh size and twisting of the original grid {Pik}, it follows directly that the grid {Qik} also has a constant mesh size

$$s' = 1/s^2 + (2l \sin 6/2)^2 \tag{21}$$

has. Thus, we also have a rectilinear rhombic lattice, but this time without planar nodes. The four rods emanating from a node Qik do not form right angles with the lattice normal Pik Qik, but they do form equal angles that are constant over the entire lattice; the deviation μ from a right angle results from sin 21 27 sin σ no

$$\omega = \quad \frac{}{2}. \tag{22}$$

Since four lattice rods converge at a point, they lie on a cone of rotation, one could appropriately speak of "conical nodes" (with constant opening).

Theorem 14: If one plots line segments of equal length in the same direction on the normal of a rectilinear rhombic lattice with planar nodes, one arrives at a rectilinear rhombic lattice with conical nodes of constant opening

From this, by crossing the boundary, we can derive A. Voss's observation that the rhombic osculating line network of a pseudospherical surface corresponds again to rhombic networks on the parallel surfaces.

These Chebyshev networks on the parallel surfaces no longer consist of osculating lines, but of an obvious generalization thereof, namely surface curves of constant normal curvature, which apparently has not yet been noticed. To prove this, let us consider the sphere 2 defined by a node Qik and its four neighbors, whose center obviously lies on the lattice normal supporting Qik

Thus, based on Theorem 14, we immediately recognize that it has a constant radius over the entire grid, the value of which is expressed by means of (21) and (22) by

$$r' = \frac{s'}{2 \sin \omega} = l + \frac{s^2}{4\, l \sin^2 \sigma/2} \tag{23}$$

After carrying out the limit s → 0 (σ/ς - τ), 2 becomes the common Meusnier sphere of the grid curves passing through the surface point; its radius

$$r_n = l + \frac{1}{l \tau^2} \tag{24}$$

is invariant for the grid curves, as is their normal curvature vn = 1/rn

Incidentally, a simple statement can also be made about the geodesic curvature of the grid curves. We consider three neighboring nodes of the original grid, say Poo, P01, P02, along with the corresponding nodes 200, 201, Qus of the derived grid. The normal projection of the triple of points onto the nodal plane of Po is related to the triple P by a rotation-scaling about Po1, whose rotation angle = ± arctg (l sin /s), while the scaling factor has the value sec. From this it follows: The oscillations of a surface of constant negative curvature K = 2 and the corresponding curves on a parallel surface at a distance l have tangents at corresponding points that form the fixed angle arctg lt, and geodesic curvatures that are in the fixed ratio : x = 1: cos.sia

Finally, it should be noted that the derived rhombic lattices {Qik} can also be realized by movable hinge systems. First, imagine a lamella model for the initial lattice {Pik} constructed according to Section VI, with the hinge axes coinciding with the lattice normals implemented as rods of length l and their ends connected by appropriately twisted and bent lamellae. This second lamella system, which is completely equivalent to the first from a kinematic point of view, can of course also exist on its own and represents in every state the

difference-geometric model of the parallel surface of a pseudospherical surface belonging to distance l, together with the distinguished system of its curves of constant normal curvature. These structures can also be connected to form Bäcklund pairs, which would only require the analogous application of the developments from the previous section

This brings the present report on a multifaceted and complex subject area to a provisional close. It seems entirely worthwhile and useful to pursue the path taken in various directions by consistently applying the difference-geometric method.

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